CHARACTERIZATIONS OF THE NON-UNIFORM IN TIME ISS PROPERTY AND APPLICATIONS

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Abstract

For time-varying control systems various characterizations of the non-uniform in time input-to-state stability property (ISS) are provided. These characterizations enable us to derive sufficient conditions for ISS and feedback stabilization for composite systems.

1. Introduction

This paper constitutes a continuation of authors' works [5,6,7,18,19] on the concept of non-uniform Input-to-State Stability (ISS) and its applicability to stability and feedback stabilization of time-varying systems:

$$\begin{aligned} &\dot{x} = f(t,x,u) \\ &x \in \Re^n, u \in \Re^m, t \geq 0 \end{aligned} \tag{1.1}$$

where f is of class $C^0\left(\Re^+ \times \Re^n \times \Re^m ; \Re^n\right)$, being locally Lipschitz with respect to (x,u) and satisfies $f(\cdot,0,0) = 0$.

In Section 2 we first provide the definitions of the non-uniform in time Robust Global Asymptotic Stability (RGAS) and Input-to-State Stability (ISS) as given in [6]. It should be pointed out that the concept of non-uniform in time ISS as proposed in [6] extends the ISS property as described in [16] for the autonomous case. In Section 3 we give the non-uniform in time extension of the familiar notion of uniform in time ISS as proposed by Sontag in [10] (see also [11-14]) concerning autonomous systems and we provide in Proposition 3.1 equivalence between this notion and the concept of ISS as suggested in [16]. Further equivalent characterizations of ISS are also given in Proposition 3.1 and links between ISS and the concepts of CICS (Converging-Input-Converging-State) and BIBS (Bounded-Input-Bounded-State) are provided in Corollary 3.4. Another important consequence of Proposition 3.1 is Corollary 3.5 concerning autonomous systems:

$$\begin{aligned} &\dot{x} = f(x,u) \\ &x \in \Re^n \ , u \in \Re^m \end{aligned} \tag{1.2}$$

Particularly, we prove that, if (1.2) is forward complete and satisfies the 0-GAS property, namely, $0 \in \Re^n$ is GAS with respect to the unforced system:

$$\dot{x} = f(x,0) \tag{1.3}$$

then (1.2) satisfies the (non-uniform in time) ISS property. It should be emphasized here that, as proved in [2], system (1.2) under the same hypotheses fails in general to satisfy the uniform in time ISS property.

In Section 4 we provide sufficient conditions for the (non-uniform in time) ISS for composite time-varying systems:

$$\dot{x} = f(t,x,y,u) \tag{1.4a}$$
$$\dot{y} = g(t,x,y,u) \tag{1.4b}$$
$$x \in \Re^n,\ y \in \Re^k,\ u \in \Re^m,\ t \ge 0$$

where $f(\cdot,0,0,0)=0$, $g(\cdot,0,0,0)=0$ and we assume that f and g are C^0 mappings being locally Lipschitz with respect to (x,y,u). It is known (see for instance [3,4,15]) that for autonomous systems the uniform ISS for (1.4a) with (y,u) as input and for (1.4b) with (x,u) as input, respectively, leads to a simple characterization of a sufficient condition under which the overall system satisfies the ISS property from the input u. For the time-varying case (1.4) Theorem 4.1 asserts that a set of additional conditions concerning qualitative behavior of the solutions of (1.4a) and (1.4b) guarantees ISS for the overall system (1.4). For the particular case of systems (1.4) when $g(\cdot)$ is independent of x, namely for the cascade interconnection:

$$\dot{x} = f(t,x,y,u) \tag{1.5a}$$
$$\dot{y} = g(t,y,u) \tag{1.5b}$$
$$x \in \Re^n,\ y \in \Re^k,\ u \in \Re^m,\ t \ge 0$$

the sufficient conditions of Theorem 4.1 are considerably simplified. Particularly, Corollary 4.2 provides sufficient conditions for ISS for the case (1.5) and generalizes a well-known result concerning the autonomous case.

Finally, by exploiting the notion of non-uniform in time Input-to-Output Stability (IOS), we study the problem of the propagation of the non-uniform in time ISS property through one integrator. Specifically, consider the system:

$$\dot{x} = f(t,x,y) \tag{1.6a}$$
$$\begin{aligned} &\dot{y} = g(t,x,y) + d(t,x,y)u \\ &x \in \Re^n,\ y \in \Re,\ t \ge 0,\ u \in \Re \end{aligned} \tag{1.6b}$$

We assume that the dynamics f, g, d are C^0, locally Lipschitz with respect to (x,y) and satisfy $f(\cdot,0,0)=0$, $g(\cdot,0,0)=0$. Theorem 4.3 provides sufficient conditions for the existence of an output static feedback stabilizer:

$$u = k(t,y) + v \tag{1.7}$$

that renders the closed-loop system (1.6) with (1.7) (non-uniformly) ISS with v as input.

