Input to state stability of pulse width modulated control systems

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Abstract

Results on stability and input-to-state stability in pulse-width modulated (PWM) control systems are presented. The results are based on a recent generalization of two time scale stability theory to differential equations with disturbances. In particular, averaging theory for systems with disturbances is used to establish the results. The nonsmooth nature of PWM systems is accommodated by working with upper semicontinuous set-valued maps, locally Lipschitz inflations of these maps, and locally Lipschitz parameterizations of locally Lipschitz set-valued maps.

1 Introduction

Pulse width modulated (PWM) control is a common paradigm in systems where the values of the actuators are discrete, like $\{0,1\}$. The idea is to maintain a constant frequency of switching between the values 0 and 1 and to use measurements from the plant to determine the duty ratio, i.e., the ratio of time spent at 0 to the time spent at 1. This kind of control strategy can be useful for systems controlled by on-off valves and/or on-off switches. PWM is a technique commonly used to control DC-DC boost and buck converters. See, for example, [2, 8, 9, 12].

The main idea used in PWM control, and many other power electronic systems [7], is to model the switching actuator by its average behavior. While the intuition behind this idea is sound, classical averaging theory is not suitable for these discontinuous systems. However, Lehman and Bass [9] have shown how averaging theory can be extended to address switching power electronic systems when the actuators don't switch infinitely often on a finite time interval, i.e., they don't chatter.

For switching power electronic systems where exogenous disturbances are considered and chattering is allowed, perhaps caused by the exogenous disturbances, there has not been a suitable version of averaging stability theory until recently [14]. (Earlier work in this direction can be found

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in [10].) In this paper, we will apply the generalized two time scale stability theory of [14] to study input-to-state stability [11] in PWM control systems.

The paper is organized as follows: In Section 2 we present mathematical preliminaries. PWM control is introduced in Section 3 and an asymptotic stability result for PWM systems without disturbances is stated in Theorem 1. Section 4 contains the proof of Theorem 1. Theorem 2 in Section 5 gives conditions for ISS of PWM systems with disturbances. The proof of Theorem 2 is sketched in Section 6.

2 Preliminaries

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing and unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{KL}$ if it is nondecreasing in it first argument, nonincreasing in its second argument and $\lim_{s\to 0^+} \beta(s,t) = \lim_{t\to\infty} \beta(s,t) = 0$.

Given an open set $\mathcal{H}$ and a compact set $\mathcal{A} \subset \mathcal{H}$, a continuous function $\omega : \mathcal{H} \to \mathbb{R}_{\geq 0}$ is said to be positive definite with respect to $\mathcal{A}$ if $\omega(x) = 0 \Leftrightarrow x \in \mathcal{A}$. It is said to be proper with respect to $\mathcal{H}$ if for each sequence x_i approaching the boundary of $\mathcal{H}$ or approaching infinity, we have $\omega(x_i) \to \infty$ as $i \to \infty$.

The open and closed unit balls are denoted $\mathcal{B}$ and $\overline{\mathcal{B}}$, respectively. A set $\mathcal{F}$ is convex if for each $x_1, x_2 \in \mathcal{F}$ we have that $\lambda x_1 + (1-\lambda)x_2 \in \mathcal{F}, \forall \lambda \in [0,1]$. Given a set $\mathcal{F}$, the closed convex hull of the set, denoted as $\overline{\text{co}}\mathcal{F}$, is the smallest closed convex set containing $\mathcal{F}$.

For a function $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^r$, we define

$$f(y, \overline{\mathcal{B}}) := \left\{ w : w = f(y,x) \ , \ x \in \overline{\mathcal{B}}) \right\} \ . \tag{1}$$

A set-valued map $F : \mathbb{R}^n \to$ (subsets of $\mathbb{R}^n$) is upper semicontinuous if for each x and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(x + \delta\mathcal{B}) \subseteq F(x) + \varepsilon\mathcal{B} \tag{2}$$

where $x + \delta\mathcal{B}$ denotes the ball of radius δ around the point x. A set-valued map $F_L : \mathbb{R}^n \to$ (subsets of $\mathbb{R}^n$) is said to be locally Lipschitz if, for each x there exists a neighborhood $\mathcal{U}$ of x and $L > 0$ such that

$$x_1, x_2 \in \mathcal{U} \qquad \Longrightarrow \qquad F_L(x_1) \subseteq F_L(x_2) + L|x_1 - x_2|\overline{\mathcal{B}} \ . \tag{3}$$

If a set-valued map is locally Lipschitz then it is upper semicontinuous. For the notion of a measurable set-valued map, we refer the reader to [1, Section 8.1]. We will use the integral of a set-valued map which corresponds to the set of integrals of integrable selections from the set-valued map. See [1, Section 8.6].

For a discontinuous differential equation

$$\dot{x} = f(x,t) \tag{4}$$

where f is locally bounded and measurable, the generalized Krasovksii, respectively Filippov, solutions of (4) correspond to the solutions of the differential inclusion

$$\dot{x} \in F(x,t) := \bigcap_{\delta > 0} \overline{\text{co}} f(x + \delta\mathcal{B}, t) \tag{5}$$

respectively

$$\dot{x} \in F(x,t) := \bigcap_{\delta > 0} \ \bigcap_{\text{meas}(N)=0} \overline{\text{co}} f((x + \delta\mathcal{B} \backslash N), t) \ . \tag{6}$$

(The set-valued maps on the right-hand sides are upper semicontinuous in x. See, e.g., [4, p. 85].) The generalized solution concepts of Filippov and Krasovskii agree for the systems that we consider in the sequel. For a general comparison, see [6].

