Pseudo Balancing for Discrete Nonlinear Systems

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Abstract

In this paper, a rationale is presented for balancing a nonlinear discrete time system. However, it is shown that even with a set of very reasonable assumptions, it is not possible to construct a globally balanced realization. The obstruction comes from certain integrability conditions which are generically not satisfied. One way around this is to relax the requirements of global balancing by restricting the balancedness conditions to a discrete set of points in the state space.

1 Introduction

A reasonable definition for a global balanced realization for a nonlinear discrete time system seems to involve the requirement that the linearized system along a nominal solution is balanced in the usual linear time-varying sense. This implies that a global nonlinear balanced system for an arbitrary realization should be defined as the one that closes the commutative diagram between linearization and balancing. Unfortunately, it was shown that such a *global* balanced realization may not exist in dimensions larger than one (or, if one relaxes the notion of balancedness to *uncorrelatedness*, in dimensions larger than two.) See [8] for the continuous, and [7] for the discrete case. In this paper, it is shown that by limiting exactness to some specific points in the state space, e.g. a periodic orbit [9, 11], or fixed nominal trajectory, balancedness or uncorrelatedness can be obtained via interpolation.

Consider thus the smooth discrete time nonlinear system

$$x_{k+1} = f(x_k, u_k), \tag{1.1}$$
$$y_k = h(x_k, u_k). \tag{1.2}$$

The rationale of our balancing philosophy is that the balancing should not make reference to just one solution of the system (the equilibrium solution), but should be defined for all nominal solutions (iterated maps), thus for all initial conditions. We shall speak of the *nominal orbit*. In addition, it will be assumed that the input consists of a nominal feedback and a (small) perturbation: $u_k = K(x_k) + \tilde{u}_k$. A special case is the constant input. This gives

$$x_{k+1} = f(x_k, K(x_k) + \tilde{u}_k) \stackrel{\text{def}}{=} F(x_k, \tilde{u}_k), \tag{1.3}$$
$$y_k = h(x_k, K(x_k) + \tilde{u}_k) \stackrel{\text{def}}{=} H(x_k, \tilde{u}_k) \tag{1.4}$$

The nominal solution, represented by the sequence $\{\overline{x}_k\}$, is governed by the map

$$\overline{x}_{k+1} = F(\overline{x}_k, 0) \tag{1.5}$$

The associated perturbation model follows from the controlled map

$$x_{k+1} = F(\overline{x}_k + \tilde{x}_k, \tilde{u}_k) \tag{1.6}$$

where $\tilde{x_k}$ in addition to $\tilde{u}_k$ is also assumed *small*. Thus:

$$\tilde{x}_{k+1} = d\overline{F}\big|_k \, \tilde{x}_k + \overline{G}\big|_k \, \tilde{u}_k \tag{1.7}$$

where

$$[d\overline{F}_k]_i \overset{\text{def}}{=} \left[\frac{\partial F_i}{\partial x_1}, \cdots, \frac{\partial F_i}{\partial x_n}\right] \quad \text{evaluated at } (\overline{x}(k), 0),$$

and

$$\overline{G}\big|_k \overset{\text{def}}{=} \frac{\partial F}{\partial u}\bigg|_k \quad \text{also evaluated at } (\overline{x}(k), 0).$$

Likewise the output perturbation equation is

$$\begin{aligned} y_k - \overline{y}_k &= H(\overline{x}_k + \tilde{x}_k, \tilde{u}_k) - H(\overline{x}_k, 0) \\ &= d\overline{H}\big|_k \, \tilde{x}_k + \left[\frac{\partial H}{\partial u}\right]_k \tilde{u}_k, \end{aligned} \tag{1.8}$$

where again $d\overline{H}$ indicates that the differential is evaluated at the nominal solution, $(\overline{x}(k), 0)$. Hence the perturbation system about the nominal feedback law, $u = K(x)$, is given by the triple $(d\overline{F}, \overline{G}, d\overline{H})$. The perturbation model (1.7-1.8) is time variant in general. The paper is organized as follows: in Section 2, local realization properties, i.e., reachability and observability of discrete time nonlinear systems are reviewed. More details may be found in [7]. In section 3, the balanced realization for a class of nonlinear systems is given. Section 4 discusses a specific interpolation for orbits of planary systems. Examples are given in Section 5.

2 Local Reachability and Observability

Let us start from the *perturbation*-model for (1.2), assuming a particular nominal solution sequence, $\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{N-1}, \ldots$ and small excursions. Introduce the k-step iterated symbol (analogous to the "Ad"-operator in continuous time), defined as

$$\left(\text{it}_F^{k-1}\, G\right)_{\ell-1} \overset{\text{def}}{=} dF|_{\ell-1} \cdots dF|_{\ell-k+1}\, G|_{\ell-k}, \tag{2.9}$$

and satisfying the recursion

$$\left(\text{it}_F^{k}\, G\right)_{\ell-1} = dF|_{\ell-1} \left(\text{it}_F^{k-1}\, G\right)_{\ell-2}$$

with $\left(\mathrm{it}_F^0\, G\right)_{\ell-1} = G|_{\ell-1}$. Note that this definition is somewhat different from the one proposed in [7], but seems more natural, since $(\mathrm{it}_F^{k-1} G)_{\ell-1}$ gives the effect of the input, $u_{\ell-k}$, k steps away, on the state at time ℓ. Indeed, the local ℓ-step reachability map for a sequence of *small inputs*, $\tilde{u}_0, \ldots, \tilde{u}_{\ell-1}$, starting at $k = 0$, with the slightly perturbed state $\overline{x}_0 + \tilde{x}_0$ is given by ("h.o.t." are higher order terms)

$$\begin{aligned} x_\ell &= F^{\circ\ell}(\overline{x}_0) + (dF|_{\ell-1} \cdots dF|_0)\ \tilde{x}_0 + (dF|_{\ell-1} \cdots dF|_1 G|_0)\ \tilde{u}_0 + \cdots \\ &\quad \cdots + G|_{\ell-1} \tilde{u}_{\ell-1} + h.o.t. \end{aligned}$$

