Controllability of a Pair of Coupled Quantum Dots

V. Ramakrishna
Department of Mathematical Sciences
and Center for Signals, Systems and Telecoomunications
University of Texas at Dallas
Richardson, TX 75083
USA

Abstract

This article studies the controllability of a pair of coupled quantum dots being interrogated by an external electromagnetic field. It is shown that this system is controllable. However, in the limit of large spatial separation between the two dots the dynamical Lie algebra of the problem degenerates to the complex representation of $so(4)$ in $su(4)$. The question of which pure and mixed states for the system are accessible from the initial condition is then studied via an ab initio approach. This is achieved by finding an explicit conjugation, within $su(4)$, between the complex representation of $so(4)$ and $su(2) \otimes su(2)$, and then using the greater structure of the group $SU(2) \otimes SU(2)$.

1 Introduction

The engineering of quantum systems is a field of intense research. Motivations come from several sources - site-specific chemistry, spectroscopy, semiconductor heterostructures and quantum information processing. The final item on the previous list has, in particular, triggered the systematic investigation of the engineering of innumerable quantum systems - atoms, molecules, Bose-Einstein condensates, Rydberg atoms, quantum wells, wires and dots etc., This article considers the control of a coupled pair of quantum dots.

Quantum dots have been suggested in [1] as vehicles of quantum computation, and therefore their control is of considerable interest. More details on this suggested architecture follows in Section 2. The interest of coupled quantum dots in this manuscript is that, under the limit of large separation between the dots, the real Lie algebra generated by the internal Hamiltonian and the external Hamiltonian degenerates into a representation of $so(4)$ by skew-Hermitian matrices. In their interesting work, Schirmer et al., have shown, [2], that for four level atomic systems, with only nearest neighbour interactions (i.e., $b_{ij} \neq 0$ iff $j = i-1, i+1$, where b_{ij} are the entries of the matrix representation of the interaction Hamiltonian), the control (or dynamical) Lie algebra cannot ever be $so(4)$. Thus, this coupled quantum dot system is interesting, since it is a concrete physical system, which though controllable, has its dynamical Lie algebra degenerating into $so(4)$.

Thus, in the limit of large separation, the system is effectively uncontrollable. Therefore, the question of determining the reachable set from the initial condition becomes interesting. It is shown in Section 3, that this representation of $so(4)$ via skew-Hermitian matrices is explicitly conjugate, within $su(4)$, to the Lie algebra $su(2) \otimes su(2)$. Thus, this problem effectively is equivalent to determining the orbits of the Lie group $SU(2) \otimes SU(2)$ on the unit sphere in C^4 (for pure states) and on the positive semidefinite states with trace one (for mixed states). Since the linear action of any subgroup of $SU(4)$ on the unit sphere is equivalent to the action of the subgroup on pure states written as density matrices, this work will concentrate on the second action. Of course, one may embed density matrices in the vector space of Hermitian matrices and then use invariant theory to determine the orbits. In fact, this has been done in [5] where the same question arises in the context of determining entanglement invariants of two qubits. Strictly speaking, this definition of entanglement invariants is not universally accepted, since it deals with only one aspect of entanglement (indeed, for mixed states there is no genuine consensus, on what ought to be the precise definition of entanglement). From the perspective of this work, the usage of invariant theory leads to criteria which do not seem to have adequate physical intuition - i.e., the invariants, whose common level sets (more precisely their connected components) the orbits are, do not seem (at least to me) to possess physical interpretation. Therefore, this manuscript will use a less highbrow technique. Specifically, an analogue of the Bloch sphere for 4×4 density matrices will be used and elementary arguments will then be used to address orbit determination. The drawback of this approach is that, while for pure states necessary and sufficient conditions are thereby obtained, the results for mixed states are only necessary conditions (pending further research). On the other hand, the criteria seem more closely related to entanglement criteria which are more "physical".

The remainder of this manuscript are organized as follows. Section 2 contains a brief introduction to the physics of quantum dots and the specific model in use. Section 3 argues that while the system is controllable, its Lie algebra degenerates into a specific representation of $so(4)$ in the limit of large separation between the dots. This Lie algebra is shown to be explicitly conjugate to $su(2) \otimes su(2)$. Section 4 presents results on the determination of reachable sets. Section 5 offers conclusions.

