Feedback Control for Multidimensional Systems and Interpolation Problems for Multivariable Functions

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Abstract

This paper examines the connections between feedback stabilization and H^∞ control, model matching problems and multivariable Nevanlinna-Pick interpolation problems for multidimensional or nD linear systems.

1 Introduction

Feedback stabilization and optimal control problems for the case of classical linear systems have been much studied over the past several decades. More recently, such problems for the case of multidimensional or nD linear systems have been drawing the attention of researchers—see [DX99], [DXZ01], [SS92], [Sul94], and [Zerz00]. While most of these authors are motivated by applications to physical situations having nD-system models, Helton [Hel01] has pointed out a connection with adaptive control for a classical 1D system. After D.Givone and R.Roesser [GR72], and E.Fornasini and G.Marchesini [FM76] and some other researchers proposed various types of multidimensional linear models in the seventies, most mathematicians and system engineers have been focusing on the development and extension of the existence theories for classical linear systems to the nD systems. New theories and notions, all of which are more generalized and complicated then those in the classical case, have been introduced to describe the behavior of the nD systems.

It is well known that for linear time invariant 1D system there are various approaches to solve the feedback stabilization problems; for instance, the pole placement, the matrix fraction description (MFD's), and the interpolation approaches. In the latter approach (see [Fra87], one goes through coprime factorization to get the Q-parameter; with Q as the new design parameter rather than the controller K, one has a model matching problem. Let F be the performance function, which is affine in Q. Then, with the performance function F as the design parameter rather than Q, one has an interpolation problem for F. One then solves an interpolation problem to get F, and then backsolves for Q and finally for K, a desired controller. A criterion for internal stability can be expressed directly in terms of K: K is internally stabilizing for the closed loop system whenever F is stable and satisfies the appropriate interpolation conditions. Incorporation of a tolerance level on the performance function then leads to an interpolation problem of Nevanlinna-Pick type.

However, this approach is much more complicated for the nD case for a number of reasons. First of all, the reduction to the model-matching form is not obvious since the notion of coprime factorization splits in several independent ways in the nD case (see [YG79]). Secondly, the multivariable analogue of Nevanlinna-Pick interpolation is much more complicated.

By using the various notions of coprime in the nD case, the matrix fraction description (MFD) approach for nD linear systems and its connection with the properties of nD polynomial and rational matrices were investigated by Z. Lin [Lin88]. He proved that, for nD case, the rational matrix function $P(\mathbf{z})$ does not always admit a minor right coprime decomposition. From this fact, he was able to produce a counterexample to illustrate that the determinant test for internal stability of 2D systems due to Humes-Jury [HJ77] may not be extended to the nD case when $P(\mathbf{z})$ does not admit a minor right coprime decomposition. Therefore, he introduced the notion of *generating polynomials* (later renamed *reduced minors*) and applied it to the stability test for nD systems.

The notion of reduced minors was introduced in connection with the feedback stabilization problem for nD systems in [Lin98], [Lin99], and [Lin00]. However, in those papers, Lin studied the (output) feedback stabilization problem, which is the special case of the standard $\mathcal{H}^\infty$ control framework, and provided the constructive method to obtain a set of all stabilizing controllers via the famous Youla parameterization. This paper uses the results of Lin to provide the connection between feedback stabilization and interpolation conditions for nD linear systems for the case where the plant P has a double-coprime factorization (see Definition 2.4) in the so-called 1-block case. When one goes on to demand performance in addition to internal stability as a design goal, there results an nD matrix Nevanlinna-Pick interpolation problem. We discuss the recent work on this problem and the remaining issues to be settled before the nD theory reaches a state comparable to the 1D case. This connection between nD matrix Nevanlinna-Pick interpolation and feedback stabilization for nD plants has previously been pointed out by Helton [Hel01] for the scalar case.

2 Preliminaries

In the following, we shall let $\mathbb{R}$ denote the field of real numbers; $\mathbb{R}[\mathbf{z}] = \mathbb{R}[z_1, \ldots, z_d]$ the polynomial ring over $\mathbb{R}$ in d indeterminates $(z_1, \ldots, z_d)$, all of which are complex variables; $\mathbb{R}(\mathbf{z}) = \mathbb{R}(z_1, \ldots, z_d)$, the field of rational functions which is equal to the quotient field of $\mathbb{R}[\mathbf{z}]$; $\mathbb{R}_s(\mathbf{z}) \subseteq \mathbb{R}(\mathbf{z})$ the subset of rational functions in $\mathbb{R}(\mathbf{z})$ having no poles in the closed unit polydisk, $\overline{\mathbb{D}^d}\} = \{(z_1, \ldots, z_d) | \, |z_1| \leq 1, \ldots, |z_d| \leq 1\}$. $\mathbb{R}^{m\times l}(\mathbf{z})$ the set of $m \times l$ matrices with entries in $\mathbb{R}(\mathbf{z})$ (i.e., entries are rational functions); $\mathbb{R}_s^{m\times l}(\mathbf{z})$ the set of $m \times l$ matrices with entries in $\mathbb{R}_s(\mathbf{z})$ (i.e., entries are stable real rational functions). The d–D polynomial is said to be stable if it has no zeros in $\overline{\mathbb{D}^d}$.

In the standard H^∞-control context, the problem is to design a controller K which minimizes the largest energy error signal z over all disturbances w of L^2-norm at most 1, subject to the additional constraint that K stabilizes the system:

$$\min_{K \text{stabilizing}} \max_{\| w \|_2 \leq 1} \| z \|_2 \tag{2.1}$$

where the L^2-norm of any signal $x(t)$ is regarded as the measure of energy of a vector-valued signal and defined by

$$\| x \|_2^2 = \int_0^\infty \| x(t) \|^2 \, dt \tag{2.2}$$

Loosely speaking, the goal of the H^∞-control problem is to find a stabilizing controller K so as to minimize the H^∞-norm of the desired performance function, say F. In other words, one needs to construct a controller K so that the closed loop system is structurally (internally) stable with norm equal to at most a given tolerance level $\gamma > 0$.