Since this is also, $\overline{x}_\ell + \tilde{x}_\ell$, we get the compact form, $((\mathrm{it}_F^{\ell-1}\, dF)_{\ell-1}\, \tilde{x}_0 = (\mathrm{it}_F^{\ell} \tilde{x}_0)_{\ell-1})$,

$$\tilde{x}_\ell = R_{\mathrm{loc}}^{(\ell)}(\overline{x}_\ell)\, \mathcal{U}_\ell + (\mathrm{it}_F^{\ell-1}\, dF)_{\ell-1}\, \tilde{x}_0 + h.o.t. \tag{2.10}$$

Here, the matrix

$$R_{\mathrm{loc}}^{(\ell)}(\overline{x}_\ell) \stackrel{\mathrm{def}}{=} \left[\, G|_{\ell-1},\ (\mathrm{it}_F^1\, G)_{\ell-1}\ \cdots\ (\mathrm{it}_F^{\ell-1}\, G)_{\ell-1} \right], \tag{2.11}$$

is the *local (ℓ-step) reachability matrix* (for reaching $\tilde{x}_\ell$), and $\mathcal{U}_\ell \stackrel{\mathrm{def}}{=} [u_{\ell-1}, \ldots, u_0]'$. Hence if $R_{\mathrm{loc}}^{(\ell)}(\overline{x}_\ell)$ is nonsingular, the sequence $\mathcal{U}_\ell = [R_{\mathrm{loc}}^{(\ell)}(\overline{x}_\ell)]^{-1} \tilde{x}_f$ will steer the event $(\overline{x}_0, 0)$ to a neighborhood of $F^{\circ\ell}(\overline{x}_0) + \tilde{x}_f$ at time ℓ. More precisely, $\tilde{x}_f$ will be the deviation, up to first order, from the nominal state $\overline{x}_\ell$ in ℓ steps. It is obvious that for a system of order n, not all perturbations in the neighborhood of $\overline{x}_k$ will be reachable unless the number of steps, $k \geq n$, and a reachability condition holds: $\mathrm{rank}\, R_{\mathrm{loc}}^{(k)}(\cdot) = n$.

Likewise, the output perturbation (deviation from the nominal output), in the absence of input perturbations, but with nonzero $\tilde{x}_0$, is found as

$$\tilde{y}_k = d\overline{H}|_k \left(\mathrm{it}_F^k \tilde{x}_0\right)_{k-1} = dH|_k\ dF|_{k-1} \cdots dF|_0\ \tilde{x}_0 \tag{2.12}$$

thus generating an output perturbation sequence, satisfying $\mathcal{Y}_\ell = [\tilde{y}_0, \ldots, \tilde{y}_{\ell-1}]'$,

$$\mathcal{Y}_\ell = O_{\mathrm{loc}}^{(\ell)}(\tilde{x}_0). \tag{2.13}$$

The matrix

$$O_{\mathrm{loc}}^{(\ell)}(\overline{x}_0) = \begin{bmatrix} dH|_0 \\ dH|_1 dF|_0 \\ \vdots \\ dH|_{\ell-1} dF|_{\ell-2} \cdots dF|_0 \end{bmatrix} \tag{2.14}$$

is the *local ℓ -step observability matrix* (observing the perturbation $\tilde{x}_0$.

Define for an n-th order system the *local gramians* at $\overline{x}_0$ as the n-step (n is the order of the system) gramians gramians *relative to the nominal model at* $\overline{x}_0$. It is also shown in [7] that the time symmetry with respect to the event $(\overline{x}_0, 0)$ requires that we *shift the reachability problem to the past*, and consider the local reachability matrix $R^{(\ell)}(\overline{x}_0)$, associated with the reachability of a neighborhood of $\overline{x}_0$ at step 0. Define now the *local reachability Gramian* at $\overline{x}_0$ as

$$\mathcal{R}(\overline{x}_0) = R^{(n)}_{\text{loc}}(\overline{x}_0) R^{(n)'}_{\text{loc}}(\overline{x}_0). \tag{2.15}$$

Note that it actually depends on $\overline{x}_{-n} = F^{\circ(-n)}(\overline{x}_0)$ where n is the system order. F^{-1} is the backwards map. See [7] for details. Likewise, the *local observability Gramian* is

$$\mathcal{O}(\overline{x}_0) = \left[O^{(n)'}_{\text{loc}}(\overline{x}_0) O^{(n)}_{\text{loc}}(\overline{x}_0) \right]. \tag{2.16}$$

The Gramians play a fundamental role as weighting matrices for the input and output perturbation energies. Observe that the gramians are necessarily singular if fewer than n steps would be considered. Hence these are truly the *minimum time* gramians.