2. Review of the Notion of RGAS and the Non-Uniform in Time ISS Property

In this section we provide the definitions of non-uniform in time Robust Global Asymptotic Stability (RGAS) and ISS as precisely given in [6]. We consider time-varying systems:

$$\begin{aligned} &\dot{x} = f(t,x,d) \\ &x \in \Re^n \ , t \geq 0 \ , d \in D \end{aligned} \tag{2.1}$$

where $D \subset \Re^m$ is a compact set, $f : \Re^+ \times \Re^n \times D \to \Re^n$ is a C^0 map being locally Lipschitz with respect to $x \in \Re^n$ and satisfies $f(t,0,d) = 0$ for all $(t,d) \in \Re^+ \times D$. By M_D we denote the set of all measurable functions from $\Re^+$ to D.

Definition 2.1 *We say that zero $0 \in \Re^n$ is (non-uniformly in time)* ***Robustly Globally Asymptotically Stable (RGAS)*** *for (2.1), if for every $t_0 \geq 0$, $d \in M_D$ and $x_0 \in \Re^n$ the corresponding solution $x(\cdot)$ of (2.1) exists for all $t \geq t_0$ and satisfies the following properties:*

P1 (Stability). *For every $\varepsilon > 0$, $T \geq 0$, it holds that*

$$\sup\left\{ |x(t)| : d \in M_D \ , t \geq t_0 \ , |x_0| \leq \varepsilon \ , t_0 \in [0,T] \ \right\} < +\infty \tag{2.2a}$$

and there exists a $\delta := \delta(\varepsilon, T) > 0$ such that

$$|x_0| \leq \delta \ , \ t_0 \in [0,T] \Rightarrow \ |x(t)| \leq \varepsilon \ , \forall t \geq t_0 , d \in M_D \tag{2.2b}$$

P2 (Attractivity). *For every $\varepsilon > 0$, $T \geq 0$ and $R \geq 0$, there exists a $\tau := \tau(\varepsilon, T, R) \geq 0$, such that*

$$|x_0| \leq R, t_0 \in [0,T] \ \Rightarrow \ |x(t)| \leq \varepsilon \ , \forall t \geq t_0 + \tau, d \in M_D \tag{2.2c}$$

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Definition 2.2 *We say that system (2.1) satisfies the 0-GAS property, if $0 \in \Re^n$ is GAS for (2.1) with $D = \{0\}$, or equivalently, for the unforced system $\dot{x} = f(t,x,0)$.*

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Definition 2.2 *Consider system (1.1) and let $\gamma(t,s) : \left(\Re^+\right)^2 \to \Re^+$ be a C^0 function, locally Lipschitz in s, such that for each fixed $t \geq 0$ the mapping $\gamma(t,\cdot)$ is positive definite. We say that (1.1) satisfies the* ***"weak" (non-uniform in time) Input-to-State Stability property (wISS)*** *with gain $\gamma(\cdot)$, if each solution $x(t) = x(t,t_0,x_0;u(\cdot))$ of (1.1) exists for all $t \geq t_0$ and satisfies Properties P1 and P2 of Definition 2.1, provided that*

$$|u(t)| \leq \gamma(t,|x(t)|), \ \textit{a.e. for } t \geq t_0 \tag{2.3}$$

If in addition for each $t \geq 0$ *the function* $\gamma(t,\cdot)$ *is of class* K_∞*, then we say that (1.1) satisfies the* ***(non-uniform in time) Input-to-State Stability property (ISS)*** *with gain* $\gamma(\cdot)$.

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The following proposition summarizes some useful equivalent descriptions of the ISS property.

Proposition 2.4 ([6]) *Let* $\gamma(t,s):\left(\Re^+\right)^2 \to \Re^+$ *be a* C^0 *function, locally Lipschitz in* s*, such that for each fixed* $t \geq 0$ *the mapping* $\gamma(t,\cdot)$ *is a positive definite function. Then the following statements are equivalent:*

(i) *System (1.1) satisfies the wISS property with gain* $\gamma(\cdot)$.

(ii) $0 \in \Re^n$ *is non-uniformly in time RGAS for the system:*

$$\begin{aligned} &\dot{x} = f(t,x,\gamma(t,|x|)d) \\ &x \in \Re^n, d \in B[0,1] \subset \Re^m, t \geq 0 \end{aligned} \tag{2.4}$$

where $B[0,1]$ *denotes the closed sphere of radius 1 around* $0 \in \Re^m$.