Now consider the (output) feedback system (see diagram from [Lin00]), where $P(\mathbf{z})$ and $K(\mathbf{z})$ denote the plant and controller, respectively. Then the closed loop transfer matrix function from the input signals, u, to the error signals, e, is given by

$$\mathcal{H}_{eu} = \begin{bmatrix} (I + PK)^{-1} & -P\,(I + KP)^{-1} \\ K\,(I + PK)^{-1} & (I + KP)^{-1} \end{bmatrix} \tag{2.3}$$

Definition 2.1. A given plant $P \in \mathbb{R}^{m \times l}(\mathbf{z})$ is said to be (output) feedback stabilizable if there exists a controller K such that the closed loop transfer matrix function $\mathcal{H}_{eu}$ in (2.3) is internally stable; i.e., each entry of $\mathcal{H}_{eu}$ has no poles in $\overline{\mathbb{D}^d}$.

As mentioned earlier, there are several definitions of the *coprimeness* for polynomial matrices in several variables which all collapse to the classical notion in the one-variable case. In this paper, we focus on the strongest notion of coprime for the multivariable case, namely the notion of *zero coprime*. Therefore, from this point on, we will use *coprime* instead of *zero coprime* unless otherwise specified.

Definition 2.2 ([YG79]). Let $A \in \mathbb{R}^{l \times l}[\mathbf{z}]$, $B \in \mathbb{R}^{m \times l}[\mathbf{z}]$, and $F = \begin{bmatrix} A^T & B^T \end{bmatrix}^T \in \mathbb{R}^{(m+l) \times l}[\mathbf{z}]$, where A^T denotes the transposed matrix of A. Then A and B are said to be *zero right coprime (ZRC)* if the $l \times l$ minors of F have no common zero in $\mathbb{R}^n$. In dual manner, $A_1 \in \mathbb{R}^{m \times m}[\mathbf{z}]$, and $B_1 \in \mathbb{R}^{m \times l}[\mathbf{z}]$ are zero left coprime (ZLC) if A_1^T and B_1^T are ZRC.

Proposition 2.3 ([Zerz00]). *The matrix $F = \begin{bmatrix} A^T & B^T \end{bmatrix}^T$ is ZRC if and only if there exists a matrix $\begin{bmatrix} X & Y \end{bmatrix}$ that solves the Bézout equation*

$$\begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = I \tag{2.4}$$

By using Definition 2.2 and Proposition 2.3, one can see that the notion of double coprime factorization defined below is analogous to that for the 1D system.

Definition 2.4 ([Lin00]). Let $P \in \mathbb{R}^{m\times l}(\mathbf{z})$ be a proper real rational matrix nD system. Then P is said to have a double coprime factorization (DCF) if

1. there exist $D_l \in \mathbb{R}_s^{m\times m}(\mathbf{z})$, $D_r \in \mathbb{R}_s^{l\times l}(\mathbf{z})$, and N_r, $N_l \in \mathbb{R}_s^{m\times l}(\mathbf{z})$;
2. there exist $X_l \in \mathbb{R}_s^{l\times l}(\mathbf{z})$, $X_r \in \mathbb{R}_s^{m\times m}(\mathbf{z})$, and Y_r, $Y_l \in \mathbb{R}_s^{l\times m}(\mathbf{z})$;
3. D_l, D_r, X_l, and X_r are all nonsingular;
4. $P = N_r D_r^{-1} = D_l^{-1} N_l$ and the following Bézout identity holds:

$$\begin{bmatrix} X_l & Y_l \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & -Y_r \\ N_r & X_r \end{bmatrix} = \mathbf{I}_{(m+l)\times(m+l)} \tag{2.5}$$

To analyze the stability issue, Lin (see [Lin88]) introduced the notion of *reduced minors* and applied this notion to the stability test. Since most parts in this section are based on the results from Lin's works, the definition of *reduced minors* will be described first and followed by some useful theorems.

Definition 2.5. Let $F = \begin{bmatrix} A^T & B^T \end{bmatrix}^T \in \mathbb{R}^{(m+l)\times l}[\mathbf{z}]$, be of normal full rank l, and let $a_1, \ldots, a_\beta$ be the $l \times l$ minors of the matrix F, with $a_1 = \det(A)$, where $\beta = \begin{bmatrix} m+l \\ l \end{bmatrix}$. Extracting the greatest common divisor (g.c.d.) d of $a_1, \ldots, a_\beta$ gives, $a_j = db_j$, for $j = 1, \ldots, \beta$. Then $b_1, \ldots, b_\beta$ are called the **generating polynomials**, (later renamed as **reduced minors**) of F.

Remark: It should be noted here that the reduced minors of $E = \begin{bmatrix} A_1 & B_1 \end{bmatrix}$ can be defined in the similar way.

Proposition 2.6. *An nD discrete-time system $P \in \mathbb{R}^{m\times l}(\boldsymbol{z})$ represented by right MFD as $P = ND^{-1}$ is structurally stable if and only if $b_1 \neq 0$ in the polydisk, $\overline{\mathbb{D}^d}$, where b_j are the reduced minors of $F = \begin{bmatrix} D^T & N^T \end{bmatrix}^T$.*

Suppose now that an nD discrete system $P = ND^{-1} \in \mathbb{R}^{m\times l}(\mathbf{z})$ is not structurally stable (i.e., b_1 has a zero in $\overline{\mathbb{D}^d}$). Then one needs to find a controller K so that the closed loop system is internally stable. However, not all Ps are feedback stabilizable. The next theorem provides the necessary and sufficient conditions for such a P to be stabilizable.

Proposition 2.7 ([Lin98]). *Let $P = ND^{-1} \in \mathbb{R}^{m\times l}(\boldsymbol{z})$ be a given nD plant which is a proper rational matrix function, and let $b_1, \ldots, b_\beta$ be the reduced minors of $F = \begin{bmatrix} D^T & N^T \end{bmatrix}^T$, with $\beta = \begin{bmatrix} m+l \\ l \end{bmatrix}$. Then P is feedback stabilizable if and only if the reduced minors b_j of F $(j = 1, \ldots, \beta)$ have no common zeros in $\overline{\mathbb{D}^d}$.*

The analogue of Proposition 2.7 for the standard H^∞-control problem in full generality (see [Fra87]) does not seem to be known.