3 Nonlinear Balanced Realizations

3.1 Local balancing

The linear variational system along the nominal trajectory is time-varying, and finite time balancing for such systems is described in [10]. Now extend:

Definition 3.1. *The discrete system* (F, G) *is said to be* locally *(at* $\overline{x}_0$*) balanced for the nominal input* $\overline{u} = K(\overline{x})$*, if the minimum time gramians satisfy* $\mathcal{R}(\overline{x_0}) = \mathcal{O}(\overline{x_0}) = \Lambda(\overline{x_0})$*, where* $\Lambda(\overline{x_0})$ *is a diagonal matrix with nonnegative elements on its diagonal.*

If the diagonal elements $\lambda_i(\overline{x}_0)$ are all distinct at $\overline{x}_0$, then a *canonical* gramian at $\overline{x}_0$ may be defined as the gramian Λ for which the values on the diagonal are ordered i.e., $\lambda_1 > \lambda_2 > \cdots > \lambda_n$.

Local balancing at $\overline{x}_0$ is performed by simultaneous diagonalization of the local reachability and observability gramians. It was shown in [7] that if the *local Hankel matrix* $\mathcal{H}(\overline{x}_0)$ is defined by

$$\mathcal{H}_{\text{loc}} = O^{(n)}_{\text{loc}} R^{(n)}_{\text{loc}}$$

and has a singular value decomposition $\mathcal{H}_{\text{loc}} = U\Lambda^2 V'$, then the local balancing transformation is given by $T_{\text{bal}} = [\Lambda V'] \, R^{(n)}_{\text{loc}}$ or equivalently by $T^{-1}_{\text{bal}} = \left(O^{(n)}_{\text{loc}} \right)^{-1} [U\Lambda]$.

3.2 Global Balancing

The rationale behind our approach to balancing in the continuous time case was the idea that balancing and linearization should commute. Adopting the same in the discrete case, we desire the following commutative diagram

$$\begin{array}{ccc} (F,G,H) & \stackrel{\text{linearization}}{\longrightarrow} & (A_P,b_P,c_P) \\ \text{global balancing} \downarrow & & \text{local balancing} \downarrow \\ (\hat{F},\hat{G},\hat{H}) & \stackrel{\text{linearization}}{\longrightarrow} & (\hat{A}_P,\hat{b}_P,\hat{c}_P) \end{array}$$

Thus the problem is to extend the local (at $\overline{x}$) balancing transformations $T(\overline{x})$ to a transformation on at least some open subset of the state space. To this effect, the equation

$$\frac{\partial \xi}{\partial x} = T(x) \tag{3.17}$$

needs to be solved. This is a set of n partial differential equations of first order in n variables. It is a special case of a Mayer-Lie system. It is known that such a system of equations is not generically solvable. The necessary and sufficient conditions for solvability are $\frac{\partial T_{ij}(x)}{\partial x_k} - \frac{\partial T_{ik}(x)}{\partial x_j} = 0$, for all $i,j,k = 1,\ldots,n$. This problem led us to define a more relaxed notion of *uncorrelated* realization [7, 11].

Definition 3.2. *A realization for which the reachability and observability gramians are both diagonal is called an* uncorrelated *realization.*

Theorem 3.1. *i) A first order minimal system can be balanced.*
ii) A second order minimal system can be brought to uncorrelated form.
iii) A higher order system can be uncorrelated if and only if integrating factors exist for which ST is integrable.

4 Pseudo-balancing and Mayer-Lie Interpolation

The above theorem shows the severe obstructions towards balancing. However, consider a nominal trajectory of interest, (e.g., a periodic one), $x_0, x_1, \ldots, x_{N-1}$, where N is assumed to exceed the system order. In this case one should only be concerned about the behavior of the system at the N discrete points $x_0, x_1, \ldots, x_{N-1}$. This constitutes an *interpolation problem*: Find a diffeomorphism ξ such that

$$\left.\frac{\partial \xi}{\partial x}\right|_{\overline{x}_i} = T(\overline{x}_i) = T^{(i)} \quad ; \quad i = 0,\ldots,N-1.$$

4.1 Mayer-Lie Interpolation for Planar Systems

We study the planar systems $n = 2$. The original coordinates are denoted as x and y, the new coordinates are ξ and η. We will show that for N odd, an interesting choice is given by the homogeneous forms

$$\begin{aligned}\xi(x,y) &= c_1^{(1)}x^{2N-1} + c_2^{(1)}x^{2N-2}y + \cdots + c_{2N}^{(1)}y^{2N-1}\\ \eta(x,y) &= c_1^{(2)}x^{2N-1} + c_2^{(2)}x^{2N-2}y + \cdots + c_{2N}^{(2)}y^{2N-1}\end{aligned}$$

The coefficients $c_j^{(i)}$ are determined by matching the Jacobian $\frac{\partial(\xi,\eta)}{\partial(x,y)}^T$ at the N interpolation points. It leads to a solvable set if no two interpolation states are colinear with the origin. This result was stated in [11], but a detailed proof was not included. It is presented in the next section.

