Explicit formulae for $J-$spectral factors for well-posed linear systems

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Abstract

The standard way to obtain explicit formulas for spectral factorization problems for rational transfer functions is to use a minimal realization and then obtain formulae in terms of the generators A, B, C and D. For well-posed linear systems with unbounded generators these formulae will not always be well-defined. Instead, we suggest another approach for the class of well-posed linear systems for which zero is in the resolvent set of A. Such a system is related to a reciprocal system having *bounded* generating operators depending on B, C, D and the inverse of A. There are nice connections between well-posed linear systems and their reciprocal systems which allow us to translate a factorization problem for the well-posed linear system into one for its reciprocal system, the latter having *bounded* generating operators. We illustrate this general approach by giving explicit solutions to the sub-optimal Nehari problem.

1 Introduction

The standard way to obtain explicit formulas for spectral factorization problems for rational transfer functions is to use a minimal realization $G(s) = D + C(sI - A)^{-1}B$ and then obtain formulae in terms of the generators A, B, C and D (see Ball and Ran [1]). These formulas typically depend on the controllability and observability Gramians L_B, L_C or on solutions of various Lyapunov equations. Such an approach has been extended to certain classes of infinite-dimensional linear systems (see Curtain and Ran [5], Curtain and Zwart [6], Kaashoek et al. [9]), but the limiting factor is the difficulty in manipulating with the unbounded operators B and C. For example, in Sasane and Curtain [11] and [10], where explicit solutions to the sub-optimal Hankel norm approximation problem for exponentially stable smooth Pritchard-Salamon systems and exponentially stable analytic systems were obtained via the solution to the appropriate $J-$spectral factorization problem using the smoothing properties of these classes. However, it was not possible to extend this technique to more general well-posed linear systems. In general, it is not clear that the candidate spectral factor is even well-posed (see Staffans [13]). While an obvious approach to get around the problems with unbounded operators would be to obtain factorizations via the discretized version obtained by the usual Cayley transform, this leads to horrible formulae. Instead we suggest translating the problem to the analogous one for reciprocal systems which we now define.

To motivate our definition we recall the definition of the transfer function of a stable well-posed linear system from Staffans [12]:

$$G(s) - G(\beta) = (\beta - s)C(\beta I - A)^{-1}(sI - A)^{-1}B \quad \text{for all} \quad \beta, s \in \mathbb{C}_0^+. \tag{1.1}$$

If $0 \in \rho(A)$, then we can substitute $\beta = 0$ to obtain the identity

$$\begin{aligned} G(s) &= G(0) - CA^{-1}\left(\frac{1}{s} - A^{-1}\right)^{-1} A^{-1}B, & (1.2)\\ &= G_-\left(\frac{1}{s}\right). & (1.3)\end{aligned}$$

We note that $G_-(s)$ is the transfer function of the linear system with *bounded* generating operators $A^{-1}, A^{-1}B, -CA^{-1}, G(0)$. We define this linear system to be the *reciprocal system* of the well-posed linear system with generating operators A, B, C with A^{-1} bounded. In addition to the relationship (1.3) between their transfer functions, the reciprocal system has the same controllability and observability Gramians.

Lemma 1.1. *Let A, B, C be generating operators of a well-posed linear system with transfer function G. Suppose that $0 \in \rho(A)$ and G_- is the transfer function of its reciprocal system with generating operators A^{-1}, $A^{-1}B$, $-CA^{-1}$, $G(0)$. Then the following hold:*

1. *C is an infinite-time admissible observation operator for A iff $-CA^{-1}$ is an infinite-time admissible observation operator for A^{-1}. If either C or $-CA^{-1}$ is infinite-time admissible, then the observability Gramians are identical.*

2. *B is an infinite-time admissible control operator for A iff $-A^{-1}B$ is an infinite-time admissible observation operator for A^{-1}. If either C or $A^{-1}B$ is infinite-time admissible, then the observability Gramians are identical.*

3. *$G \in H_\infty(\mathcal{L}(U, Y))$ iff $G_- \in H_\infty(\mathcal{L}(U, Y))$.*

Proof

1. From Hansen and Weiss [8] (see also Grabowski [7]), we know that C is an infinite-time admissible observation operator iff the Lyapunov equation

$$\langle Az_1, L_C z_2\rangle + \langle L_C z_1, Az_2\rangle = -\langle Cz_1, Cz_2\rangle \tag{1.4}$$

for all z_1 and z_2 in $D(A)$, has a nonnegative definite solution $L_C = L_C^* \geq 0$. The equation (1.4) is clearly equivalent to the Lyapunov equation

$$\left\langle x_1, L_C A^{-1}x_2\right\rangle + \left\langle L_C A^{-1}x_1, x_2\right\rangle = -\left\langle CA^{-1}x_1, CA^{-1}x_2\right\rangle \tag{1.5}$$

for all x_1 and x_2 in X, which establishes the equivalence. Moreover, the observability Gramians are the smallest positive solution and so the Gramians are identical.

2. This is dual to part 1 above.

3. This follows from (1.3).

■

The idea is then to translate a factorization problem for the well-posed linear system with transfer function G into one for the system with transfer function G_-, the latter having *bounded* generating operators. We illustrate this general approach by giving explicit solutions to the sub-optimal Nehari problem for the class of well-posed linear systems with B, C finite rank, A^{-1} bounded which are input, output and input-output stable. This class includes exponentially stable well-posed linear systems with finite rank B, C.