3 Pulse width modulated control

In pulse width modulated control systems, the closed-loop system has the form

$$\dot{x} = \varepsilon \left[f(x) + \sum_{i=1}^{m} g_i(x) u(h_i(x) - p_i(t)) \right] \tag{7}$$

where ε is a small positive parameter, $u : \mathbb{R} \to [0,1]$ is the unit step function ($u(0) = 1$), the functions $h_i : \mathbb{R}^n \to [0,1]$, f and g_i are continuous, and the functions $p_i : \mathbb{R} \to [0,1]$ are measurable, bounded and periodic with period one. Often they have the form $p_i(t) = t \bmod 1$, as in Figure 1. (Actually, the period of p_i is typically parameterized by ε and the system (7) represents the original system in a transformed time scale.)

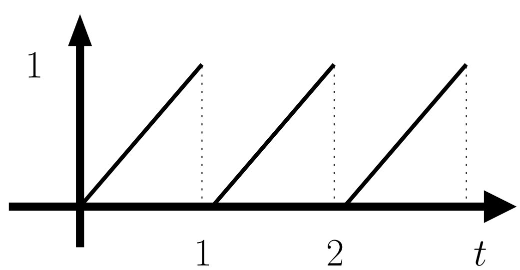

Figure 1: A typical (triangle) switching signal $p_i(t)$ in PWM control systems.

The behavior of the closed-loop system hinges on the nondecreasing, possibly discontinuous functions

$$v \mapsto \sigma_i(v) := \int_0^1 u(v - p_i(t))dt = \text{meas}\,\{t \in [0,1] : v \geq p_i(t)\} \tag{8}$$

which takes values in $[0,1]$, and its corresponding upper semicontinuous set-valued map

$$v \mapsto S_i(v) := \bigcap_{\mu > 0} \overline{\text{co}}\ \sigma_i(v + \mu\mathcal{B}) \ . \tag{9}$$

At points v where $\sigma_i(\cdot)$ is continuous, we have $S_i(v) = \{\sigma_i(v)\}$. The upper semicontinuity of S_i is standard, since S_i is constructed in the same way that a Filippov or Krasovskii differential inclusion is constructed.

If $p_i(t) \equiv 0$ then $\sigma_i(v) = u(v) = \frac{1}{2}\left[\text{sgn}(v) + 1\right]$ and the closed-loop system (7) can be related to sliding mode control. If, on the other hand, $p_i(t) = t \bmod 1$, as in Figure 1, we have $S_i(v) = \{v\}$ for all $v \in [0,1]$. For example, on the interval $[0,1]$, $u(0.8 - (t \bmod 1))$ spends 0.8 seconds at the value 1 and 0.2 seconds at the value 0, and the integral over the period $[0,1]$ is equal to 0.8. This situation is illustrated in Figure 2.

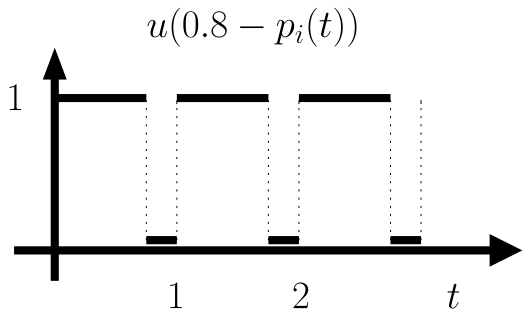

Figure 2: Duty ratio for $v = 0.8$ using $p_i(t) = t \bmod 1$.

The main result in pulse width modulated control systems of the form (7) is the following: Since $\varepsilon > 0$ is small, the state x changes slowly compared to p_i and so the effect of p_i on (7) can be averaged, as in (8), and the analysis reduced to the analysis of the system

$$\dot{x} \in f(x) + \sum_{i=1}^{m} g_i(x) S_i(h_i(x)) \tag{10}$$

which becomes simply

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) h_i(x) \tag{11}$$

when $p_i(t) = t \bmod 1$, using that h_i takes values in $[0, 1]$.

The following formal statement is enabled by the results in [14]:

Theorem 1 *Suppose the functions f, g_i, h_i are continuous and that for (10) the compact set $\mathcal{A}$ is asymptotically stable with basin of attraction $\mathcal{H}$. Under these conditions, the set $\mathcal{H}$ is open and*

- *for each continuous function $\omega : \mathcal{H} \to \mathbb{R}_{\geq 0}$ that is positive definite with respect to $\mathcal{A}$ and proper with respect to $\mathcal{H}$, there exists $\beta \in \mathcal{KL}$,*
- *and, for each $\delta > 0$ and compact $\mathcal{K} \subset \mathcal{H}$, there exists $\varepsilon^* > 0$*

such that

$$\varepsilon \in (0, \varepsilon^*] \, , \quad x(t_\circ) \in \mathcal{K} \qquad \Longrightarrow$$

the (generalized Krasovskii/Filippov) solutions of (7) exist for all $t \geq t_\circ$ and satisfy

$$\omega(x(t)) \leq \beta\left(\omega(x(t_\circ)), \varepsilon(t - t_\circ)\right) + \delta, \qquad \forall t \geq t_\circ \, . \tag{12}$$