What makes the given choice interesting is the fact (due to the homogeneity) that the straight lines $\alpha x + \beta y = 0$ are mapped into straight lines $\gamma\xi + \delta\eta = 0$ for some (γ, δ). In fact if the slope of the first is $\tan t = y/x$, then the slope of the transformed line will be

$$\tan\theta(t) = \frac{\eta(\cos t, \sin t)}{\xi(\cos t, \sin t)}.$$

Hence the map of the unit circle, i.e., the parametrized form, $[\xi = \xi(\cos t, \sin t), \eta = \eta(\cos t, \sin t)]$, essentially determines the new frame. The real roots of the polynomial $\xi(1,t) = 0$ determine the angles of the asymptotes (if any) for the plots of $\xi = c$ where c is any nonzero constant, and likewise for η. The mapping $(x,y) \to (\xi,\eta)$ is not necessarily a homeomorphism (and thus also not a diffeomorphism) of the full state space. However, the state space can be partitioned into wedges where the mapping $(x,y) \to (\xi,\eta)$ is one to one. The boundaries of the wedges are determined by the angles for which $\frac{d\theta(t)}{dt} = 0$. The latter condition determines 'turning points' on the map of the unit circle. Hence, a *maximal injective restriction* is given by the subintervals of t where

$$[\xi,\eta]_x \sin t - [\xi,\eta]_y \cos t \neq 0,$$

where $[\xi,\eta]_x$ is the Lie bracket with respect to the variable x. Both Lie brackets are evaluated at $x = \cos t, y = \sin t$. Finally, remark that the map, $\xi = x^2 + y^2$, $\eta = xy$, presents a fine example, illustrating turning points for $x = \pm y$, or $\tan t = \pm 1$, i.e., the $\pm 45^o$ lines.

4.2 Interpolation Condition

Theorem 4.1. *If no two interpolation points are collinear with the origin, then the proposed Mayer-Lie interpolation is solvable.*

Proof: Matching the Jacobian with the local balancing transformation at a point (x,y)

yields the condition

$$\frac{\partial(\xi,\eta)}{\partial(x,y)}^T =$$

$$\begin{bmatrix} (2N-1)x^{2N-2} & (2N-2)x^{2N-3}y & \cdots & y^{2N-2} & 0 \\ 0 & x^{2N-2} & \cdots & (2N-2)xy^{2N-3} & (2N-1)y^{2N-2} \end{bmatrix} \begin{bmatrix} c_1^{(1)} & c_1^{(2)} \\ \vdots & \vdots \\ c_{2N}^{(1)} & c_{2N}^{(2)} \end{bmatrix}$$

$$= \mathcal{Z}(x,y)\, C$$

Since it must hold at the N chosen points, the solvability of the coefficients is equivalent to the nonsingularity of the coefficient matrix. Thus, let $\rho_i \neq 0; i = 1, \ldots N$. In this section, we compute the determinant of the $2N \times 2N$ matrix

$$\begin{bmatrix} (2N-1)\rho_1^{2N-2} & (2N-2)\rho_1^{2N-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ (2N-1)\rho_N^{2N-2} & (2N-2)\rho_N^{2N-3} & \cdots & 1 & 0 \\ 0 & 1 & \cdots & (2N-2)/\rho_1^{2N-3} & (2N-1)/\rho_1^{2N-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & (2N-2)/\rho_N^{2N-3} & (2N-1)/\rho_N^{2N-2} \end{bmatrix}$$

Denote this determinant by $D(\rho_1, \ldots, \rho_N) = D(\underline{\rho})$. First note that

$$D(\underline{\rho}) = \frac{\partial^N}{\partial\rho_1 \cdots \partial\rho_N} \det \begin{bmatrix} \rho_1^{2N-1} & \rho_1^{2N-2} & \cdots & \rho_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \rho_N^{2N-1} & \rho_N^{2N-2} & \cdots & \rho_2 & 1 \\ 0 & 1 & \cdots & (2N-2)\lambda_1^{2N-3} & (2N-1)\lambda_1^{2N-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & (2N-2)\lambda_N^{2N-3} & (2N-1)\lambda_N^{2N-2} \end{bmatrix}_{\underline{\lambda}=1/\underline{\rho}}$$

where $\underline{\lambda} = 1/\underline{\rho}$ indicates the evaluation of the partial derivatives for $\lambda_1 = 1/\rho_1; i = 1 \ldots N$. Hence, also

$$D(\underline{\rho}) = \frac{\partial^N}{\partial\rho_1 \cdots \partial\rho_N} \frac{\partial^N}{\partial\lambda_1 \cdots \partial\lambda_N} \det \begin{bmatrix} \rho_1^{2N-1} & \rho_1^{2N-2} & \cdots & \rho_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \rho_N^{2N-1} & \rho_N^{2N-2} & \cdots & \rho_2 & 1 \\ 1 & \lambda_1 & \cdots & \lambda_1^{2N-2} & \lambda_1^{2N-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_N & \cdots & \lambda_N^{2N-2} & \lambda_N^{2N-1} \end{bmatrix}_{\underline{\lambda}=1/\underline{\rho}}$$