(iii) *There exist a function* σ *of class* KL *and a function* $\beta:\Re^+ \to \Re^+$ *of class* K^+ *such that the following property holds for all* $t \geq t_0$*:*

$$|u(\tau)| \leq \gamma(\tau,|x(\tau)|) \text{ a.e. for } \tau \in [t_0,t] \Rightarrow |x(t)| \leq \sigma\left(\beta(t_0)|x_0|\,,\, t-t_0\right) \tag{2.5}$$

(iv) *There exist a* C^∞ *function* $V:\Re^+ \times \Re^n \to \Re^+$*, functions* a_1*,* $a_2 \in K_\infty$ *and* $\beta \in K^+$ *such that the following hold for all* $(t,x,u) \in \Re^+ \times \Re^n \times \Re^m$*:*

$$a_1(|x|) \leq V(t,x) \leq a_2(\beta(t)|x|) \tag{2.6a}$$

$$|u| \leq \gamma(t,|x|) \Rightarrow \dot{V}(t,x,u)\Big|_{(1.1)} \leq -V(t,x) \tag{2.6b}$$

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Remark 2.5: The precise description of property (iii), which is given in [6], is as follows:

(v) *There exist functions* a_1 *,* a_2 *of class* K_∞ *and a function* $\widetilde{\beta}:\Re^+ \to \Re^+$ *of class* K^+ *such that the following property holds for all* $t \geq t_0$*:*

$$|u(\tau)| \leq \gamma(\tau,|x(\tau)|) \text{ a.e. for } \tau \in [t_0,t] \Rightarrow\ a_1(|x(t)|) \leq \exp(-(t-t_0))\widetilde{\beta}(t_0)a_2(|x_0|) \tag{2.7}$$

It can be easily established that (2.5) and (2.7) are equivalent.

The following proposition extends a well-known result concerning the autonomous case (see for instance [4,17]).

Proposition 2.6 *System (1.1) satisfies the wISS property from the input* u, *if and only if it satisfies the 0-GAS property.*

The proof of Proposition 2.6 is an immediate consequence of the following lemma, which is a direct extension of Lemma IV.10 in [2] and constitutes a powerful tool for the results of next section.

Lemma 2.7 *Consider the system (1.1) and suppose that it satisfies the 0-GAS property. Then, for every function* $\mu(\cdot)$ *of class* K^+, *there exists a* C^∞ *map* $V:\Re^+ \times \Re^n \to \Re^+$, *functions* $a_i(\cdot)$ $(i=1,...,4)$ *of class* K_∞ *and* $p(\cdot)$, $\kappa(\cdot)$ *of class* K^+, *such that*

$$a_1(|x|) \le V(t,x) \le a_2(p(t)|x|)\,,\ \forall (t,x) \in \Re^+ \times \Re^n \tag{2.8}$$

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x,u) \le -V(t,x) + \exp(-2t)\, a_3\left(\frac{|x|}{\mu(t)}\right) a_4\big(\kappa(t)|u|\big)$$
$$\forall (t,x,u) \in \Re^+ \times \Re^n \times \Re^m \tag{2.9}$$

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Similarly with ISS we may extend the notion of uniform in time Input-to-Output Stability (IOS) (as proposed in [12]) to the non-uniform case.

Definition 2.8 *Consider the system (2.1) with output* $y = h(t,x)$, *where the map* $h(\cdot)$ *is of class* $C^0(\Re^+ \times \Re^n; \Re^k)$ *and satisfies* $h(\cdot,0)=0$. *We say that (2.1) is (non-uniformly in time)* ***Robustly Globally Asymptotically Output Stable (RGAOS)****, if for every* $t_0 \ge 0$, $d \in M_D$ *and* $x_0 \in \Re^n$ *the corresponding solution* $x(\cdot)$ *of (2.1) exists for all* $t \ge t_0$ *and satisfies the following properties:*

P1 (Output Stability). *For every* $\varepsilon > 0$, $T \ge 0$, *it holds that*

$$\sup\left\{ |h(t,x(t))| : d \in M_D\,, t \ge t_0\,, |x_0| \le \varepsilon\,, t_0 \in [0,T] \right\} < +\infty \tag{2.10a}$$

and there exists a $\delta := \delta(\varepsilon, T) > 0$ *such that*

$$|x_0| \le \delta\,,\ t_0 \in [0,T] \Rightarrow\ |h(t,x(t))| \le \varepsilon\,, \forall t \ge t_0, d \in M_D \tag{2.10b}$$

P2 (Output Attractivity). *For every* $\varepsilon > 0$, $T \ge 0$ *and* $R \ge 0$, *there exists a* $\tau := \tau(\varepsilon, T, R) \ge 0$, *such that*

$$|x_0| \le R, t_0 \in [0,T]\ \Rightarrow\ |h(t,x(t))| \le \varepsilon\,, \forall t \ge t_0 + \tau, d \in M_D \tag{2.10c}$$