A notion somewhat weaker than stability is causality.

Definition 2.8 ([Lin98]). A rational function $\dfrac{n(\mathbf{z})}{d(\mathbf{z})}$ with $n, d \in \mathbb{R}[\mathbf{z}]$ is said to be *causal* if $d(\mathbf{0}) = d(0, \ldots, 0) \neq 0$. It is called *strictly causal* if in addition $n(\mathbf{z}) = 0$. A rational matrix function $P \in \mathbb{R}^{m\times l}$ is said to be *causal* if all its entries are causal. It is called *strictly causal* if all its entries are strictly causal.

Proposition 2.9 ([Lin98]). *If $P \in \mathbb{R}^{m\times l}(\boldsymbol{z})$ is causal (strictly causal), there exists a right MFD $P = ND^{-1}$ such that* $\det D(\mathbf{0}) \neq 0$ *(in addition, $N(\mathbf{0}) = 0_{m\times l}$). On the other hand, if $P = ND^{-1} \in \mathbb{R}^{m\times l}(\boldsymbol{z})$, and* $\det D(\mathbf{0}) \neq 0$*, then P is causal. If in addition $N(\mathbf{0}) = 0_{m\times l}$, then P is strictly causal.*

Suppose that the plant P is feedback stabilizable; i.e., P satisfies the condition given in Proposition 2.7. Then the following theorem provides the sufficient condition so that P admits the double coprime factorization, (DCF).

Proposition 2.10 ([Lin00]). *Let $P \in \mathbb{R}^{m\times l}(\boldsymbol{z})$ represent a causal feedback stabilizable MIMO nD systems. Let $P = ND^{-1}$ be a right MFD of P (not necessarily coprime), and $F = \begin{bmatrix} D^T & N^T \end{bmatrix}^T \in \mathbb{R}^{(m+l)\times l}[\boldsymbol{z}]$. If there exists a unimodular matrix $U \in \mathbb{R}^{(m+l)\times(m+l)}[\boldsymbol{z}]$ such that some single reduced minor of the polynomial matrix $F_1 = UF$ is devoid of any zeros in the closed unit polydisk, $\overline{\mathbb{D}^d}$, then P has a DCF satisfying the Bézout Identity.*

Corollary 2.11. *Suppose P admits a DCF. Then the set of all stabilizing controllers is given by*

$$\begin{aligned} K &= (X_l - QN_l)^{-1}(Y_l + QD_l) && \textit{where } \det(X_l - QN_l) \neq 0 \qquad (2.6)\\ &= (D_rQ + Y_r)(-N_rQ + X_r)^{-1} && \textit{where } \det(-N_rQ + X_r) \neq 0 \qquad (2.7)\\ &\textit{where } Q \in \mathbb{R}_s^{l\times m}(\boldsymbol{z}) \end{aligned}$$

Consider the transfer matrix functions in (2.3). If P is given as in the Proposition 2.10 with the set of all feedback stabilizing controllers given by (2.6), or (2.7), then it is easy to verify that the closed loop system can be rewritten in the model matching formulation via the Youla parameter Q, i.e.,

$$\begin{aligned} &\begin{bmatrix} (I+PK)^{-1} & -P(I+KP)^{-1} \\ K(I+PK)^{-1} & (I+KP)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} X_rD_l - N_rQD_l & -N_rX_l + N_rQN_l \\ Y_rD_l + D_rQD_l & D_rX_l - D_rQN_l \end{bmatrix} \end{aligned} \qquad (2.8)$$

Obviously, each entry in (2.8) is in the form $T_1 - T_2QT_3$, i.e., in the model matching form. For example, by letting $T_1 = X_rD_l$, $T_2 = N_r$, and $T_3 = D_l$, the first entry of the closed loop transfer matrix function $X_rD_l - N_rQD_l$ can be rewritten as $T_1 - T_2QT_3$.

3 Model Matching and Interpolation Problems

Consider the model matching problem in general stated as follows: given stable rational matrix functions T_1, T_2, and T_3 of compatible sizes, find the stable Q so as to achieve

$$\min_{Q} \| T_1 - T_2 Q T_3 \| \tag{3.1}$$

where the norm is the supremum norm over $\mathbb{D}^d$. Here $T_1 \in \mathbb{R}_s^{l\times m}(\mathbf{z})$, $T_2 \in \mathbb{R}_s^{l\times l}(\mathbf{z})$, and $T_3 \in \mathbb{R}_s^{m\times m}(\mathbf{z})$. We shall focus on the so-called *1-block case* (see [Fra87], i.e., we shall assume that T_2 and T_3 are invertible matrix functions. The performance function F is given by

$$F = T_1 - T_2 Q T_3, \text{ where } Q \in \mathbb{R}^{l\times m}(\mathbf{z}) \tag{3.2}$$

In terms of the Q parameter, If $Q \in \mathbb{R}_s^{l\times m}(\mathbf{z})$ (stable rational matrix function), then so is F. Conversely, if $F \in \mathbb{R}_s^{l\times m}(\mathbf{z})$, then one can backsolve for Q,

$$Q = T_2^{-1}(T_1 - F)T_3^{-1} \tag{3.3}$$

Since T_2^{-1} and T_3^{-1} may or may not be stable, to obtain a stable Q, this case leads to the interpolation conditions given in the Theorem 3.3. In this section, for convenience, we shall only demand that Q be holomorphic on the open polydisk $\mathbb{D}^d$. Before stating such a theorem, some preliminary lemmas are provided here.