4 Proof of main result

4.1 The function β

As we have asserted above, the set valued map

$$x \mapsto f(x) + \sum_{i=1}^{m} g_i(x) S_i(h_i(x)) =: F(x) \tag{13}$$

is upper semicontinuous, and for each x, is nonempty, compact and convex. According to [13, Proposition 3], the set $\mathcal{H}$ is open and for each $\omega : \mathcal{H} \to \mathbb{R}_{\geq 0}$ that is continuous, positive definite with respect to $\mathcal{A}$ and proper with respect to $\mathcal{H}$, there exists $\beta_\circ \in \mathcal{KL}$ such that all solutions of the system (10) that start in $\mathcal{H}$ satisfy

$$\omega(x(t)) \leq \beta_\circ(\omega(x(0)), t) \qquad \forall t \geq 0 \ . \tag{14}$$

The general averaging results in [14] give conditions under which the bound (14), which holds for the averaged system, also holds with an arbitrarily small offset for the actual system. To use the explicit sufficient conditions given in [14], $F(\cdot)$ should be locally Lipschitz rather than just upper semicontinuous. The next computations are aimed at obtaining a bound like (14) for a system $\dot{x} \in F_L(x)$ where $F(x) \subseteq F_L(x)$ and F_L is locally Lipschitz.

According to the combination of [13, Proposition 2, Theorem 3, and Theorem 1], there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a smooth function $V : \mathcal{H} \to \mathbb{R}_{\geq 0}$ such that, for all $x \in \mathcal{H}$,

$$\alpha_1(\omega(x)) \leq V(x) \leq \alpha_2(\omega(x)) \tag{15}$$

and

$$\max_{w \in F(x)} \langle \nabla V(x), w \rangle \leq -V(x) \ . \tag{16}$$

Define

$$\beta(s,t) := \alpha_1^{-1}\left(\alpha_2(s) e^{-\frac{t}{2}}\right) \ . \tag{17}$$

4.2 The value ε^*

4.2.1 Embedding (10) in a Lipschitz system

Let $\delta > 0$ and the compact set $\mathcal{K} \subset \mathcal{H}$ be given. Using that ω is proper with respect to $\mathcal{H}$ and perhaps by enlarging $\mathcal{K}$ and thereby considering a larger set of initial conditions, we can assume without loss of generality that, with the definition

$$c := \max_{x \in \mathcal{K}} \omega(x) \ , \tag{18}$$

we have

$$\{x : \omega(x) \leq \delta\} \subseteq \mathcal{K} \ , \qquad \& \qquad \delta \leq 2\alpha_1^{-1} \circ \alpha_2(c) \ . \tag{19}$$

Using (17), it follows that $\delta/2 \leq \beta(c, 0)$.

Following the proof of [13, Lemma 18], which relies on the continuity of $x \mapsto \nabla V(x)$, there exist two strictly positive real numbers ρ_1 and ρ_2 such that, with the definitions

$$F_1(x) := \overline{\text{co}} F(x + \rho_1 \overline{\mathcal{B}}) + \rho_1 \overline{\mathcal{B}} \tag{20}$$

and

$$F_2(x) := \overline{\text{co}} F_1(x + \rho_2 \overline{\mathcal{B}}) + \rho_2 \overline{\mathcal{B}} \ , \tag{21}$$

we have

$$\alpha_1\left(\frac{\delta}{2}\right) \leq V(x) \leq \alpha_2(c) \qquad \Longrightarrow \qquad \max_{w_2 \in F_2(x)} \langle \nabla V(x), w_2 \rangle \leq -\frac{1}{2} V(x) \ . \tag{22}$$

Using [13, Lemma 8], there exists a set-valued map $F_L : \mathbb{R}^n \to$ (subsets of $\mathbb{R}^n$) with nonempty compact, convex values that is locally Lipschitz and satisfies

$$F_1(x) \subseteq F_L(x) \subseteq F_2(x) \qquad \forall x \; . \tag{23}$$

Combining (22) and (23) we have

$$\alpha_1\left(\frac{\delta}{2}\right) \leq V(x) \leq \alpha_2(c) \qquad \Longrightarrow \qquad \max_{w\in F_L(x)} \langle \nabla V(x), w\rangle \leq -\frac{1}{2}V(x) \; . \tag{24}$$

It follows, like in [11, p. 441], that the solutions of

$$\dot{x} \in \varepsilon F_L(x) \tag{25}$$

starting in $\mathcal{K}$ satisfy

$$\omega(x(t)) \leq \max\left\{ \beta(\omega(x(0)), \varepsilon t) \; , \; \frac{\delta}{2} \right\} \qquad \forall t \geq 0 \; . \tag{26}$$

4.2.2 Embedding (7) in a continuous system with average contained in F_L

We define $U(s) := \bigcap_{\mu>0} \overline{\mathrm{co}}\; u\,(s + \mu\mathcal{B})$, i.e.,

$$U(s) = \begin{cases} 1 & s > 0 \\ 0 & s < 0 \\ [0,1] & s = 0 \end{cases} \tag{27}$$

and note that the Krasovskii/Filippov solutions of (7) correspond to the solutions of

$$\dot{x} \in \varepsilon\left[f(x) + \sum_{i=1}^{m} g_i(x) U(h_i(x) - p_i(t)) \right] =: \varepsilon \widetilde{F}(x,t) \; . \tag{28}$$