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Definition 2.9 *Consider system (1.1) with output* $y = h(t,x)$, *where* $h(\cdot) \in C^0(\Re^+ \times \Re^n; \Re^k)$ *with* $h(\cdot,0)=0$ *and let* $\gamma(t,s):\left(\Re^+\right)^2 \to \Re^+$ *be a* C^0 *function, locally Lipschitz in* s, *such that for each fixed* $t \ge 0$ *the mapping* $\gamma(t,\cdot)$ *is of class* K_∞. *We say that (1.1) satisfies the* *(non-uniform in time) **Input-to-Output Stability property (IOS)** with gain* $\gamma(\cdot)$, *if each solution* $x(t)=x(t,t_0,x_0;u(\cdot))$ *of (1.1) exists for all* $t\geq t_0$ *and satisfies Properties P1 and P2 of Definition 2.8, provided that (2.3) holds.*

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3. Characterizations of the Non-Uniform in Time ISS Property

The following proposition provides equivalent characterizations of the (non-uniform in time) ISS property.

Proposition 3.1 *The following statements are equivalent:*

(i) *System (1.1) satisfies the ISS property.*

(ii) *There exist functions* $\rho(\cdot)\in K_\infty$, $\phi(\cdot)$, $\beta(\cdot)\in K^+$ *and* $\sigma(\cdot)\in KL$ *such that*

$$\rho\big(\phi(t)|u(t)|\big)\leq|x(t)| \text{ a.e. for } t\geq t_0 \Rightarrow |x(t)|\leq\sigma\big(\beta(t_0)|x_0|\,,t-t_0\big),\ \forall t\geq t_0 \tag{3.1}$$

(iii) *There exist functions* $\rho(\cdot)\in K_\infty$, $\phi(\cdot)$, $\beta(\cdot)\in K^+$ *and* $\sigma(\cdot)\in KL$ *such that, for every* $(t_0,x_0)\in\Re^+\times\Re^n$ *and for every input* $u=u(t)$ *of class* $\mathbf{L}^\infty_{loc}([t_0,+\infty))$, *the corresponding solution* $x(t)$ *of (1.1) with* $x(t_0)=x_0$ *exists for all* $t\geq t_0$ *and satisfies:*

$$|x(t)|\leq\max\left\{\sigma(\beta(t_0)|x_0|\,,t-t_0),\ \sup_{t_0\leq\tau\leq t}\sigma\big(\beta(\tau)\rho\big(\phi(\tau)|u(\tau)|\big),t-\tau\big)\right\} \tag{3.2}$$

(iv) *There exist functions* $\zeta(\cdot)\in K_\infty$, $\beta(\cdot),\delta(\cdot)\in K^+$ *and* $\sigma(\cdot)\in KL$ *such that, for every* $(t_0,x_0)\in\Re^+\times\Re^n$ *and for every input* $u=u(t)$ *of class* $\mathbf{L}^\infty_{loc}([t_0,+\infty))$, *the corresponding solution* $x(t)$ *of (1.1) with* $x(t_0)=x_0$ *exists for all* $t\geq t_0$ *and satisfies:*

$$|x(t)|\leq\max\left\{\sigma(\beta(t_0)|x_0|\,,t-t_0),\ \sup_{t_0\leq\tau\leq t}\zeta\big(\delta(\tau)|u(\tau)|\big)\right\} \tag{3.3}$$

(v) *There exist a function* $\theta(\cdot)\in K_\infty$, *being locally Lipschitz on* $\Re^+$, *and a function* $\delta(\cdot)\in K^+$ *such that* $0\in\Re^n$ *is RGAS for the system:*

$$\dot{x}=f(t,x,\frac{\theta(|x|)}{\delta(t)}d),\ d(\cdot)\in M_{B[0,1]} \tag{3.4}$$

(vi) *System (1.1) satisfies the 0-GAS Property and there exist functions* $\bar{\sigma}(\cdot),\bar{\zeta}(\cdot)\in K_\infty$ $\bar{\delta}(\cdot),\mu(\cdot),\bar{\beta}(\cdot)\in K^+$ *such that for every* $(t_0,x_0)\in\Re^+\times\Re^n$ *and* $u=u(t)$ *of class* $\mathbf{L}^\infty_{loc}([t_0,+\infty))$ *the corresponding solution* $x(t)$ *of (1.1) with* $x(t_0)=x_0$ *exists for all* $t\geq t_0$ *and satisfies:*

$$|x(t)| \le \mu(t)\left(\bar{\sigma}\left(\bar{\beta}(t_0)|x_0|\right) + \sup_{t_0 \le \tau \le t} \bar{\zeta}\left(\bar{\delta}(\tau)|u(\tau)|\right)\right) \tag{3.5}$$