Lemma 3.1 (Implicit Function Theorem (see [Sha92])). *If functions $f_1, \ldots, f_k, (k < n)$ are holomorphic in a neighborhood of a point $\boldsymbol{z}^0 \in \mathbb{C}^n$ and also $\det\left(\frac{\partial f_i}{\partial z_j}\right) \neq 0$ in that neighborhood $(i, j = 1, \ldots, k)$, then the system of equations $f_1(\boldsymbol{z}) = \cdots = f_k(\boldsymbol{z}) = 0$ is locally solvable relative to the points $z_1, \ldots, z_k$ and the solution $z_j = g_j(z_{k+1}, \ldots, z_n)$ for $j = 1, \ldots, k$ is holomorphic in a neighborhood of the point $(z_{k+1}^0, \ldots, z_n^0)$.*

Theorem 3.2. *Suppose that we are given an irreducible polynomial $g(\boldsymbol{z})$ in $\boldsymbol{z} = (z_1, \ldots, z_d)$ and that k is a given positive integer. Then a necessary condition for a holomorphic function f on the polydisk $\mathbb{D}^d$ to have the form*

$$f(\boldsymbol{z}) = g^k(\boldsymbol{z})\varphi(\boldsymbol{z}); \quad \boldsymbol{z} \in \mathbb{D}^d \tag{3.4}$$

for some function φ holomorphic on $\mathbb{D}^d$ is that f satisfy the interpolation conditions

$$\left.\frac{\partial^{|j|} f}{\partial \boldsymbol{z}^j}\right|_{V(g)} = 0 \quad \text{for } |j| = 0, 1, \ldots, k-1 \tag{3.5}$$

(where $V(g) = \left\{\boldsymbol{z}^0 \in \mathbb{D}^d \colon g(\boldsymbol{z}^0) = 0\right\}$) in a neighborhood of each smooth point $\boldsymbol{z}^0$ of $V(g)$ inside $\mathbb{D}^d$.

Conversely, if the interpolation conditions (3.5) are satisfied, then the function f can be written as in (3.4) in a neighborhood of each smooth point $\boldsymbol{z}^0$ of $V(g)$ in $\mathbb{D}^d$.

Proof. Suppose that f has the representation $f(\mathbf{z}) = g^k(\mathbf{z})\varphi(\mathbf{z})$. Then $f|_{V(g)} = 0$. Also, all partial derivatives of f with respect to all variables $z_1, \ldots, z_d$ of order l are equal to zero along $V(g)$ for $l = 1, \ldots, k-1$ since each such derivative necessarily contains a factor of $g(\mathbf{z})$. Hence the interpolation conditions (3.5) hold.

Assume now that f satisfies the interpolation conditions (3.5) in a neighborhood of the smooth point $z^0 \in V(g) \cap \mathbb{D}^d$. Let $U(\mathbf{z}^0, \delta) = \left\{ \mathbf{z} \in \mathbb{D}^d \middle| \, |\mathbf{z} - \mathbf{z}^0| < \delta \right\} \subset \mathbb{D}^d$ be a neighborhood around a point $\mathbf{z}^0 \in \mathbb{D}^d$ for small $\delta > 0$. Since f is holomorphic in $\mathbb{D}^d$, for any $\mathbf{z} \in U(\mathbf{z}^0, \delta)$ f admits a multivariable power series representation; i.e.,

$$f(\mathbf{z}) = f(z_1, \ldots, z_d) = \sum_{|j|=0}^{\infty} C_j \left(\mathbf{z} - \mathbf{z}^0\right)^j \tag{3.6}$$

where $C_j = C_j(\mathbf{z}^0) = \dfrac{1}{j!} \dfrac{\partial^{|j|} f}{\partial \mathbf{z}^j}\bigg|_{\mathbf{z}=\mathbf{z}^0}$

Since any partial derivative of a holomorphic function is again holomorphic, C_j is also a holomorphic function since f is. Now for any $z_i \in \mathbb{C}$, we denote $\grave{z}_i$ the remaining variables; i.e., $\grave{z}_i = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_d)$. By using this notation, the equation (3.6) may be rewritten as

$$f(\mathbf{z}) = f(\grave{z}_i, z_i) = \sum_{\substack{\grave{j} \in N^{d-1} \\ j \in N}} C_{\grave{j},j} (\grave{z}_i - \grave{z}_i^0)^{\grave{j}} (z_i - z_i^0)^j \tag{3.7}$$

where $C_{\grave{j},j} = C_{\grave{j},j}(\grave{z}_i^0, z_i^0)$
In particular, when $\grave{z}_i = \grave{z}_i^0$, the equation (3.7) becomes a function of one complex variable.

$$f(z_i) = f(\grave{z}_i^0, z_i) = \sum_{j \in N} C_{\grave{0},j} (z_i - z_i^0)^j \tag{3.8}$$

where $C_{\grave{0},j} = C_{\grave{0},j}(\grave{z}_i^0, z_i^0)$

Now let $V(g)$ be the zero variety of a holomorphic function g given by

$$V(g) = \left\{ \mathbf{z}^0 \in \mathbb{D}^d : \; g(\mathbf{z}^0) = 0 \right\}.$$

$V(g)$ is also irreducible since g is. Then for any $\mathbf{z}^0 = (\grave{z}_i^0, z_i^0) \in V(g) \cap U(\mathbf{z}^0, \delta)$ and by assuming that $\dfrac{\partial g}{\partial z_i}(\mathbf{z}_0) \neq 0$ (such an i exists if $\mathbf{z}_0$ is a smooth point of $V(g)$), the Implicit Function Theorem 3.1 implies that there exists a holomorphic function h defined on $U(\grave{z}_i^0, \delta)$ so that $z_i = h(\grave{z}_i)$. Such a z_i is unique in a neighborhood of z_i^0 so that $(\grave{z}_i, z_i) \in V(g)$ for $\grave{z}_i \in U(\grave{z}_i^0, \delta)$

Define $\tilde{g}(\mathbf{z}) = \tilde{g}(\grave{z}_i, z_i) = z_i - h(\grave{z}_i)$ for $\mathbf{z}^0 \in U(\mathbf{z}^0, \delta)$. Then $V(\tilde{g}) = \{\mathbf{z} \in U(\mathbf{z}^0, \delta) | \, z_i = h(\grave{z}_i)\} = V(g) \cap U(\mathbf{z}^0, \delta)$. As a result, $\tilde{g}(\mathbf{z}) = g(\mathbf{z})\psi(\mathbf{z})$, where ψ is holomorphic on $U(\mathbf{z}^0, \delta)$