or,

$$D(\underline{\rho}) = \frac{\partial^N}{\partial\rho_1\cdots\partial\rho_N}\frac{\partial^N}{\partial\lambda_1\cdots\partial\lambda_N}$$
$$\left\{(\lambda_1\ldots\lambda_N)^N \det\begin{bmatrix} \rho_1^{2N-1} & \rho_1^{2N-2} & \cdots & \rho_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \rho_N^{2N-1} & \rho_N^{2N-2} & \cdots & \rho_2 & 1 \\ 1/\lambda_1^{2N-1} & 1/\lambda_1^{2N-2} & \cdots & 1/\lambda_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1/\lambda_N^{2N-1} & 1/\lambda_N^{2N-2} & \cdots & 1/\lambda_N & 1 \end{bmatrix}\right\}_{\underline{\lambda}=1/\underline{\rho}}$$
$$= \frac{\partial^N}{\partial\rho_1\cdots\partial\rho_N}\frac{\partial^N}{\partial\lambda_1\cdots\partial\lambda_N}\left\{(\lambda_1\ldots\lambda_N)^N \det V(\underline{\rho}, 1/\underline{\lambda})\big|\right\}_{\underline{\lambda}=1/\underline{\rho}}.$$

where $V(\underline{\rho}, 1/\underline{\lambda})$ denotes the Vandermonde matrix $V(\rho_1,\ldots,\rho_N, 1/\lambda_1,\ldots,1/\lambda_N)$. This reduces further to

$$D(\underline{\rho}) = (\lambda_1\cdots\lambda_N)^{2N-2}\frac{\partial^N}{\partial\rho_1\cdots\partial\rho_N}$$
$$\Bigg[(2N-1)^N + (2N-1)^{N-1}\left(\lambda_1\frac{\partial}{\partial\lambda_1}+\cdots+\lambda_N\frac{\partial}{\partial\lambda_N}\right)+\cdots$$
$$+(2N-1)\left(\lambda_2\frac{\partial}{\partial\lambda_2}\cdots\lambda_N\frac{\partial}{\partial\lambda_N}+\cdots+\lambda_1\frac{\partial}{\partial\lambda_1}\cdots\lambda_{N-1}\frac{\partial}{\partial\lambda_{N-1}}\right)+$$
$$+\left(\lambda_1\frac{\partial}{\partial\lambda_1}\cdots\lambda_N\frac{\partial}{\partial\lambda_N}\right)\Bigg]\det V(\underline{\rho}, 1/\underline{\lambda})\Big|_{\underline{\lambda}=1/\underline{\rho}}$$

Upon setting $\theta_i = 1/\lambda_i$ for $i = 1,\ldots,N$ and noting that

$$\lambda_i\frac{\partial}{\partial\lambda_i} = -\theta_i\frac{\partial}{\partial\theta_i}$$

one gets

$$D(\underline{\rho}) = \frac{1}{(\rho_1\cdots\rho_N)^{2N-2}}\frac{\partial^N}{\partial\rho_1\cdots\partial\rho_N}$$
$$\big[(2N-1)^N - (2N-2)^{N-1}\mathcal{T}_1 + \cdots$$
$$\cdots$$
$$(-1)^{N-1}(2N-1)\mathcal{T}_{N-1} + (-1)^N\mathcal{T}_N\big]\det V(\underline{\rho}, \underline{\theta})\big|_{\underline{\theta}=\underline{\rho}}$$

where the differential operators $\mathcal{T}_i; i = 1, \ldots, N$ are defined by

$$\begin{aligned}
\mathcal{T}_1 &= \sum_{i=1}^{N} \theta_i \frac{\partial}{\partial \theta_i} \\
\mathcal{T}_2 &= \sum \prod_{2 \text{ factors}} \left(\theta_i \frac{\partial}{\partial \theta_i} \right) \\
&\vdots \\
\mathcal{T}_{N-1} &= \sum \prod_{(N-1) \text{ factors}} \left(\theta_i \frac{\partial}{\partial \theta_i} \right) \\
\mathcal{T}_N &= \prod_{i=1}^{N} \left(\theta_i \frac{\partial}{\partial \theta_i} \right),
\end{aligned}$$

the products are taken over nonrepeating indices, and the sums include all such possibilities.

Consider now the term

$$R_1(\underline{\rho}) = \frac{\partial^N}{\partial \rho_1 \cdots \partial \rho_N} V(\underline{\rho}, \underline{\theta}) \Big|_{\underline{\theta} = \underline{\rho}}$$

Expressing the Vandermonde determinant as

$$\det V(\underline{\rho}, \underline{\theta}) = (\rho_1 - \theta_1) \cdots (\rho_N - \theta_N) F$$

where F collects all other factors

$$F = \prod_{i<j} (\rho_i - \rho_j) \prod_{k<l} (\theta_k - \theta_l) \prod_{m \neq n} (\rho_m - \theta_n)$$

one readily finds

$$\begin{aligned}
R_1(\underline{\rho}) &= \frac{\partial^{N-1}}{\partial \rho_2 \cdots \partial \rho_N} \left\{ [F + (\rho_1 - \theta_1) F_{\rho_1}] \left[(\rho_2 - \theta_2) \cdots (\rho_N - \theta_N) \right] \right\} \Big|_{\underline{\theta} = \underline{\rho}} \\
&= \frac{\partial^{N-1}}{\partial \rho_2 \cdots \partial \rho_N} \frac{V(\underline{\rho}, \underline{\theta})}{\rho_1 - \theta_1} \Big|_{\underline{\theta} = \underline{\rho}} + \\
&\quad + (\rho_1 - \theta_1) \frac{\partial^{N-1}}{\partial \rho_2 \cdots \partial \rho_N} \left((\rho_2 - \theta_2) \cdots (\rho_N - \theta_N) F_{\rho_1} \right) \Big|_{\underline{\theta} = \underline{\rho}} \\
&= \frac{\partial^{N-1}}{\partial \rho_2 \cdots \partial \rho_N} \frac{V(\underline{\rho}, \underline{\theta})}{\rho_1 - \theta_1} \Big|_{\underline{\theta} = \underline{\rho}} = \frac{\partial^{N-1}}{\partial \rho_2 \cdots \partial \rho_N} \left[(\rho_2 - \theta_2) \cdots (\rho_N - \theta_N) F \right] \Big|_{\underline{\theta} = \underline{\rho}}
\end{aligned}$$