(vii) *There exist a* C^∞ *function* $V : \Re^+ \times \Re^n \to \Re^+$*, functions* $a_1(\cdot)$, $a_2(\cdot)$, $a_3(\cdot)$ *of class* K_∞ *and functions* $p(\cdot)$, $q(\cdot)$ *of class* K^+ *such that for all* $(t,x,u) \in \Re^+ \times \Re^n \times \Re^m$ *we have:*

$$a_1(|x|) \le V(t,x) \le a_2(p(t)|x|) \tag{3.6a}$$

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x,u) \le -V(t,x) + a_3(q(t)|u|) \tag{3.6b}$$

$\lhd$

Remark 3.2: When the functions $\phi(\cdot)$ and $\beta(\cdot)$ are bounded, then (3.1) is equivalent to the uniform in time ISS property, as given in [16,17]. Likewise, when $\beta(\cdot)$ and $\delta(\cdot)$ are bounded, then (3.3) is equivalent to ISS property, as originally proposed in [10] by Sontag. The equivalence between (3.1) and (3.3) generalizes the well known fact that for the autonomous case and when $\gamma(\cdot)$ is independent of t, namely, $\gamma(\cdot)$ is of class K_∞, the uniform in time ISS property as given by Sontag is equivalent to the corresponding characterization given in [16,17]. Finally, we note that, when $p(\cdot)$ and $q(\cdot)$ are bounded, then (3.6a,b) coincides with the Lyapunov characterization in [11] for the uniform in time ISS property. $\lhd$

Remark 3.3: By exploiting the result of Proposition 3.1 and particularly the equivalence between (i) and (iv), it can be established that the notion of non-uniform in time ISS remains invariant under:

S1 Scaling of time $t = \int_0^\tau a(s)ds$, where $a(\cdot) \in C^0(\Re^+)$ with $a(s) > 0$, for all $s \ge 0$ and $\int_0^{+\infty} a(s)ds = +\infty$.

S2 Transformations $x = \Phi(t,z)$ of the state, where $\Phi(\cdot) \in C^2(\Re^+ \times \Re^n ; \Re^n)$, with $\Phi(t,0) = 0$, $\Phi(t,\Re^n) = \Re^n$ and $\det\left(\frac{\partial \Phi}{\partial z}(t,z)\right) \ne 0$ for all $(t,z) \in \Re^+ \times \Re^n$ and such that there exists a pair of functions $a_i(\cdot) \in K_\infty$ ($i = 1,2$) with $a_1(|z|) \le |\Phi(t,z)| \le a_2(|z|)$ for all $(t,z) \in \Re^+ \times \Re^n$.

S3 Input transformations $u = q(t,v)$, where $q(\cdot) \in C^0(\Re^+ \times \Re^m ; \Re^m)$ satisfying $q(t,0) = 0$ for all $t \ge 0$, and in such a way that there exists a function $q^{-1}(\cdot) \in C^0(\Re^+ \times \Re^m ; \Re^m)$ with $u = q(t, q^{-1}(t,u))$ for all $(t,u) \in \Re^+ \times \Re^m$ and such that $f(t,x,q(t,v))$ is locally Lipschitz with respect to $v \in \Re^m$.

It should be emphasized here that the uniform in time ISS, does not in general remain invariant under S1 or S3.

An immediate consequence of Proposition 3.1 is the following corollary, which provides links between non-uniform in time ISS and the concepts of BIBS and CICS.

Corollary 3.4 *Suppose that system (1.1) satisfies the ISS property, and in particular assume that (3.3) holds for certain functions* $\zeta \in K_\infty$, $\sigma \in KL$, $\beta, \delta \in K^+$. *Let* $u = u(t)$ *be an input of class* $\mathbf{L}^\infty_{loc}\left(\Re^+\right)$ *such that:*

$$\delta(t)|u(t)| \text{ is bounded over } \Re^+$$

Then for every $(t_0, x_0) \in \Re^+ \times \Re^n$, *the solution* $x(\cdot, t_0, x_0; u(\cdot))$ *is bounded over* $\Re^+$ *as well. If in addition*

$$\lim_{t \to +\infty} \delta(t)|u(t)| = 0$$

then for every $(t_0, x_0) \in \Re^+ \times \Re^n$, *it holds that* $\lim_{t \to +\infty} x(t, t_0, x_0; u(\cdot)) = 0$.

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Statement (vii) of Proposition 3.1 shows that, under a special type of forward completeness, the 0-GAS Property for (1.1) is equivalent to (non-uniform in time) ISS for (1.1). For the autonomous case we establish below that 0-GAS plus forward completeness is equivalent to ISS, which in general is not true for the uniform in time ISS (see [2]).

Corollary 3.5 *Consider the autonomous system (1.2), where* f *is locally Lipschitz with respect to* (x,u) *and satisfies* $f(0,0) = 0$. *Suppose that:*

(i) $0 \in \Re^n$ *is GAS for the unforced system (1.3) (0-GAS Property).*

(ii) *System (1.2) is forward complete with* u *as input.*

Then the solutions of (1.2) satisfy (3.5) (statement (vii) of Proposition 3.1), and therefore (1.2) fulfills the ISS property.