Since f satisfies the interpolation conditions (3.5), $C_{\grave{0},j} = C_{\grave{0},j}(\grave{z}_i^0, z_i^0) = 0$ for $j = 0, 1, \ldots, k-$

1 for each $\mathbf{z}^0 = (\grave{z}_i^0, z_i^0) \in V(g)$. Hence, (3.8) becomes

$$\begin{aligned} f(z_i) &= f(\grave{z}_i^0, z_i) \\ &= \sum_{j=k}^{\infty} C_{\grave{0},j}(z_i - z_i^0)^j \\ &= (z_i - z_i^0)^k \sum_{j=0}^{\infty} C_{\grave{0},j+k}(z_i - z_i^0)^j \end{aligned} \tag{3.9}$$

In particular, since $z_i^0 = h(\grave{z}_i^0)$ for $\mathbf{z}^0 = (\grave{z}_i^0, z_i^0) \in V(g) \cap U(\mathbf{z}^0, \delta)$,

$$f(\grave{z}_i^0, z_i) = \left(z_i - h(\grave{z}_i^0)\right)^k \sum_{j=0}^{\infty} C_{\grave{0},j+k}\left(z_i - h(\grave{z}_i^0)\right)^j$$

We now consider $\grave{z}_i^0$ as a variable and therefore replace $\grave{z}_i^0$ by $\grave{z}_i$. So,

$$\begin{aligned} f(\mathbf{z}) &= f(\grave{z}_i, z_i) \\ &= (z_i - h(\mathbf{z}))^k \sum_{j=0}^{\infty} C_{\grave{0},j+k}\,(z_i - h(\grave{z}_i))^j \\ &= \tilde{g}^k(\mathbf{z})\Phi(\mathbf{z}) \end{aligned} \tag{3.10}$$

$$\text{where} \quad \begin{aligned} C_{\grave{0},j+k} &= C_{\grave{0},j+k}\,(\grave{z}_i, h(\grave{z}_i)) \\ \Phi(\mathbf{z}) &= \sum_{j=0}^{\infty} C_{\grave{0},j+k}\,(\grave{z}_i, h(\grave{z}_i))^j \end{aligned}$$

Then, $f(\mathbf{z}) = g^k(\mathbf{z})\varphi(\mathbf{z})$ on $V(g) \cap U(\mathbf{z}^0, \delta)$, where $\varphi(\mathbf{z}) = \psi^k(\mathbf{z})\Phi(\mathbf{z})$ is holomorphic since ϕ and Φ are. Now combine together all these localized φ's to obtain (3.4) in a neighborhood of $\mathbf{z}^0$. □

We now explain the type of interpolation problem to which the model matching problem can be converted in the 1-block case. For $u = 1, \ldots, \eta$, assume that we are given distinct irreducible (scalar) polynomials q_u with zero variety $V(q_u)$ having nontrivial intersection with $\mathbb{D}^d$, holomorphic matrix functions G_u and $\widetilde{G}_u$ (of compatible sizes for the interpolation conditions to follow to make sense) and positive integers k_u. For $v = 1, \ldots, \mu$ assume that similarly we are given distinct irreducible polynomials s_v together with holomorphic matrix functions H_v and $\widetilde{H}_v$ (of compatible sizes) and positive integers ℓ_v. For each pair of indices (u, v) for which $q_u = s_v =: h_{uv}$, assume that we are given an additional matrix function R_{uv}. The whole aggregate

$$\omega = \{q_u, G_u, \widetilde{G}_u, k_u; s_v, H_v, \widetilde{H}_v, \ell_v; R_{uv}\} \tag{3.11}$$

we call a *1-block interpolation data set*. We say that a matrix function F holomorphic on $\mathbb{D}^d$ satisfies the interpolation conditions associated with ω (denoted by $F \in \mathcal{I}(\omega)$) if

$$\left\{\frac{\partial^{|i|}}{\partial z_i} G_u(\mathbf{z})F(z)\right\}\Big|_{V(q_u)} = \{\frac{\partial^{|i|}}{\partial z_i}\widetilde{G}_u(\mathbf{z})\}\Big|_{V(q_u)} \text{ for } u = 1,\dots,\eta \text{ and } i = 0,1,\dots,k_u-1, \tag{3.12}$$

$$\left\{\frac{\partial^{|j|}}{\partial \mathbf{z}_j} F(\mathbf{z}))H_v(\mathbf{z})\right\}\Big|_{V(s_v)} = \left\{\frac{\partial^{|j|}}{\partial \mathbf{z}_j}\widetilde{H}_v(\mathbf{z})\right\}\Big|_{V(s_v)} \text{ for } v = 1,\dots,\mu \text{ and } j = 0,1,\dots,\ell_v-1, \text{ and} \tag{3.13}$$

$$\left\{\frac{\partial^{|l|}}{\partial \mathbf{z}_\ell} G_u(\mathbf{z})F(\mathbf{z}))H_v(\mathbf{z})\right\}\Big|_{V(h_{uv})} = \left\{\frac{\partial^{|l|}}{\partial \mathbf{z}_\ell}R_{uv}(\mathbf{z})\right\}\Big|_{V(h_{uv})} \text{ for all pairs of indices } u, v$$
$$\text{with } q_u = s_v \text{ and for } l = 0,1,\dots,k_u+\ell_v-1 \tag{3.14}$$