We define

$$\mathcal{X} := \{x : \omega(x) \leq \beta(c,0)\} \; . \tag{29}$$

It follows from the properties of ω that $\mathcal{X}$ is a compact subset of $\mathcal{H}$, and thus there exists $\nu > 0$ be such that

$$\mathcal{C} := \mathcal{X} + \nu\overline{\mathcal{B}} \subset \mathcal{H} \; . \tag{30}$$

Claim 1 *There exist $M > 0$ and a set-valued map $(x,t) \mapsto \widetilde{F}_c(x,t)$ such that*

1.
$$(x,t) \in \mathcal{C}\times\mathbb{R} \qquad \Longrightarrow \qquad \widetilde{F}(x,t) \subseteq \widetilde{F}_c(x,t) \subseteq M\overline{\mathcal{B}} \; ; \tag{31}$$

2. *$\widetilde{F}_c(x,t)$ is nonempty, compact and convex for each $(x,t) \in \mathcal{C}\times\mathbb{R}$;*

3. *the mapping $t \mapsto \widetilde{F}_c(x,t)$ is measurable for each $x \in \mathcal{C}$;*

4. for each $\widetilde{\varepsilon} > 0$ there exists $\widetilde{\delta} > 0$ such that

$$x_1, x_2 \in \mathcal{C}\ ,\ |x_1 - x_2| \le \widetilde{\delta} \qquad \Longrightarrow \qquad \widetilde{F}_c(x_1, t) \subseteq \widetilde{F}_c(x_2, t) + \widetilde{\varepsilon}\overline{\mathcal{B}}\ ; \tag{32}$$

5. there exists $T^* > 0$ such that

$$(x, t_\circ) \in \mathcal{C} \times \mathbb{R}\ , \quad T \ge T^* \quad \Longrightarrow \quad \frac{1}{T}\int_0^T \widetilde{F}_c(x, t + t_\circ)dt \subseteq F_L(x)\ . \tag{33}$$

Proof of claim: We define

$$M := \max_{x \in \mathcal{C}} \left(|f(x)| + \sum_{i=1}^m |g_i(x)| \right)\ . \tag{34}$$

Recall the value ρ_1 used in (20). We define

$$U_c(s) = \begin{cases} 0 & s < -\frac{\rho_1}{2} \\ \left[0, \frac{2s}{\rho_1} + 1\right] & s \in \left[-\frac{\rho_1}{2}, 0\right] \\ \left[\frac{2s}{\rho_1}, 1\right] & s \in \left[0, \frac{\rho_1}{2}\right] \\ 1 & s > \frac{\rho_1}{2} \end{cases} \tag{35}$$

and

$$\widetilde{F}_c(x, t) := f(x) + \sum_{i=1}^m g_i(x) U_c(h_i(x) - p_i(t))\ . \tag{36}$$

It is clear that items 2 and 3 of the claim hold. Since

$$U(s) \subseteq U_c(s) \subseteq U\left(s + \frac{\rho_1}{2}\overline{\mathcal{B}}\right) \qquad \forall s \tag{37}$$

it follows from (28) and (34) that (31) holds. Moreover, since the set-valued mapping U_c is globally Lipschitz with Lipschitz constant $2/\rho_1$ and the functions f, g_i and h_i are continuous, and thus uniformly continuous on $\mathcal{C}$, it follows that item 4 holds.

It remains to establish item 5. Since

$$\frac{1}{T}\int_0^T \widetilde{F}_c(x, t + t_\circ)dt = f(x) + \sum_{i=1}^m g_i(x) \frac{1}{T}\int_0^T U_c(h_i(x) - p_i(t + t_\circ))dt \tag{38}$$

and (13), (20) and (23) hold, it is enough to show that there exists $T^* > 0$ such that for all $t_\circ \in \mathbb{R}$, $v \in \mathbb{R}$ and $T \ge T^*$,

$$\frac{1}{T}\int_0^T U_c(v - p_i(t + t_\circ))dt \subseteq S_i(v + \rho_1\overline{\mathcal{B}}) + \rho_1\overline{\mathcal{B}}\ . \tag{39}$$

We claim that this condition holds with $T^* = 2\rho_1^{-1}$. To see this we first observe, using the second inclusion in (37), the definition of $\sigma_i(\cdot)$ in (8), and the definition of S_i in (9) that

$$\begin{aligned} \int_0^1 U_c(v - p_i(t))dt &\subseteq \int_0^1 U\left(v - p_i(t) + \frac{\rho_1}{2}\overline{\mathcal{B}}\right)dt \\ &\subseteq \left[\sigma_i\left(v - \frac{3}{4}\rho_1\right), \sigma_i\left(v + \frac{3}{4}\rho_1\right)\right] \\ &\subseteq S_i(v + \rho_1\overline{\mathcal{B}})\ . \end{aligned} \tag{40}$$