since the last term vanishes due to $\rho_1 = \theta_1$. Now perform the partial derivation with respect to ρ_2 likewise to obtain

$$R_1(\underline{\rho}) = \frac{\partial^{N-2}}{\partial \rho_3 \cdots \partial \rho_N} \left[(\rho_3 - \theta_3) \cdots (\rho_N - \theta_N) F \right] \Big|_{\underline{\theta} = \underline{\rho}}$$

Keep turning the crank, until

$$\begin{aligned} R_1(\underline{\rho}) &= \left.\frac{\det V(\underline{\rho},\underline{\theta})}{(\rho_1-\theta_1)\cdots(\rho_N-\theta_N)}\right|_{\underline{\theta}=\underline{\rho}} \\ &= \left.\prod_{i<j}(\rho_i-\rho_j)\prod_{k<l}(\theta_k-\theta_l)\prod_{m\neq n}(\rho_m-\theta_n)\right|_{\underline{\theta}=\underline{\rho}} \\ &= (-1)^{N(N-1)/2}\prod_{i<j}(\rho_i-\rho_j)^4. \end{aligned}$$

Consider also the general term involved in $\mathcal{T}_k$ of the form

$$R_k(\sigma;\underline{\rho}) = \frac{\partial^{N-k}}{\partial\rho_{\sigma(k+1)}\cdots\partial\rho_{\sigma(N)}}\frac{\partial^2}{\partial\rho_{\sigma(1)}\partial\theta_{\sigma(1)}}\cdots\frac{\partial^2}{\partial\rho_{\sigma(k)}\partial\theta_{\sigma(k)}}\det V(\underline{\rho},\underline{\theta})\Big|_{\underline{\theta}=\underline{\rho}},$$

where σ is an arbitrary permutation. Denote for brevity the partial differential operator by $\mathcal{D}_{\sigma(1\ldots,k;k+1,\ldots,N)}$. Then:

$$R_k(\sigma;\underline{\rho}) = \mathcal{D}\left\{\frac{\det V(\underline{\rho},\underline{\theta})}{(\rho_{\sigma(1)}-\theta_{\sigma(1)})\cdots(\rho_{\sigma(k)}-\theta_{\sigma(k)})}\,(\rho_{\sigma(1)}-\theta_{\sigma(1)})\cdots(\rho_{\sigma(k)}-\theta_{\sigma(k)}).\right\}_{\underline{\theta}=\underline{\rho}},$$

Setting

$$F = \frac{\det V(\underline{\rho},\underline{\theta})}{(\rho_{\sigma(1)}-\theta_{\sigma(1)})\cdots(\rho_{\sigma(k)}-\theta_{\sigma(k)})},$$

then

$$\begin{aligned} R_k(\sigma;\underline{\rho}) = \mathcal{D}\Big\{&\left(-F_{\rho_{\sigma(k)}}+F_{\theta_{\sigma(k)}}\right)\,(\rho_{\sigma(1)}-\theta_{\sigma(1)})\cdots(\rho_{\sigma(k-1)}-\theta_{\sigma(k-1)})+ \\ &+(\rho_{\sigma(1)}-\theta_{\sigma(1)})\cdots(\rho_{\sigma(k)}-\theta_{\sigma(k)})F_{\theta_{\sigma(k)}\rho_{\sigma(k)}}\Big\}_{\underline{\theta}=\underline{\rho}} \end{aligned}$$

Now, by successively "peeling off" the second order partials with respect to $\rho_{\sigma(i)}$ and $\theta_{\sigma(i)}$, we find

$$\begin{aligned} &R_k(\sigma;\underline{\rho}) = \\ &\quad \mathcal{D}_{\sigma(1),\ldots,\sigma(k-2);\sigma(k+1),\ldots,\sigma(n)}\left\{\left[\left(-\frac{\partial}{\partial\rho_{\sigma(k)}}+\frac{\partial}{\partial\theta_{\sigma(k)}}\right)\left(-\frac{\partial}{\partial\rho_{\sigma(k)}}+\frac{\partial}{\partial\theta_{\sigma(k)}}\right)F\right]\cdot\right. \\ &\qquad\qquad \left.\cdot\left(\rho_{\sigma(1)}-\theta_{\sigma(1)}\right)\cdots\left(\rho_{\sigma(k-2)}-\theta_{\sigma(k-2)}\right)\right\}_{\underline{\rho}=\underline{\theta}} \\ &\quad = \cdots \\ &\quad = \frac{\partial^{n-k}}{\partial\rho_{\sigma(k+1)}\cdots\rho_{\sigma(n)}}\left[\left(-\frac{\partial}{\partial\rho_{\sigma(1)}}+\frac{\partial}{\partial\theta_{\sigma(1)}}\right)\left(-\frac{\partial}{\partial\rho_{\sigma(k)}}+\frac{\partial}{\partial\theta_{\sigma(k)}}\right)F\right]_{\underline{\rho}=\underline{\theta}}. \end{aligned}$$