Given T_1, T_2 and T_3 as in the 1-block case of the model matching problem, we associate an interpolation data set ω as follows. Consider the set of unstable entries of T_2^{-1}, say $\left\{\frac{p_{i_a,j_a}}{q_{i_a,j_a}}(\mathbf{z})\right\}$ for $a = 1,\dots,\alpha$. Let $q(\mathbf{z})$ be the least common multiple of $\{q_{i_1,j_1}(\mathbf{z}),\dots,q_{i_\alpha,j_\alpha}(\mathbf{z})\}$. Also let $T_3^{-1}(\mathbf{z}) = \frac{r_{ij}}{s_{ij}}(\mathbf{z}), i,j = 1,\dots,m$ be an $m\times m$ rational matrix valued function in d variables, and consider the set of unstable entries of T_3^{-1}, say $\left\{\frac{r_{i_b,j_b}}{s_{i_b,j_b}}(\mathbf{z})\right\}$ for $b = 1,\dots,\beta$. Let $s(\mathbf{z})$ be the least common multiple of $\left\{s_{i_1,j_1}(\mathbf{z}),\dots,s_{i_\beta,j_\beta}(\mathbf{z})\right\}$. Suppose now that $q(\mathbf{z})$ and $s(\mathbf{z})$ can be factored into irreducible polynomials, say $q(\mathbf{z}) = q_1^{k_1}(\mathbf{z})\cdots q_\eta^{k_\eta}(\mathbf{z})$, where $k_i > 0$ for $i = 1,\dots,\eta$, and $s(\mathbf{z}) = s_1^{\ell_1}(\mathbf{z})\cdots s_\mu^{\ell_\mu}(\mathbf{z})$, where $\ell_i > 0$ for $i = 1,\dots,\mu$, respectively. Let $V(g)$ be the zero variety of $g(\mathbf{z}) = \left\{\mathbf{z}^0 \in \mathbb{D}^d \middle|\, g(\mathbf{z}^0) = 0\right\}$ contained in $\mathbb{D}^d$. Then for each $u \in \{1,\dots,\eta\}$, $T_2^{-1}(\mathbf{z}) = \frac{G_u}{q_u^{k_u}}(\mathbf{z})$, where $G_u(\mathbf{z})$ is a meromorphic matrix function in $\mathbb{D}^d$ evaluated along the variety $V(q_u)$, and $q_u^{k_u}$ is an unstable irreducible polynomial with multiplicity k_u. In addition we set $\widetilde{G}_u(\mathbf{z}) = G_u(\mathbf{z})T_1(\mathbf{z})$. Analogously, for each $v \in \{1,\dots,\mu\}$, $T_3^{-1}(\mathbf{z}) = \frac{H_v}{s_v^{\ell_v}}(\mathbf{z})$, where $H_v(\mathbf{z})$ is a meromorphic matrix function in $\mathbb{D}^d$ along the variety $V(s_v)$. Set $\widetilde{H}_v(\mathbf{z}) = T_1(\mathbf{z})H_v(\mathbf{z})$. In addition, if q and s have some common factors, say $q_u = s_v$ for some pair of indices u and v, set $h_{uv} = q_u = s_v$ and $R_{uv}(\mathbf{z}) = G_u(\mathbf{z})T_1(\mathbf{z})H_v(\mathbf{z})$. In this way we have formed an interpolation data set ω as in (3.11). When ω is formed in this way from T_1, T_2, T_3, let us write $\omega = \omega_{T_1,T_2,T_3}$.

Now we are ready to state the main theorem, which gives the connection between the model matching and interpolation problems

Theorem 3.3. *Let T_1, T_2, T_3 be the data set for a 1-block model matching problem, and let ω_{T_1,T_2,T_3} be the associated interpolation data set as delineated in the previous paragraph.*

Then a necessary condition for a given function F holomorphic on $\mathbb{D}^d$ to have the model matching form $F = T_1 - T_2QT_3$ for a stable Q is that F satisfy the interpolation conditions

(3.12), (3.13) *and* (3.14) *associated with the data set* ω_{T_1,T_2,T_3} *(i.e.,* $F \in \mathcal{I}(\omega_{T_1,T_2,T_3})$*). Conversely, if* F *satisfies the interpolation conditions* (3.12), (3.13) *and* (3.14) *for the data set* ω_{T_1,T_2,T_3} *then* F *has the model matching form* (3.2) *locally in a neighborhood of all points of* $\mathbb{D}^d$ *with the possible exception of the singular points of some* $V(q_u)$ *or* $V(s_v)$*.*

Proof. The proof is separated into four cases depending on the location of zeros of T_2 and T_3.

Case 1: *Both* T_2 *and* T_3 *do not contain any zero in the* $\mathbb{D}^d$*.* Then there are no interpolation conditions since both T_2^{-1} and T_3^{-1} are stable.

Case 2: *Either* T_2 *or* T_3 *contains at least one zero in the* $\mathbb{D}^d$. Without loss of generality, suppose T_3 contains unstable zeros. This implies T_3^{-1} becomes unstable, and hence (3.3) becomes

$$(T_1 - F)T_3^{-1} = T_2 Q = \tilde{Q} \tag{3.15}$$

Follow the construction as stated in the theorem yields, $(T_1(\mathbf{z}) - F(\mathbf{z}))H_v(\mathbf{z}) = s_v^{\ell_v}(\mathbf{z})\tilde{Q}(\mathbf{z})$ for each $v \in \{1, \ldots, \mu\}$, and then apply Theorem 3.2. In this case, (3.12) and (3.14) can be dropped.

Case 3: *Both* T_2 *and* T_3 *contain zeros in the* $\mathbb{D}^d$*, and the denominators of unstable entries of* T_2^{-1} *and* T_3^{-1} *have no common factors.* We can consider the interpolation conditions for T_2 and T_3 separately since they do not have any factors in common, and then apply the theorem 3.2. This case, the interpolation conditions (3.14) can be dropped.

Case 4: *Both* T_2 *and* T_3 *contain zeros in the* $\mathbb{D}^d$*, and the denominators of unstable entries of* T_2^{-1} *and* T_3^{-1} *have some common factors.* First, we consider the interpolation conditions for T_2 and T_3 separately to get (3.12) and (3.13). Since T_2^{-1} and T_3^{-1} contain some common factors of unstable entries, we define an unstable polynomial $h = \{h_{uv}^{k_u+\ell_v}\}$ for some pair of indices u and v such that $q_u = s_v$ as stated in the theorem. Then for each h_{uv}, equation (3.3) can be rewritten as

$$\frac{1}{q_u^{k_u}} G_u(\mathbf{z})(T_1(\mathbf{z}) - F(\mathbf{z}))H_v(\mathbf{z})\frac{1}{s_v^{\ell_v}} = Q(\mathbf{z}), \tag{3.16}$$

or, $G_u(\mathbf{z})(T_1(\mathbf{z}) - F(\mathbf{z}))H_v(\mathbf{z}) = h_{uv}^{k_u+\ell_v}(\mathbf{z})Q(\mathbf{z})$.
Then apply theorem 3.2 to obtain the interpolation condition (3.14). □