2 Quantum Dots

In semiconductor heterostructures, by imposing designed potential barries via techniques such as electron beam lithography or molecular beam epitaxy, the motion of an electron can be confined to a plane (quantum wells), or to a line (quantum wires) or to behave as a stationary particle (quantum dots). Thus, to a certain extent, a quantum dot behaves like a virtual atom. It is also possible to arrange several quantum dots in a line. For more details about the physics of such heterostructures, the reader could do no better than consult [3]. Furthermore, such a configuration of quantum dots can be made to interact with external electromagnetic fields. This interaction has been proposed in [1] as an architecture for displaying the conditional dynamics needed for quantum computation. The following Hamiltonian describes (under simplifying assumptions which do not consider the effects of holes in the valence band of the semiconductors):

$$H = H_1 + H_2 + V_{12} - du(t) \tag{2.1}$$

This takes into account only the ground and first excited states of each single electron quantum dot. With this clarified, the various terms on the RHS of Equation (2.1) are as follows. $H_i = \hbar\omega_i, i = 1, 2$ where ω_i is the energy difference between the two states in the ith dot. V_{12}'s matrix elements are determined according to $(-1)^{\epsilon_1+\epsilon_2}\hbar(\frac{-d_1 d_2}{4\pi\epsilon_0 R^3})$. We refer the reader to [1] for the specifics about ϵ_1, ϵ_2 - in brief they are the labels of the computational basis in each dot. $d_i, i = 1, 2$ are the dipole moments in the ground and first excited states in the ith dot and R is the separation between the two dots. Finally, dd is the dipole coupling between the ground and first excited states and $u(t)$ is the external optical field. Note that in [1] the last term in the RHS of Equation (2.1) is not explicitly mentioned since the external field there is only used in a static, adiabatic, on-off fashion. However, for more general purposes it is useful to consider more general fields.

With this the associated control system for the unitary generator is given by

$$\dot{U} = AU + BUu(t) \tag{2.2}$$

where

$$A = -i\mathrm{d}iag\ (-\omega_1 - \omega_2 - D, \omega_2 - \omega_1 + D, \omega_1 - \omega_2 + D, \omega_1 + \omega_2 - D), D = \frac{d_1 d_2}{4\pi\epsilon_0 R^3}$$

and

$$B = \begin{pmatrix} 0 & id & id & 0 \\ id & 0 & 0 & id \\ id & 0 & 0 & id \\ 0 & id & id & 0 \end{pmatrix}$$

3 Controllability and Degeneration to $so(4)$

It can be shown that the real Lie algebra generated by A and B is $su(4)$ under the following conditions i) $\mid \omega_1 \mid \neq \mid \omega_2 \mid$; and ii) $D = \frac{d_1 d_2}{4\pi\epsilon_0 R^3} \neq 0$ and iii) $d \neq 0$. It is conceivable that condition some other combination of i) and ii) would still ensure this Lie algebra equalling $su(4)$, but this is to be investigated. Thus, under i), ii) and iii) the system is fully controllable. An explicit calculation of this Lie algebra is omitted here, since the principal interest of this article is the obtainment of $so(4)$.

If $D = 0$, but i) and ii) hold then this Lie algebra is indeed a representation of $so(4)$ by imaginary matrices in $su(4)$. Before delving into this further, we note that D could be zero if either i) one of the $d_i = 0$ or ii) $R \to \infty$. While the former is theoretically possible, such a system would never be chosen by the experimentalist because then there would be no dipole-dipole interaction between the two single-electron dots and thus, such a system would be of no use in quantum information processing. The latter condition too may seem artificial - indeed, it would be if the "computer" consisted of just two qubits, since there would be no need to place the two dots very far apart from each other. However, in a configuration consisting of many such dots, the members of a certain pair will eventually be at a great separation. Thus, if a local two qubit transformation involving this pair is desired, then we are lead to a system for which $R \to \infty$. Of course, this separation must be such that the two dots can be addressed by the laser and the field must be such that intermediate dots are not excited (this is a question of the profile of the field - a question which can be addressed by imposing amplitude, pulse area and frequency content restrictions on the field, see e.g., [6, 7, 8]). The correct manner in which to view this large separation condition is then the following. When the separation is large the Lie algebra generated is still $su(4)$ and thus, the system is theoretically controllable. But with R increasing the system gets closer to being uncontrollable and this Lie algebra is approximated by an imaginary representation of $so(4)$.

Specifically, when $D = 0$, A is approximated by $\tilde{A} = -i diag\ (-\omega_1 - \omega_2, \omega_2 - \omega_1, \omega_1 - \omega_2, \omega_1 + \omega_2)$ and B remains unchanged. The real Lie algebra generated by $\tilde{A}$ and B is a Lie subalgebra of $su(4)$, denoted L_0 (the zero stands for $D = 0$) which is conjugate to $su(2) \otimes su(2)$ by the following element of $SU(4)$:

$$V = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \end{pmatrix}$$

, i.e.,

$$VL_0V^* = su(2) \otimes su(2)$$

Thus, to check if two mixed states $\rho_i, i = 1, 2$ are reachable from one another in the limit $R \to \infty$ it suffices to check if the mixed states $V^*\rho_i V, i = 1, 2$ belong to the same orbit of the conjugate action of $SU(2) \otimes SU(2)$ on the set of 4×4 mixed states (it is easy to check that this is indeed an action).