Proof Since $0 \in \Re^n$ is GAS for the system (1.3), then by Lemma IV.10 in [2] and Theorem 3 in [14], there exists a smooth function $V : \Re^n \to \Re^+$, functions a_1, a_2, λ, δ of class K_∞, such that for all $(x,u) \in \Re^n \times \Re^m$ we have:

$$a_1\left(|x|\right) \le V(x) \le a_2\left(|x|\right) \tag{3.7a}$$

$$\frac{\partial V}{\partial x}(x) f(x,u) \le -V(x) + \lambda\left(|x|\right)\delta\left(|u|\right) \tag{3.7b}$$

Clearly, inequalities (3.7a,b) give the following estimate for the solution $x(\cdot)$ of (1.2) initiated from $x_0 \in \Re^n$ at time $t_0 \geq 0$ and corresponding to some input $u = u(t)$ of class $\mathbf{L}_{loc}^{\infty}([t_0,+\infty))$:

$$a_1(|x(t)|) \leq \exp(-(t-t_0))a_2(|x_0|) + \int_{t_0}^{t} \exp(-(t-\tau))\lambda(|x(\tau)|)\delta(|u(\tau)|)d\tau \,,\ \forall t \geq t_0 \qquad (3.7c)$$

Furthermore, since (1.2) is forward complete, Corollary 2.11 in [1] guarantees the existence of a smooth and proper function $W : \Re^n \to \Re^+$, functions a_3 , a_4 , σ of class K_∞ and a constant $R > 0$ such that for all $(x,u) \in \Re^n \times \Re^m$ it holds:

$$a_3(|x|) \leq W(x) \leq a_4(|x|) + R \qquad (3.8a)$$

$$\frac{\partial W}{\partial x}(x) f(x,u) \leq W(x) + \sigma(|u|) \qquad (3.8b)$$

It follows from (3.8a,b) that the solution $x(\cdot)$ of (1.2) satisfies

$$a_3(|x(t)|) \leq \exp(t-t_0)(a_4(|x_0|) + R) + \int_{t_0}^{t} \exp(t-\tau)\sigma(|u(\tau)|)d\tau \ ,\ \forall t \geq t_0 \qquad (3.9)$$

Define $\tilde{\lambda}(s) := \lambda\left(2a_3^{-1}(2s)\right)$. It then follows from (3.7c) and (3.9) that for all $t \geq t_0$ it holds:

$$\begin{aligned} a_1(|x(t)|) \leq{}& \exp(-(t-t_0))a_2(|x_0|) + \int_{t_0}^{t} \tilde{\lambda}\left(\int_{t_0}^{\tau} \exp(\tau-s)\sigma(|u(s)|)ds \right)\delta(|u(\tau)|)d\tau \\ &+ \int_{t_0}^{t} \tilde{\lambda}\left(\exp(\tau-t_0)(a_4(|x_0|)+R)\right)\delta(|u(\tau)|)d\tau \end{aligned} \qquad (3.10)$$

Corollary 10 and Remark 11 in [14] guarantee the existence of a function $q(\cdot) \in K_\infty$, such that

$$\sigma(rs) + \delta(rs) + \tilde{\lambda}(rs) \leq q(r)q(s) \,,\ \forall r,s \geq 0 \qquad (3.11)$$

and let $\phi \in K^+$ be a function that satisfies the following inequality for all $t \geq 0$:

$$q\left(\frac{1}{\phi(t)}\right) \leq \frac{\exp(-t)}{1 + q(\exp(t))} \qquad (3.12)$$

By virtue of (3.11) and (3.12) it follows:

$$\int_{t_0}^{\tau} \exp(\tau - s)\sigma(|u(s)|)ds \overset{(3.11)}{\leq} \exp(\tau) \sup_{t_0 \leq s \leq \tau} q(\phi(s)|u(s)|) \int_{t_0}^{\tau} q\left(\frac{1}{\phi(s)}\right)ds \overset{(3.12)}{\leq} \exp(\tau) \sup_{t_0 \leq s \leq \tau} q(\phi(s)|u(s)|)$$

$$\text{for all } \tau \geq t_0 \tag{3.13}$$

By exploiting (3.10), (3.11), (3.12) and (3.13) we obtain the following estimation for the solution $x(t)$ of system (1.2) initiated at $x(t_0) = x_0$ and corresponding to some input $u(\cdot) \in \mathbf{L}_{loc}^{\infty}([t_0, +\infty))$:

$$\begin{aligned} a_1(|x(t)|) &\overset{(3.11),\,(3.13)}{\leq} \exp(-(t - t_0))a_2(|x_0|) \\ &+ \sup_{t_0 \leq \tau \leq t} q(\phi(\tau)|u(\tau)|)\left[\sup_{t_0 \leq \tau \leq t} q(q(\phi(\tau)|u(\tau)|)) + q(a_4(|x_0|) + R)\right]\int_{t_0}^{t} q(\exp(\tau))q\left(\frac{1}{\phi(\tau)}\right)d\tau \\ &\overset{(3.12)}{\leq} a_2(|x_0|) + \sup_{t_0 \leq \tau \leq t} q(\phi(\tau)|u(\tau)|)\left[\sup_{t_0 \leq \tau \leq t} q(q(\phi(\tau)|u(\tau)|)) + q(2a_4(|x_0|)) + q(2R)\right] \end{aligned} \tag{3.14}$$

We define:

$$a(s) := a_1^{-1}\left(2a_2(s) + (q(2a_4(s)))^2\right) \tag{3.15a}$$

$$\zeta(s) := a_1^{-1}\left(4q(2R)q(s) + 4(q(s) + q(q(s)))^2\right) \tag{3.15b}$$

It follows from estimation (3.14) and definitions (3.15a,b) that

$$|x(t)| \leq a(|x_0|) + \sup_{t_0 \leq \tau \leq t} \zeta(\phi(\tau)|u(\tau)|), \ \forall t \geq t_0$$

The last inequality asserts, by recalling statement (vi) of Proposition 3.1, that system (1.2) satisfies the ISS property. ◁

4. Applications

I Small-Gain Theorem

We first provide sufficient conditions for ISS for composite systems (1.4). The next theorem constitutes a generalization of the well-known small gain theorem for autonomous systems under the presence of (uniform in time) ISS (see [3,4]).

Theorem 4.1 *For the system (1.4) we assume:*

A1 *Subsystem (1.4a) satisfies the ISS property from the input (y,u). Particularly, there exist functions $\sigma_1(\cdot) \in KL$, $\rho_1(\cdot) \in K_\infty$, $\phi_1(\cdot) \in K^+$, $\beta_1(\cdot) \in K^+$ such that for every $(t_0, x_0) \in \Re^+ \times \Re^n$ and for every input $(y,u) = (y(t),u(t))$ of class $\mathbf{L}_{loc}^{\infty}([t_0, +\infty))$, the* *solution $x(t)$ of (1.4a) with $x(t_0) = x_0$ exists for all $t \ge t_0$ and the following property holds:*

$$|x(t)| \le \max\left\{ \sigma_1(\beta_1(t_0)|x_0|, t-t_0),\ \sup_{t_0 \le \tau \le t} \sigma_1\left(\beta_1(\tau)\rho_1\left(\phi_1(\tau)\max\left\{|y(\tau)|, |u(\tau)|\right\}\right), t-\tau\right) \right\} \tag{4.1}$$

A2 *Subsystem (1.4b) satisfies the ISS property from the input (x,u). Particularly, there exist a constant $\lambda \ge 0$, functions $\sigma_2(\cdot) \in KL$, $\rho_2(\cdot) \in K_\infty$, $\phi_2(\cdot) \in K^+$, $\beta_2(\cdot) \in K^+$ such that for every $(t_0, y_0) \in \Re^+ \times \Re^k$ and for every input $(x,u) = (x(t), u(t))$ of class $\mathbf{L}^\infty_{loc}([t_0, +\infty))$, the solution $y(t)$ of (1.4b) with $y(t_0) = y_0$ exists for all $t \ge t_0$ and it holds that*

$$|y(t)| \le \max\left\{ \sigma_2(\beta_2(t_0)|y_0|, t-t_0),\ \sup_{t_0 \le \tau \le t} \sigma_2\left(\beta_2(\tau)\rho_2\left(\phi_2(\tau)\max\left\{\lambda|x(\tau)|, |u(\tau)|\right\}\right), t-\tau\right) \right\} \tag{4.2}$$

A3 *The following properties hold for all $t_0 \ge 0$, $s \ge 0$:*

$$\lim_{t \to +\infty} \beta_1(t)\rho_1\left(\phi_1(t)\sigma_2(s, t-t_0)\right) = 0 \tag{4.3a}$$

$$\lim_{t \to +\infty} \beta_2(t)\rho_2\left(\phi_2(t)\ \lambda\sigma_1(s, t-t_0)\right) = 0 \tag{4.3b}$$

A4 *There exists a function $a(\cdot)$ of class K_∞ with*

$$a(s) < s\ ,\ \forall s > 0 \tag{4.4}$$

such that the following inequalities are satisfied for all $t_0 \ge 0$:

$$\sup_{t \ge t_0} \sigma_1\left(\beta_1(t)\rho_1\left(\phi_1(t)\sigma_2\left(\beta_2(t_0)\rho_2\left(\phi_2(t_0)\lambda s\right), t-t_0\right)\right), 0\right) \le a(s)\ ,\ \forall s \ge 0 \tag{4.5a}$$

$$\sup_{t \ge t_0} \sigma_2\left(\beta_2(t)\rho_2\left(\phi_2(t)\lambda\,\sigma_1\left(\beta_1(t_0)\rho_1\left(\phi_1(t_0)s\right), t-t_0\right)\right), 0\right) \le a(s)\ ,\ \forall s \ge 0 \tag{4.5b}$$

Then system (1.4) satisfies the ISS property from the input u.