Remark 3.4. Theorem 3.3 is a multivariable analogue of Theorem 16.9.3 in [BGR90]. To be consistent with the terminology given in [BGR90], the equations in (3.12), (3.13), and (3.14) are called respectively the left-, right- and two-sided interpolation conditions for a tangential interpolation problem. The proof for the single-variable case relies heavily in the end on the existence of a local Smith-McMillan form for rational matrix functions; as the Smith-McMillan form is unavailable in the multivariable setting, our proof here relies exclusively on making use of the notion of factor coprime. Theorem 16.9.3 in [BGR90] is more precise in that the statement there asserts that one can take the interpolation data set ω to be in canonical, minimal form. Specifically, in the 1-variable case, the data functions G_u and H_V are constructed from a canonical set of left null functions for T_2 (respectively, right null functions for T_3) and one can in general reduce the number of rows of G_u (respectively, the number of columns of H_v). To implement a similar reduction in the multivariable case, the following weaker form of the local Smith-McMillan form would be of interest: *Given a polynomial $m \times n$ matrix function $W(\boldsymbol{z})$ and an irreducible (scalar) polynomial q such that*

$$\underset{i:\,|i|\le k-1}{span}\left\{ im\, \frac{\partial^{|i|}}{\partial z_i} W(z)|_{V(q)} \right\} = r,$$

then there exists $m \times r$, $r \times n$ and $n \times n$ matrix polynomials P, Q, S so that

$$W(z) = P(z)Q(z) + q(z)^k S(z).$$

Example 1. Let $T_3(z)^{-1} = \begin{bmatrix} \frac{1}{z_1^2} & \frac{1}{z_2+2} \\ \frac{z_3}{z_1 z_2 - 0.5} & \frac{1}{z_1} \end{bmatrix}$ Obviously, the set of unstable entries of T_3^{-1} is $\left\{ \frac{1}{z_1^2}, \frac{z_3}{z_1 z_2 - 0.5}, \frac{1}{z_1} \right\}$. Let $q(\mathbf{z}) = \text{l.c.m. of } \{z_1^2,\ z_1 z_2 - 0.5,\ z_1\} = z_1^2\,(z_1 z_2 - 0.5)$. Set $q_1 = z_1$ with multiplicity 2, and $q_2 = z_1 z_2 - 0.5$. Then, the corresponding G_1 and G_2, respectively are given by

$$\begin{aligned} G_1(\mathbf{z}) &= \begin{bmatrix} 1 & \frac{z_1^2}{z_2+2} \\ \frac{z_1^2 z_3}{z_1 z_2 - 0.5} & z_1 \end{bmatrix}, \\ \text{and}\quad G_2(\mathbf{z}) &= \begin{bmatrix} \frac{z_1 z_2 - 0.5}{z_1^2} & \frac{z_1 z_2 - 0.5}{z_2+2} \\ z_3 & \frac{z_1 z_2 - 0.5}{z_1} \end{bmatrix}. \end{aligned}$$

Consider first when $q_1 = z_1$, then $G_1(\mathbf{z})$ can be written as $G_1(\mathbf{z}) = z_1^2 \tilde{Q}(\mathbf{z})$ and the zero variety $V(q_1) = \{(z_1,\ z_2,\ z_3) \in \mathbb{D}^3 |\ z_1 = 0\}$. Then the interpolation conditions are:

$$(T_1 - F)(\mathbf{z})\, G_1(\mathbf{z})|_{V(q_1)} = (T_1 - F)(\mathbf{z})|_{V(q_1)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \underline{0} \tag{3.17}$$

$$\begin{aligned} \frac{\partial}{\partial z_1} (T_1 - F)(\mathbf{z})\, G_1(\mathbf{z})|_{V(q_1)} &= \frac{\partial}{\partial z_1} (T_1 - F)(\mathbf{z})|_{V(q_1)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &\quad + (T_1 - F)(\mathbf{z})|_{V(q_1)} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \underline{0} \end{aligned} \tag{3.18}$$

When $q_2 = z_1 z_2 - 0.5$, then $G_2(\mathbf{z})$ can be written as $G_2(\mathbf{z}) = (z_1 z_2 - 0.5)\tilde{Q}(\mathbf{z})$ and the zero variety $V(q_2) = \{(z_1,\ z_2,\ z_3) \in \mathbb{D}^3 |\ z_1 z_2 = 0.5\}$. Then the interpolation condition is:

$$(T_1 - F)(\mathbf{z})\, G_2(\mathbf{z})|_{V(q_2)} = (T_1 - F)(\mathbf{z})|_{V(q_2)} \begin{bmatrix} 0 & 0 \\ z_3 & 0 \end{bmatrix} = \underline{0} \tag{3.19}$$

Hence, the interpolation problem may be restated as: *find* $F \in \mathbb{R}_s^{l\times m}(\boldsymbol{z})$ *satisfying the interpolation conditions* (3.17)–(3.19) *so that* $Q = T_2^{-1}(T_1 - F)T_3^{-1}$ *is stable.*

Remark 3.5. If one loosens the 1-block assumption on (T_1, T_2, T_3), the model matching form for F is equivalent to interpolation conditions for F on subvarieties of higher codimension, or alternatively, to interpolations conditions on the whole of $\mathbb{D}^d$. For the single-variable case ($d = 1$), there are only the two possibilities of codimension equal to 1 (interpolation at isolated points) or interpolation along the whole unit disk—see [BR92, BR94] for a thorough treatment.