Clearly, all these terms are zero. Hence,

$$\begin{aligned} D &= \frac{1}{(\theta_1\cdots\theta_N);2N-2}\frac{\partial^N}{\partial\rho_1\cdots\rho_N}(2N-1)^N\det V(\underline{\rho},\underline{\theta})\Big|_{\underline{\theta}=\underline{\rho}} \\ &= \frac{(2N-1)^N}{(\rho_1\cdots\rho_N)^{2N-2}}\left.\frac{\det V(\underline{\rho},\underline{\theta})}{(\rho_1-\theta_1)\cdots(\rho_N-\theta_N)}\right|_{\underline{\theta}=\underline{\rho}} \end{aligned}$$

Finally, with

$$\left.\frac{\det V(\underline{\rho}, \underline{\theta})}{\prod_{i=1}^{N} \rho_i - \theta_i}\right|_{\underline{\theta}=\underline{\rho}} = (-1)^{N(N-1)/2} \prod_{i<j} (\rho_i - \rho_j)^4$$

we get

$$D_N = (-1)^{N(N-1)/2} (2N-1)^N \frac{\prod_{i<j}(\rho_i - \rho_j)^4}{(\rho_1 \cdots \rho_N)^{2N-2}}$$

and thus proved the assertion, since D_N is nonzero if all $\rho_i = y_i/x_i$ are disjoint.

5 Examples

First we give the exact balanced form for a scalar system, and apply the result to the logistic equation. We correct a (minor) error in [7]. The second example involves the delayed logistic equation, and a third one is a discrete model for enzyme kinetics [3].

5.1 Scalar case

The affine scalar system

$$\begin{aligned} x_{k+1} &= f(x_k) + g(x_k) u_k \\ y_k &= h(x_k) \end{aligned}$$

for *small* inputs u_k, nominally identically equal to 0, yields for the gramians

$$\begin{aligned} \mathcal{R}_{\text{loc}}(x) &= g^2(f^{(-1)}(x)) \\ \mathcal{O}_{\text{loc}}(x) &= [h'(x)']^2 \end{aligned}$$

from which the canonical gramian and the *local* balancing transformation are respectively $\Lambda(x) = |h'(x)\, g(f^{(-1)}(x))|$ and $T(x) = \left|\frac{h'(x)}{g(f^{(-1)}(x))}\right|$. The *global* balancing transformation exists, and follows by integrating

$$\frac{d\xi}{dx} = \left|\frac{h'(x)}{g(f^{(-1)}(x))}\right|$$

Consider the well studied logistic equation $x_{k+1} = \mu x_k(1 - x_k)$, on which we shall assume that the perturbation input consists in a variation of the parameter μ, and x is directly observed. The reachability matrix is simply $g(x) = x_{-1}(1 - x_{-1})$, where x_{-1} is the inverse image of x under the nominal map. But this gives $R_{\text{loc}}(x) = g(x) = x/\mu$. Since $h(x) = x$ the observability matrix, $O_{\text{loc}}(x) = 1$ and the local balancing transformation is $T(x) = \sqrt{\mu/|x|}$. The resulting global balancing transformation follows from

$$\frac{d\xi}{dx} = \sqrt{\frac{\mu}{x}}.$$

which gives $\xi(x) = 2\sqrt{\mu x}$. Finally, the globally balanced realization is

$$\begin{aligned} \xi_{k+1} &= \sqrt{\mu}\,\xi_k \sqrt{1 - \frac{\xi_k^2}{4}} \\ y &= \frac{\xi^2}{4\mu}. \end{aligned}$$

The corresponding minimum time canonical gramian is $\Lambda(\xi) = \frac{\xi^2}{4\mu^2}$.

5.2 Delayed Logistic Equation

Consider here the planar system, with observation of x, used by Spencer in the prediction of the influenza outbreak in England and Wales [1].

$$\begin{aligned} x_{k+1} &= \mu x_k(1 - y_k) \\ y_{k+1} &= x_k. \end{aligned}$$

For $\mu = 2.1$, the system exhibits an attracting limit cycle enclosed in $[0,1]^2$. (Starting at a point on the limit cycle, the seventh iterate overtakes it). The local reachability gramian and observability gramian are respectively

$$\mathcal{R}(x,y) = \begin{bmatrix} 2\frac{x^2}{\mu^2} & \frac{xy}{\mu^2} \\ \frac{xy}{\mu^2} & \frac{y^2}{u^2} \end{bmatrix} \quad ; \quad \mathcal{O}(x,y) = \begin{bmatrix} 1 + \mu^2(1-y)^2 & -\mu^2(1-y)x \\ -\mu^2(1-y)x & \mu^2 x^2 \end{bmatrix}.$$

The dominant value $\lambda_1(x,y)$ of the canonical gramain is displayed in Figure 2. The height of