4 A "Bloch Sphere" Approach to Determining Orbits

Since mixed states are, in particular, Hermitian matrices one could as well use methods of invariant theory to solve the problem arising at the end of the previous section. For reasons, mentioned in the introduction, this section will develop a different technique.

The matrices, $I_4, \sigma_i \otimes I_2, I_2 \otimes \sigma_i, \sigma_i \otimes \sigma_k, i, k = 1, \ldots, 3$ form a basis for $M_4(C)$ and a basis for the real vector space of 4×4 Hermitian matrices too. Thus, every mixed state can be written as $4\rho = \alpha I_4 + \sum_{i=1}^{3} \beta_i \sigma_i \otimes I_2 + \sum_{i=1}^{3} \gamma_i I_2 \otimes \sigma_i + \sum i = 1^3 \sum_{k=1}^{3} \delta_{ik} \sigma_i \otimes \sigma_k$, with all coefficients being real. Let $\beta = (\beta_1, \beta_2, \beta_3)$ and similarly let γ be the vector of coefficients, γ_i and finally, let Δ be the 3×3 real matrix with coefficient δ_{ik}.

Since ρ has to have trace 1, it follows that $\alpha = 1$. Furthermore, the vectors β and γ have to belong to the convex set $\{x \in R^3, || \ x \ || \leq 1\}$. This follows from taking partial trace of a mixed state ρ. This yields $\frac{1}{2}(I_2 + \sum_{i=1}^{3} \beta_i \sigma_i)$ in one case and $\frac{1}{2}(I_2 + \sum_{i=1}^{3} \gamma_i \sigma_i)$ in the other. Since these partial traces have to be 2×2 states (pure or mixed), the conclusion follows from the usual Bloch sphere picture in two dimensions. However, it does not follow that the pure states are given by the extremen points of this set. Both to determine when a trace 1 Hermitian matrix is a state and which such states are pure, it is useful to compute the square of a Hermitian, trace 1 matrix in terms of its expansion given above (since, positive semidefinite matrices are squares of Hermitian matrices and since pure states are projections).u The details of this simple but laborious calculation are omitted for reasons of brevity. Only the following results will be recorded here:

Lemma 4.1. *Every mixed state satsifies* $\alpha = 1$, $\beta = \frac{1}{2}(k\beta_0 + D\gamma_0)$, $\gamma = \frac{1}{2}(k\gamma_0 + D^T\beta_0)$, $\Delta = \frac{1}{2}(kD - adj\ (D) - \beta_0\gamma_0^T)$. *Here,* β_0, γ_0 *are in* R^3, D *is a* 3×3 *matrix and* k *is a constant which satisfy the following:* $|| \ \beta_0 \ ||^2 + || \ \gamma_0 \ ||^2 + Tr(D^T D) \leq 4$ *and* k^2 *is difference between the RHS and the LHS of this inequality.*

Every pure state can be represented by $\alpha = 1$, $\beta \in R^3$ *satisfying* $|| \ \beta \ || \leq 1$, $\gamma = D^T\beta$, *where* D *satisfies the same relationship with beta,* γ *as in the paragraph above. Furthermore,* $\Delta = D$. *Hence,* $detT = || \ \beta \ ||^2 - 1 \leq 0$.

The relevance of the last statement above will be obvious soon. Now let us compute the effect of an element $U \otimes V \in SU(2) \otimes SU(2)$ acting by conjugation on a state. This is easy and is given by the following elegant formula: $\phi_{U \otimes V}(S)$ corresponds to $\beta \rightarrow U_R\beta; \gamma \rightarrow V_R\gamma; \Delta \rightarrow U_R \Delta V_R^T$, where U_R, V_R are the $SO(3)$ matrices corresponding to U, V respectively under the standard homomorphism from $SU(2)$ to $SO(3)$.. From this simple formula a few obvious necessary conditions for belonging to the same orbit follow: the vectors β, γ of the initial state and the final state must have the same length. Furthermore, from the appearance of $\Delta \rightarrow U_R \Delta V_R^T$ one is inexorably lead to the singular values of Δ playing an important role. Obviously the initial state's Δ and final state's Δ must have the same singular values. For pure states one can using the characterization above do more. Before proceeding, it is useful to note that even though the Δ's of the initial and final states are not conjugate, they will have equal determinants. The relevance of this is the following well known result on canonical forms:

Lemma 4.2. *Suppose,* Δ *is a Hermitian matrix with* $det(\Delta) < 0$. *Then there exist two* $SO(3)$ *matrices,* X, Y *such that* $X\Delta Y^T = -diag(\sigma_1, \ldots, \sigma_n)$, *where the* σ_i *are the singular values of* Δ.