Next, we break the integral on the left-hand side of (39) into integration over the intervals $[0,1]$, $[1,2]$, …, $[n-1,n]$, $[n,T]$ where n is the largest integer not larger than T, and we denote the value of the jth integral by w_j for $j = 1,\dots,n+1$. Since $U_c(s) \subseteq U\left(s+\frac{1}{2}\rho_1\overline{\mathcal{B}}\right) \subseteq [0,1]$ for all s, it follows that $|w_j| \le 1$ for all $j = 1,\dots,n+1$. Also, using (40) and that $p_i(\cdot)$ is periodic with period one, $w_j \in S_i(v+\rho_1\overline{\mathcal{B}})$ for all $j = 1,\dots,n$. Since $S_i(v+\rho_1\overline{\mathcal{B}})$ is convex by definition, the convex combination $n^{-1}\sum_{j=1}^{n} w_i \in S_i(v+\rho_1\overline{\mathcal{B}})$. So, using $T \ge 2\rho_1^{-1}$, the integral on the left-hand side of (39) can be written as

$$\begin{aligned}
T^{-1}\sum_{j=1}^{n+1} w_i &= n^{-1}\sum_{j=1}^{n} w_i + (T^{-1}-n^{-1})\sum_{j=1}^{n} w_i + T^{-1}w_{n+1} \\
&\in S_i(v+\rho_1\overline{\mathcal{B}}) + \left(T^{-1}(T-n)+T^{-1}\right)\overline{\mathcal{B}} \qquad (41)\\
&\subseteq S_i(v+\rho_1\overline{\mathcal{B}}) + 2T^{-1}\overline{\mathcal{B}} \\
&\subseteq S_i(v+\rho_1\overline{\mathcal{B}}) + \rho_1\overline{\mathcal{B}}\ ,
\end{aligned}$$

i.e., (39) holds. ■

4.2.3 Working with enlarged systems

We now have arrived at the situation where we have that:

1. the trajectories of (7) that remain in $\mathcal{C} = \mathcal{X} + \nu\overline{\mathcal{B}}$ are subsumed by the trajectories of the system
$$\dot{x} \in \varepsilon\widetilde{F}_c(x,t) \subseteq \varepsilon M\overline{\mathcal{B}} \qquad (42)$$
where $\widetilde{F}_c(\cdot,t)$ is continuous uniformly in t, and $\widetilde{F}_c(x,\cdot)$ is measurable.

2. the trajectories of
$$\dot{x}_{avg} \in \varepsilon F_L(x_{avg})\ , \qquad (43)$$
where F_L is locally Lipschitz, satisfy
$$x_{avg}(0) \in \mathcal{K} \qquad \Longrightarrow \qquad \omega(x_{avg}(t)) \le \max\left\{\beta(\omega(x_{avg}(0)),\varepsilon t)\ ,\ \frac{\delta}{2}\right\}\ . \qquad (44)$$
In particular, the trajectories of (43) don't leave the compact set $\mathcal{X}$;

3. on $\mathcal{C}$ and for sufficiently large T, the set-valued maps $\widetilde{F}_c$ and F_L are related by
$$\frac{1}{T}\int_0^T \widetilde{F}_c(x,t+t_\circ)dt \subseteq F_L(x)\ . \qquad (45)$$

These conditions lead to the following statement:

Proposition 1 *There exists $T^* > 0$ and for each $T \ge T^*$ there exists $\varepsilon^* > 0$ such that for each $\varepsilon \in (0,\varepsilon^*]$, each $x_\circ \in \mathcal{K}$, each $t_\circ \in \mathbb{R}$ and each solution $x(\cdot)$ of (42) satisfying $x(t_\circ) = x_\circ$ there exists a solution $x_{avg}(\cdot)$ of (43) satisfying $x(t_\circ) = x_\circ$ and*

$$\omega(x(t)) \le \omega(x_{avg}(t)) + \frac{\delta}{2} \qquad \forall t \in \left[t_\circ, t_\circ + \frac{T}{\varepsilon}\right] \qquad (46)$$

and

$$x(t) \in \mathcal{K} \qquad \forall t \in \left[t_\circ + \frac{T^*}{\varepsilon}, t_\circ + \frac{T}{\varepsilon} \right] \ . \tag{47}$$

We defer the proof of Proposition 1. First we will show how it gets used to prove the bound (12) for the trajectories of (42) starting in $\mathcal{K}$, and hence enables the proof of Theorem 1.

Claim 2 *For sufficiently small ε, the bound (12) holds for the trajectories of (42) starting in $\mathcal{K}$.*

Proof. Let $T^* > 0$ come from Proposition 1, and recall the definition of c in (18). Let $T \geq T^*$ be such that

$$\beta(c, \tau) \leq \frac{\delta}{2} \qquad \forall \tau \in \left[\frac{T}{2}, \infty \right) \ . \tag{48}$$

Let $\varepsilon^* > 0$ come from Proposition 1 for this T. Let $\varepsilon \in (0, \varepsilon^*]$. Now we have, for all $t \in [t_\circ, t_\circ + T/\varepsilon]$,

$$\begin{aligned} \omega(x(t)) &\leq \omega(x_{avg}(t)) + \frac{\delta}{2} \\ &\leq \max\left\{ \beta(\omega(x_\circ), \varepsilon(t - t_\circ)) \ , \ \frac{\delta}{2} \right\} + \frac{\delta}{2} \ . \end{aligned} \tag{49}$$

Using (48) it also follows that for $t \in [t_\circ + T/(2\varepsilon), t_\circ + T/\varepsilon]$,

$$\omega(x(t)) \leq \delta \tag{50}$$

and $x(t_\circ + T/(2\varepsilon)) \in \mathcal{K}$. The above argument can now be applied repeatedly to obtain

$$\omega(x(t)) \leq \delta \qquad \forall t \in [t_\circ + T/(2\varepsilon), \infty) \ . \tag{51}$$