Remark 3.6. We now consider the special case where $f(\mathbf{z}) = g^k(\mathbf{z})\varphi(\mathbf{z})$ where $k = 1$. Then the interpolation conditions (3.12) – (3.14) simplify to

$$G_u(\mathbf{z})\big(T_1(\mathbf{z}) - F(\mathbf{z})\big)\big|_{V(q_u)} = 0 \text{ for } u = 1, \ldots, \eta, \tag{3.20}$$

$$\big(T_1(\mathbf{z}) - F(\mathbf{z})\big)H_v(\mathbf{z})\big|_{V(s_v)} = 0 \text{ for } v = 1, \ldots, \mu, \text{ and} \tag{3.21}$$

$$\frac{\partial}{\partial z_i}\big[G_u(\mathbf{z})\big(T_1(\mathbf{z}) - F(\mathbf{z})\big)H_v(\mathbf{z})\big]\Big|_{V(h_{uv})} = 0 \text{ for } i = 1, \ldots, d, \text{ and}$$
$$\text{for all pairs of indices } u, v \text{ with } q_u = s_v \tag{3.22}$$

Note that all these formulations of interpolation conditions depend heavily on a particular choice of coordinates for the various varieties $V(q_u)$ and $V(s_v)$. It is of interest to note that conditions (3.20) and (3.21) can be expressed in a more coordinate-free form by using the Poincaré residue map (see [GH78, page 147]). Indeed, in connection with (3.20) e.g., application of the Poincaré residue map to the d-form

$$T_2(\mathbf{z})^{-1}(F(\mathbf{z}) - T_1(\mathbf{z}))\ dz_1 \wedge \cdots \wedge dz_d = \frac{G_u(\mathbf{z})}{q_u(\mathbf{z})}(F(\mathbf{z}) - T_1(\mathbf{z}))\ dz_1 \wedge \cdots \wedge dz_d$$

yields the $(d-1)$-form on $V(q_u)$

$$(-1)^{i-1}G_u(\mathbf{z})(F(\mathbf{z}) - T_1(\mathbf{z}))\frac{dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_d}{\partial q_u/\partial z_i}\Bigg|_{V(q_u)}$$

for any i such that $\frac{\partial q_u}{\partial z_i} \neq 0$. Thus the interpolation condition (3.20) on F can be expressed as the vanishing of the Poincaré residue of the d-form $T_2(\mathbf{z})^{-1}(F(\mathbf{z}) - T_1(\mathbf{z}))\ dz_1 \wedge \cdots \wedge dz_d$ along the variety $V(q_u)$.

Theorem 3.3 characterizes internal stability in terms of satisfaction of interpolation conditions by the performance function F. In the H^∞-control problem, we seek such a performance function F which in addition has norm less than or equal to given tolerance level γ. By scaling, we may assume without loss of generality that we have set $\gamma = 1$. Let us say that a matrix-valued function F is in the d-variable Schur-class, denoted by $\mathcal{S}_d$, if F is holomorphic on $\mathbb{D}^d$ with $\|F(\mathbf{z})\| \le 1$ for all points $\mathbf{z} \in \mathbb{D}^d$. The H^∞ problem (with tolerance level γ normalized to $\gamma = 1$) then becomes: *For given interpolation data* $\mathcal{I} = \{\{q_u, k_u, G_u, s_v, \ell_v, H_v, T_1\}$, *find a matrix function* F *(of the appropriate size) in the Schur class* $\mathcal{S}_d$ *which meets the interpolation conditions* (3.12), (3.13) *and* (3.14).

It turns out that it is not so convenient to solve this multivariable Nevanlinna-Pick interpolation problem in the Schur class but rather in a somewhat restricted class which we call the *Schur-Agler class* $\mathcal{SA}_d$ defined as follows: *the matrix-valued function* F *is in the* d*-variable Schur-Agler class* $\mathcal{SA}_d$ *if* F *is holomorphic on* $\mathbb{D}^d$ *and* $\|F(rT_1, \dots, rT_d)\| \le 1$ *for all* $r < 1$ *and for any* d*-tuple of commuting contraction operators* $(T_1, \dots, T_d)$ *on a Hilbert space* $\mathcal{H}$. It turns out that the $\mathcal{SA}_d = \mathcal{S}_d$ for $d = 1, 2$ but $\mathcal{SA}_d \underset{\neq}{\subset} \mathcal{S}_2$ for $d > 2$.

The scalar case of this interpolation problem, with the interpolation nodal varieties taken to have dimension zero, is simply: given interpolation nodes $\mathbf{z}^1, \dots, \mathbf{z}^n \in \mathbb{D}^d$ and interpolation values $w_1, \dots, w_n \in \mathbb{C}$, find a scalar function $F \in \mathcal{SA}_d$ satisfying the interpolation conditions

$$F(\mathbf{z}^i) = w_i \text{ for } i = 1, \dots, n. \tag{3.23}$$

The original result of Agler [Agl87] on this problem is: *there exist a scalar function* F *in* $\mathcal{SA}_d$ *meeting the interpolation conditions* (3.23) *if and only if there exist* d *positive-semidefinite* $n \times n$ *matrices* Λ^1, ... , Λ^d *so that*

$$1 - w_i \overline{w}_j = \sum_{k=1}^{d} (1 - \mathbf{z}_k^i \overline{\mathbf{z}_k^j}) \Lambda_{i,j}^k \text{ for } i, j = 1, \dots, n. \tag{3.24}$$

This condition now known as a Linear Matrix Inequality (LMI) is a practical solution to the problem. This result was extended to the matrix-valued setting (with the interpolation nodal varieties still assumed to be zero-dimensional and without consideration of two-sided interpolation conditions) in [BT99, AMcC99]. A contour integral formulation which incorporated higher-order interpolation conditions but still at isolated points was solved in [ABB00]. There now has appeared some work which applies to the case of interpolation along higher dimensional varieties (but with the multiplicities k_u and ℓ_v all taken equal to 1)—see [BB1, BB2]; to apply this work one must first obtain a parametrization of the various varieties $V(q_u)$ and $V(s_v)$ and then the solution takes the form of an infinite LMI.

There are a number of remaining issues which must be resolved before this interpolation approach to H^∞-control for multivariable systems comes close to being as successful as the 1-variable case; two such issues are:

1. Formulation and solution of the interpolation problem in terms of state-space coordinates.

2. A reliable analysis of how to approximate the solution of an infinite LMI via solving finite LMIs.

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