Similar statements for the cases of positive and zero determinants exits, but they are omitted here for brevity. Obviously, by choosing $U, V \in SU(2)$ satisfying $\Phi(U) = X, \Phi(V) = Y$ ($\Phi : SU(2) \rightarrow SO(3)$ being the canonical homomorphism obtained by identifying $SU(2)$

with the group of unit quaternions), a canonical form for density matrices under the conjugate action of $SU(2) \otimes SU(2)$, is thereby obtained.

Returning to pure states and using the characterization in Lemma (4.1) yields $DD^T + (\| \beta \|^2 - 1)I = \beta\beta^T$. This yields the singular values of D to be $1, \sqrt{1 - \| \beta \|^2}, \sqrt{1 - \| \beta \|^2}$. From Lemma (4.1) an obvious necessary condition for the initial pure state and the final pure state to belong to the same $SU(2) \otimes SU(2)$ is that their β's have the same length. The calculation of the singular values above reveals this to be sufficient too. Indeed, now a canonical form for the pure state with these singular values is given by a pure state with $\beta = -(\| \beta \|, 0, \ldots, 0)$, $\gamma = (\| \beta \|, 0, \ldots, 0)$, $\Delta = -\mathrm{d}iag(1, \sqrt{1 - \| \beta \|^2}), \sqrt{1 - \| \beta \|^2})$.

Thus, for pure states a necessary and sufficient condition for equivalence under $SU(2) \otimes SU(2)$, and thus for reachability under the imaginary representation of $so(4)$, is obtained (note pure states remain pure under the mapping $\rho \rightarrow V^* \rho V$). For other states one can state a not very useful necessary and sufficient condition (in terms of a stabilizer group). Certain states, such as those with $\beta = \gamma = 0$, obviously are equivalent iff their Δ's have the same singular values. Such states intuitively seem the very opposite of the so-called separable states. However, note that, under one definition of separable states (convex combinations of Kronecker product states), it is possible for a separable state to have non-zero β's and γ's. Since the intention of this work is not to contribute bandwidth to the yet unsettled definition of separability, this work will not go further into this issue. Clearly states which have $\Delta = 0$ are equivalent iff their β's and γ's have the same norms.

Finally, note that the imaginary representation of $so(4)$ is also conjugate, within $SU(4)$, to the usual representation of $so(4)$ by real, skew-symmetric matrices. Thus, one can obtain some necessary conditions for reachability. However, even for pure states, it is difficult to get sufficient conditions in this fashion.

5 Conclusions

This work studied a concrete system, which though controllable, tends towards uncontrollability when a certain parameter increases. Furthermore, the limiting dynamical Lie algebra was identified with a concrete realization of $so(4)$. This representation was then shown to be explicitly conjugate to $su(2) \otimes su(2)$ within $SU(4)$. Thus, the probelm of reachability in the limit was reduced to determining orbits of a group action of $SU(2) \otimes SU(2)$ on 4×4 density matrices. This problem was then studied by expanding every density matrix in terms of the basis for Hermitian matrices consisting of the identity matrix and the Pauli matrices together with their Kronecker products with the identity and themselves. This lead to a simple solution for pure states and some other classes of states. Further work is needed for arbitrary mixed states.

6 Acknowledgements

This work was supported by a grant from the NSF - DMS 0072415. Thanks are due to Sonia Schirmer for useful discussions on the topic. In particular she pointed out the conjugation, not used here, viz., the one with the representation by real antisymmetric matrices of $so(4)$.

References

[1] A. Barenco et al., Phys. Rev. Lett, **74**, 4087, (1995).

[2] See e.g., S. Schirmer, I. Pullen and A. Solomon, arXiv, quant-ph/0203104.

[3] G. Bastard, *Wave Mechanics Applied to Semiconductor Heterostructures*, John-Wiley, 1991

[4] C. Weisbuch and B. Vinter, *Quantum Semiconductor Structures*, Academic Press, 1991.

[5] See e.g., Y. Makhlin, arXiv quant-ph 0002045

[6] V. Ramakrishna, R. J. Ober, K. L. Flores and H. Rabitz, in press Phys. Rev. A, see arXiv, quant-ph 0012019.

[7] V. Ramakrishna, K. L. Flores, H. Rabitz and R. J. Ober, Phys. Rev. A, 053409, Vol 62, 2001.

[8] V. Ramakrishna et al., Phys. Rev. A, 032106, Vol 61, 2000.