The conclusion then follows by combining (49) and (51). ■

Remark 4.1 *This proof technique is illustrated graphically in Figure 3. The solid line, in both plots, is the graph of $\omega(x(t))$ (on a time scale with $\varepsilon = 1$ and time indicated relative to $t_\circ$). Over the initial T seconds, $\omega(x(t))$ is compared to $\omega(x_{avg}(t))$, indicated by the dashed graph in the upper plot, where $x_{avg}(\cdot)$ is a solution of the averaged system that is close to $x(\cdot)$. After information about $\omega(x(t))$ over the first T seconds is derived, a new close solution of the average system is introduced and considered for T seconds starting $T/2$ seconds after the original initial time. This solution is indicated by the dashed curve in the lower plot. It gives information about $\omega(x(t))$ between T and $3T/2$. This process is continued iteratively, as suggested by the vertical dashed line at the bottom of the figure, to arrive at the conclusion about $\omega(x(t))$ for all time.* ■

4.3 Proof of Proposition 1

We make use of the following result, which is a combination of [1, Theorem 9.6.2, Theorem 9.7.1].

Lemma 1 *If $F : \mathbb{R}^n \times \mathbb{R} \to$ (subsets of $\mathbb{R}^n$) has compact, convex values and is such that $F(\cdot, t)$ is continuous (respectively, locally Lipschitz) uniformly in t and $F(x, \cdot)$ is measurable then there exists a function $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that*

1. *$f(x, t, \overline{\mathcal{B}}) = F(x, t)$;*

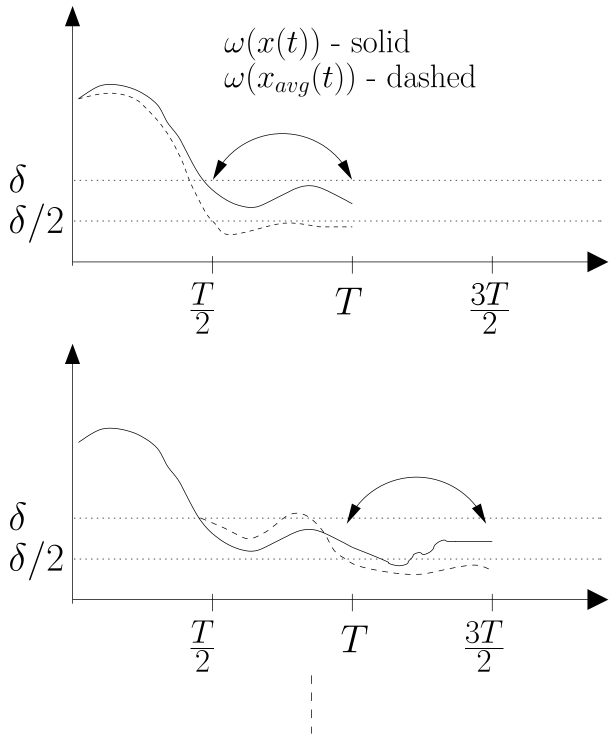

Figure 3: Graphical illustration of proof technique for Claim 2.

2. *$f(\cdot, t, d)$ is continuous (respectively, locally Lipschitz) uniformly in $d \in \overline{\mathcal{B}}$ and t;*

3. *$f(x, \cdot, d)$ is measurable;*

4. *$f(x, t, \cdot)$ is continuous.*

Via what is known as Filippov's lemma (see [3, Exercise 3.7.20] or [1, Theorem 8.2.10]), it can be shown that the solutions of $\dot{x} \in F(x,t)$ are the same as the solutions of $\dot{x} = f(x, t, d(t))$, where $d(\cdot)$ ranges over the set of measurable functions taking values in $\overline{\mathcal{B}}$.

Lemma 2 (Filippov (special case)) *Assume $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ has the properties in the conclusion of Lemma 1. Let $x(\cdot)$ be absolutely continuous and satisfy, for almost all t,*

$$\dot{x}(t) \in f(x(t), t, \overline{\mathcal{B}}) \ . \tag{52}$$

Then there exists a measurable function $d : \mathbb{R} \to \overline{\mathcal{B}}$ such that, for almost all t,

$$\dot{x}(t) = f(x(t), t, d(t)) \ . \tag{53}$$

Remark 4.2 *In the notation of [3, Exercise 3.7.20], $\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is given by $\varphi(t, u) = \dot{x}(t) - f(x(t), t, u)$ (with $\dot{x}(t)$ defined arbitrarily in $f(x(t), t, \overline{\mathcal{B}})$ for times (a set, if nonempty, of measure zero) where it is not otherwise defined), and $\Gamma = \overline{\mathcal{B}}$.* ■

Using the above results, we now know that the enlarged problem is the same as the problem where we have that:

1. the trajectories of (7) that remain in $\mathcal{C} = \mathcal{X} + \nu\overline{\mathcal{B}}$ are subsumed by the trajectories of

$$\dot{x} = \varepsilon f_c(x, t, d) \subseteq \varepsilon M \overline{\mathcal{B}} \tag{54}$$

   where $d : \mathbb{R} \to \overline{\mathcal{B}}$ is measurable, $f_c(\cdot, t, d)$ is continuous, uniformly in t and d, $f_c(x, \cdot, d)$ is measurable, $f_c(x, t, \cdot)$ is continuous;

2. the trajectories of

$$\dot{x}_{avg} = \varepsilon f_\ell(x_{avg}, e) \tag{55}$$

   where $e : \mathbb{R} \to \overline{\mathcal{B}}$ is measurable, f_ℓ is locally Lipschitz, satisfy (44); in particular, the trajectories of (55) don't leave the compact set $\mathcal{X}$;

3. on $\mathcal{C}$ and for T sufficiently large, f_c and f_ℓ are related by: for each measurable $d : \mathbb{R} \to \overline{\mathcal{B}}$, T and $t_\circ$, there exists $e \in \overline{\mathcal{B}}$ such that

$$\frac{1}{T} \int_0^T f_c(x, t + t_\circ, d(t)) dt = f_\ell(x, e) \ . \tag{56}$$

In this situation, the result of Proposition 1 follows from (the proof of) [14, Proposition 8].