$\triangleleft$

II Cascade Connections

It is known that the cascade connection of two autonomous independent ISS subsystems satisfies the ISS property. This is not in general true for the time-varying case under the assumption of the non-uniform in time ISS property. Particularly, by specializing Theorem 4.1 to the case 1.5 we obtain the following result, which constitutes a generalization of recent results concerning time-varying systems (see [9, 20]).

Corollary 4.2 *Consider the system (1.5) and suppose that the following hold:*

A1 *The subsystem (1.5a) satisfies the (non-uniform in time) ISS property from the input* (y,u) *and that there exist functions* $\sigma_1(\cdot)\in KL$, $\rho_1(\cdot)\in K_\infty$ *and* $\phi_1(\cdot)$, $\beta_1(\cdot)$ *of class* K^+ *such that for every* $(t_0,x_0)\in\Re^+\times\Re^n$ *and for every input* $(y,u)=(y(t),u(t))$ *of class* $\mathbf{L}^\infty_{loc}([t_0,+\infty))$ *the corresponding solution* $x(t)$ *of (1.5a) with* $x(t_0)=x_0$ *exists for all* $t\geq t_0$ *and inequality (4.1) holds.*

A2 *The subsystem (1.5b) satisfies the (non-uniform in time) ISS property from the input* u *and there exist functions* $\sigma_2(\cdot)\in KL$, $\rho_2(\cdot)\in K_\infty$ *and* $\phi_2(\cdot)$, $\beta_2(\cdot)$ *of class* K^+ *such that for every* $(t_0,y_0)\in\Re^+\times\Re^k$ *and for every input* $u=u(t)$ *of class* $\mathbf{L}^\infty_{loc}([t_0,+\infty))$ *the corresponding solution* $y(t)$ *of (1.5b) with* $y(t_0)=y_0$ *exists for all* $t\geq t_0$ *and the following property holds:*

$$|y(t)|\leq\max\left\{\sigma_2(\beta_2(t_0)|y_0|,t-t_0)\,,\ \sup_{t_0\leq\tau\leq t}\sigma_2\big(\beta_2(\tau)\rho_2\big(\phi_2(\tau)|u(\tau)|\big),t-\tau\big)\right\}\tag{4.6}$$

A3 *The following property holds for all* $(t_0,s)\in(\Re^+)^2$*:*

$$\lim_{t\to+\infty}\beta_1(t)\rho_1\big(\phi_1(t)\sigma_2(s,t-t_0)\big)=0\tag{4.7}$$

Then system (1.5) exhibits the non-uniform in time ISS property from the input u.

◁

III Output Feedback Stabilization

We derive sufficient conditions for robust output static feedback stabilization for systems in feedback form. The following result extends the corresponding results in [3,16].

Theorem 4.3 *Suppose that:*

B1 *There exists a* C^2 *function* $k:\Re^+\times\Re^n\to\Re$, *with* $k(\cdot,0)=0$, *such that system (1.6a) with* $y=k(t,x)+z$ *satisfies the non-uniform in time ISS property from the input* z *for certain gain function* $\gamma(\cdot)$.

B2 *System (1.6a) with* $y=k(t,x)+z$ *satisfies the non-uniform in time IOS property from the input* z *with* $k(\cdot,\cdot)$ *as output function and the same gain* $\gamma(\cdot)$ *as in B1.*

B3 *There exists a function* φ *of class* K^+ *such that*

$$\varphi(t)d(t,x,y)\geq 1,\ \forall(t,x,y)\in\Re^+\times\Re^n\times\Re$$

Then for every function $\bar{\gamma}(t,s):(\Re^+)^2\to\Re^+$, *which is* C^0, *locally Lipschitz in* s, *with* $\bar{\gamma}(t,\cdot)\in K_\infty$ *for each* $t\geq 0$, *there exists a* C^∞ *function* $\tilde{k}:\Re^+\times\Re\to\Re$ *with* $\tilde{k}(\cdot,0)=0$ *such* *that system (1.6) with* $u=\tilde{k}(t, y-k(t,x))+v$ *satisfies the ISS property from the input* v *with gain* $\bar{\gamma}(t,s)$.

$\triangleleft$

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