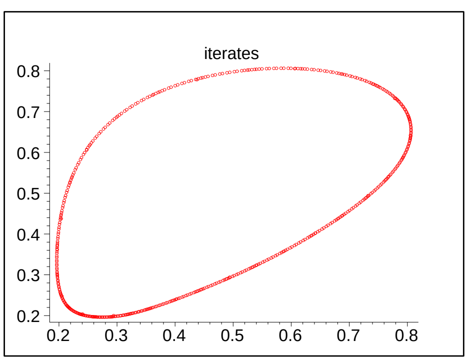

Figure 1: Limit cycle for delayed logistic equation

the plot indicates the value, the shading is modulated by the angle of the dominant direction (mapped back to the original (x,y) coordinates). If T_{loc} is the local balancing transformation, this is the direction of the first column of T_{loc}^{-1}, i.e., the 'jointly most observable and reachable' direction. With three interpolation points: $\{(0.2,0.2),(0.5,0.2),(0.2,0.5)\}$, we obtained the

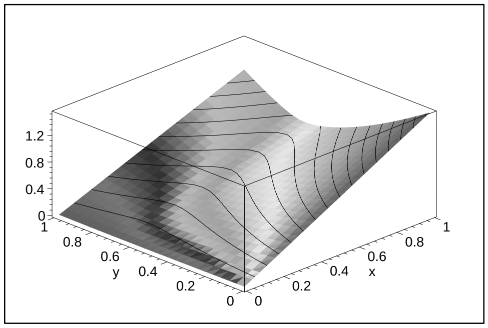

Figure 2: Dominant λ and corresponding direction

pseudo-balancing transformation

$$\begin{aligned}
\xi(x,y) &= 2115x^5 - 13823x^4y + 31269x^3y^2 + \\
&\quad -29100x^2y^3 + 11596xy^4 - 1649y^5 \\
\eta(x,y) &= 1048x^5 - 7187x^4y + 17472x^3y^2 + \\
&\quad -18101x^2y^3 + 7821xy^4 - 1180y^5.
\end{aligned}$$

As expected, along the limit cycle, the dynamics is almost one dimensional (a rough approximation being $\eta + 1.077\xi = 0$).

5.3 Enzyme Kinetics

We condider the Briggs Haldane [3] model for the enzyme reactions

$$\begin{aligned}
\frac{d\sigma}{dt} &= -\sigma + x(\sigma + \alpha) \\
\epsilon\frac{dx}{dt} &= \sigma - x(\sigma + \kappa)
\end{aligned}$$

Simple discretization yields

$$\begin{aligned}
\sigma_{k+1} &= (1-\tau)\sigma_k + \tau\alpha x_k + \tau x_k\sigma_k \\
x_{k+1} &= \left(1 - \frac{\kappa\tau}{\epsilon}\right)x_k + \frac{\tau}{\epsilon}\sigma_k - \frac{\tau}{\epsilon}x_k\sigma_k
\end{aligned}$$

The quasi steady state approximation [3] restricts the dynamics to the manifold $\sigma - x\sigma - \kappa x = 0$. To set the ideas we choose the numerical values so that the dynamical equations are

$$\begin{aligned}
\sigma_{k+1} &= 0.5\sigma_k + 0.9x_k + 0.5x_k\sigma_k \\
x_{k+1} &= x_k + u(\sigma_k - x_k(\sigma_k + 2))
\end{aligned}$$

where the nominal parameter u (considered as the input) is 0.5. Some iterates of this map are shown in Figure 3. The initial point was [0.9,0.9]. The initial fluctuations (switching

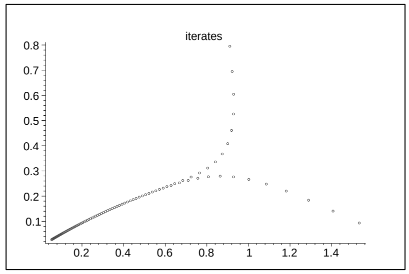

Figure 3: Iterates for Enzyme Dynamics

between the two 'branches') dampen quickly, clearly displaying the (attracting) manifold. The canonical gramian and local balancing transformation were numerically computed. A fieldplot indicating the magnitude and principal directions, transformed back to the original coordinate system is shown in Figure 4. The presence of the attracting manifold is clearly

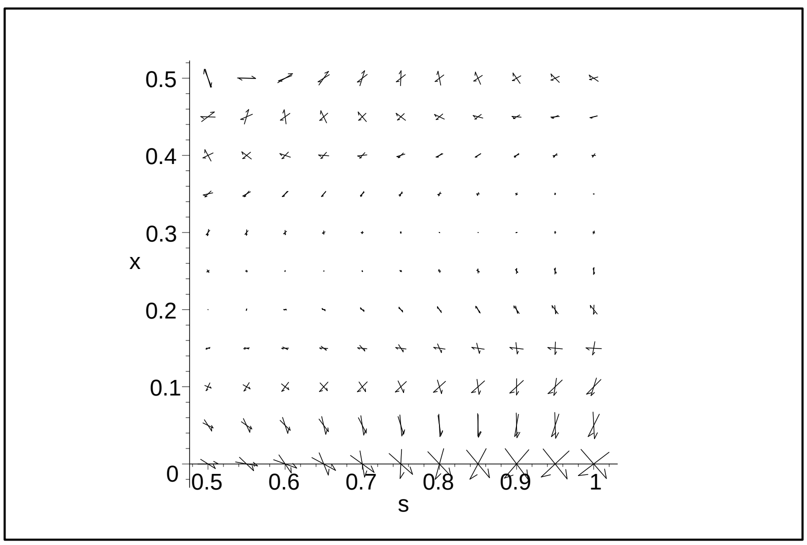

Figure 4: Fieldplot of Canonical Gramian and Principal Directions

visible as states from which perturbations are hard to detect and hard to reach.

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