5 Input-to-state stability results

It is clear from the development of the proof that it should be straightforward to handle the situation where certain types of disturbances d appear explicitly in the differential equation (7). For example, we could consider

$$\dot{x} = \varepsilon \left[f(x, d_f, d_s) + \sum_{i=1}^{m} g_i(x, d_s) u(h_i(x, d_s) - p_i(t)) \right] \tag{57}$$

where $d_f : \mathbb{R} \to \mathcal{V}_f$ represents an arbitrary, measurable function and $d_s : \mathbb{R} \to \mathcal{V}_s$ is "slowly varying". We will denote these classes of functions $\mathcal{D}_f$ and $\mathcal{D}_{s,\varepsilon}$ respectively. More precisely (cf. [14, Assumptions 1 and 2]):

Assumption 1 *The sets $\mathcal{V}_s$ and $\mathcal{V}_f$ are compact.*

Assumption 2 *The set $\mathcal{D}_{s,\varepsilon}$ is such that*

1. *it is shift invariant, i.e., if $t \mapsto d_s(t)$ belongs to $\mathcal{D}_{s,\varepsilon}$ then $t \mapsto d_s(t + t_\circ)$ belongs to $\mathcal{D}_{s,\varepsilon}$ for all $t_\circ$.*

2. *For each $T > 0$ and $\rho > 0$ there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$,*

$$d_s \in \mathcal{D}_{s,\varepsilon} \qquad \Longrightarrow \qquad |d_s(t) - d_s(0)| \leq \rho \qquad \forall t \in [0, T] \ . \tag{58}$$

For simplicity, we will assume:

Assumption 3 *The functions f, g_i and h_i are locally Lipschitz and $p_i(t) = t \bmod 1$.*

In this case, the natural "averaged" system to consider is

$$\dot{x} = \varepsilon \left[f(x, d_f, d_s) + \sum_{i=1}^{m} g_i(x, d_s) \mathrm{sat}_{01}(h_i(x, d_s)) \right] . \tag{59}$$

Here we enforce that h_i takes values in $[0, 1]$ by passing h_i through the function sat_{01} defined to be the nondecreasing function that is linear on $[0, 1]$ and with range $[0, 1]$. In fact, the actual average system that we will consider is

$$\dot{x} = \varepsilon \left[f(x, d_f, d_s) + \sum_{i=1}^{m} g_i(x, d_s) \mathrm{sat}_{01}(h_i(x, d_s) + e) \right] =: \varepsilon f_{avg}(x, d_f, d_s, e) \tag{60}$$

where e is measurable and takes values in $\rho_3 \overline{\mathcal{B}}$ where $\rho_3 > 0$ is arbitrarily small. We will use $\mathcal{E}$ to denote this class of functions. We note that f_{avg} is locally Lipschitz. We assume:

Assumption 4 *Given continuous, positive semidefinite functions ω_o and ω_i and the function $\beta \in \mathcal{KL}$ satisfy*

1. *ω_o is proper with respect to $\mathcal{H}$;*
2. *the trajectories of the system (60) satisfy (compare with (26) or (44))*

$$\omega_o(x(t)) \leq \max \{\beta(\omega_o(x(0)), \varepsilon t), ||\omega_i(d_f, d_s)||_\infty\} \tag{61}$$

for all initial conditions in an open set $\mathcal{H}$ and all measurable $(d_f, d_s, e) \in \mathcal{D}_f \times \mathcal{D}_{s,\varepsilon} \times \mathcal{E}$.

Remark 5.1 *We note that $\omega_i(\cdot, \cdot)$ may be strictly positive to account for the fact that e does not appear explicitly in the bound (61). E.g., we may have*

$$\omega_i(d_f, d_s) = \widetilde{\omega}_i(d_f, d_s) + \gamma(\rho_3) \tag{62}$$

where $\gamma(\cdot)$ and $\widetilde{\omega}_i(\cdot, \cdot)$ are positive definite, and ρ_3 represents the worst case infinity norm for $e \in \mathcal{E}$. ■

We can then make the following statement:

Theorem 2 *Under the Assumptions 1-4, for each $\delta > 0$ and compact set $\mathcal{K} \subset \mathcal{H}$ there exists $\varepsilon^* > 0$ such that*

$$\varepsilon \in (0, \varepsilon^*] \ , \ x(t_\circ) \in \mathcal{K} \ , \ (d_f, d_s) \in \mathcal{D}_f \times \mathcal{D}_{s,\varepsilon} \quad \Longrightarrow \tag{63}$$

the (generalized Krasovskii/Filippov) solutions of (57) exist for all $t \geq t_\circ$ and satisfy

$$\omega_o(x(t)) \leq \max \{\beta(\omega_o(x(t_\circ)), \varepsilon(t - t_\circ)), ||\omega_i(d_f, d_s)||_\infty\} + \delta \ . \tag{64}$$

6 Proof of Theorem 2

We follow the proof of Theorem 1, however ω (which now is ω_o) and β are already given and the averaged system is already locally Lipschitz. Therefore, after taking the definition of c from (18), we pick up the proof at Section 4.2.2. We define $U(s)$ as in (27) and note that the Krasovskii/Filippov solutions of (57) correspond to the solutions of (cf. (28))

$$\dot{x} \in \varepsilon \left[f(x, d_f, d_s) + \sum_{i=1}^{m} g_i(x, d_s) U(h_i(x, d_s) - p_i(t)) \right] =: \varepsilon \widetilde{F}(x, t, d_f, d_s) \tag{65}$$

We define (cf. (29))

$$\mathcal{X} := \left\{ x : \omega_o(x) \leq \max \left\{ \beta(c, 0), \max_{(d_f, d_s) \in \mathcal{V}_f \times \mathcal{V}_s} \omega_i(d_f, d_s) \right\} \right\} . \tag{66}$$

Like before, it follows from the properties of $\omega_\circ$ that $\mathcal{X}$ is a compact subset of $\mathcal{H}$, and thus there exists $\nu > 0$ be such that (cf. (30))

$$\mathcal{C} := \mathcal{X} + \nu \overline{\mathcal{B}} \subset \mathcal{H} . \tag{67}$$

We define Φ to be the set of measurable functions taking values in $\overline{\mathcal{B}}$. Then we have the following claim, which parallels Claim 1:

Claim 3 *There exist $M > 0$ and a map $(x, t, d_f, d_s, \phi) \mapsto \widetilde{f}_c(x, t, d_f, d_s, \phi)$ such that*

1.

$$(x, t, d_f, d_s) \in \mathcal{C} \times \mathbb{R} \times \mathcal{V}_f \times \mathcal{V}_s \quad \Longrightarrow \quad \widetilde{F}(x, t, d_f, d_s) \subseteq \widetilde{f}_c(x, t, d_f, d_s, \overline{\mathcal{B}}) \subseteq M\overline{\mathcal{B}} \tag{68}$$

2. *the mapping $(x, d_f, d_s, \phi) \mapsto \widetilde{f}_c(x, t, d_f, d_s, \phi)$ is continuous for each t;*

3. *the mapping $t \mapsto \widetilde{f}_c(x, t, d_f, d_s, \phi)$ is measurable for each $(x, d_f, d_s, \phi) \in \mathcal{C} \times \mathcal{V}_f \times \mathcal{V}_s \times \overline{\mathcal{B}}$;*

4. *for each $\widetilde{\varepsilon} > 0$, there exists $\widetilde{\delta} > 0$ such that*

$$(x_1, x_2, d_f, d_s, \phi) \in \mathcal{C} \times \mathcal{C} \times \mathcal{V}_f \times \mathcal{V}_s \times \overline{\mathcal{B}} , \quad |x_1 - x_2| \leq \widetilde{\delta} \quad \Longrightarrow \quad \left| \widetilde{f}_c(x_1, t, d_f, d_s, \phi) - \widetilde{f}_c(x_2, t, d_f, d_s, \phi) \right| \leq \widetilde{\varepsilon} ; \tag{69}$$

5. *for each $\rho > 0$ there exists $T^* > 0$ such that for each*

$$(x, t_\circ, d_s, d_f, \phi) \in \mathcal{C} \times \mathbb{R} \times \mathcal{V}_s \times \mathcal{D}_f \times \Phi , \quad T \geq T^*$$

there exists $e \in \mathcal{E}$ such that

$$\left| \int_0^T \left[\widetilde{f}_c(x, t + t_\circ, d_f(t), d_s, \phi(t)) - f_{avg}(x, d_f(t), d_s, e(t)) \right] dt \right| \leq \rho T . \tag{70}$$

Proof. The proof of this claim is like the proof of Claim 1 with the extra step that we explicitly parameterize the set-valued map $U_c(s)$ defined in (35) using the function

$$u_c(s,\phi) = \frac{1}{2}\left[\mathrm{sat}_{01}\left(\frac{2s}{\rho_1}\right)+1\right] + \frac{1}{2}\max\left\{1-\frac{2|s|}{\rho_1},0\right\}\phi \ . \tag{71}$$

We note that $s \mapsto u_c(s,\phi)$ is globally Lipschitz uniformly in $\phi \in \overline{\mathcal{B}}$, and

$$U(s) \subseteq u_c(s,\overline{\mathcal{B}}) \subseteq U\left(s+\frac{\rho_1}{2}\overline{\mathcal{B}}\right) \ . \tag{72}$$

We define

$$f_c(x,t,d_f,d_s,\phi) := f(x,d_f,d_s) + \sum_{i=1}^{m} g_i(x,d_s)u_c(h_i(x,d_s)-p_i(t),\phi) \tag{73}$$

and

$$M = \max_{(x,d_f,d_f)\in\mathcal{C}\times\mathcal{V}_f\times\mathcal{V}_s}\left(|f(x,d_f,d_s)| + \sum_{i=1}^{m}|g_i(x,d_s)|\right) \ . \tag{74}$$

The first four conditions of the claim follow immediately from Assumption 3 and the properties of u_c.

Finally, the last condition of the claim is established by following the calculations (38)-(41) at the end of the proof of Claim 1, using the fact that $u_c(s,\overline{\mathcal{B}}) = U_c(s)$. ■

At this point we have the analogy of Proposition 1, which follows from (the proof of) [14, Proposition 8], and then the analogy of Claim 2 to complete the proof of Theorem 2. ■

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