The assumption that A^{-1} is not essential and can be removed. As mentioned in Curtain [2], for any $i\omega$ in the resolvent set of A, it is always possible to define a reciprocal system based on $A_\omega = A - i\omega I$. For then the new reciprocal system with generating operators $A_\omega^{-1}, A_\omega^{-1}B, -CA_\omega^{-1}, G(i\omega)$ and transfer function G_-^ω satisfies

$$G_-^\omega\left(\frac{1}{s}\right) = G(s + i\omega),$$

and an analogous version of Lemma 1.1 holds.

2 The Nehari problem

We solve this problem for the well-posed linear system on a Hilbert space X with generating operators (A, B, C) under the following assumptions:

A0. The input and output spaces are finite-dimensional, that is, $U = \mathbb{C}^{\mathtt{m}}$ and $Y = \mathbb{C}^{\mathtt{p}}$.

A1. $0 \in \rho(A)$.

A2. B is an infinite-time admissible control operator for $\{T(t)\}_{t\geq 0}$.

A3. C is an infinite-time admissible observation operator for $\{T(t)\}_{t\geq 0}$.

A4. $G(\cdot) \in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$.

The *sub-optimal Nehari problem* is the following: If $\sigma > \|H_G\|$, the Hankel norm of the system, then find all $K(-\cdot) \in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ such that $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$. K is then called a *solution* of the sub-optimal Nehari problem.

Now

$$\begin{aligned} G(s)+K(s) &= G_{-}\left(\frac{1}{s}\right)+K(s) \\ &= -CA^{-1}\left(\frac{1}{s}I-A^{-1}\right)^{-1}A^{-1}B+K(s)+G(0) \\ &= -CA^{-1}\left(\frac{1}{s}I-A^{-1}\right)^{-1}A^{-1}B+K_{-}\left(\frac{1}{s}\right), \end{aligned}$$

where $K_{-}(s):=G(0)+K\left(\frac{1}{s}\right)$. Clearly, $G\in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ iff $G_{-}\in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ and $K_{-}(-\cdot)\in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ iff $K(-\cdot)\in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$, and in the L_∞-norm

$$\|G+K\|_\infty=\|G_{-}+K_{-}\|_\infty.$$

This means that the Hankel norm of G is equal to that of G_{-}. So instead of solving the suboptimal Nehari problem for G, we solve the suboptimal Nehari problem for the reciprocal system with the bounded generating operators A^{-1}, $A^{-1}B$, $-CA^{-1}$. This system satisfies all the conditions in Curtain and Oostveen [4]: its transfer function is in $H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ and the operators $A^{-1}B$ and $-CA^{-1}$ are infinite-time admissible with the same Gramians L_B and L_C as the original system. So we translate the results to this system. Let $N_\sigma=\left(I-\frac{1}{\sigma^2}L_BL_C\right)^{-1}\in\mathcal{L}(X)$. Then all solutions K_{-} to the sub-optimal Nehari problem

$$\left\|-CA^{-1}\left(i\cdot I-A^{-1}\right)^{-1}A^{-1}B+K_{-}(i\cdot)\right\|_\infty\le\sigma$$

are given by

$$K_{-}(-\cdot)=R_1(-\cdot)R_2(-\cdot)^{-1},$$

where

$$\begin{bmatrix} R_1(-\cdot) \\ R_2(-\cdot)\end{bmatrix}=\Lambda(-\cdot)^{-1}\begin{bmatrix} Q(-\cdot) \\ I_{\mathtt{m}}\end{bmatrix},$$

$Q(-\cdot)\in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ satisfies $\|Q\|_\infty\le 1$ and

$$\Lambda(\cdot)=\begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & \sigma I_{\mathtt{m}}\end{bmatrix}+\frac{1}{\sigma^2}\begin{bmatrix} CA^{-1}L_B \\ \sigma\left(A^{-1}B\right)^*\end{bmatrix}N_\sigma^*\left(\cdot I+A^*\right)^{-1}\begin{bmatrix} -\left(CA^{-1}\right)^* & L_CA^{-1}B\end{bmatrix}.$$

Appealing to Theorem 11.1 in Weiss and Weiss [14], we see that

$$\Lambda(-\cdot)-\begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & \sigma I_{\mathtt{m}}\end{bmatrix}\in H_2(\mathbb{C}^{(\mathtt{p}+\mathtt{m})\times(\mathtt{p}+\mathtt{m})}),$$

but not all components need be in H_∞. $\Lambda_{11}(-s)$ is invertible for every $s\in\mathbb{C}_0^+$ and $\Lambda_{11}(-\cdot)^{-1}-I_{\mathtt{p}}\in H_2\cap H_\infty(\mathbb{C}^{\mathtt{p}})$, and $\Lambda(-s)$ is invertible for every $s\in\mathbb{C}_0^+$. Furthermore, $\Lambda(\cdot)$ satisfies the following $J-$spectral factorization problem

$$\begin{bmatrix} I_{\mathtt{p}} & 0 \\ G_{-}(i\omega)^* & I_{\mathtt{m}}\end{bmatrix}\begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & -\sigma^2 I_{\mathtt{m}}\end{bmatrix}\begin{bmatrix} I_{\mathtt{p}} & G_{-}(i\omega) \\ 0 & I_{\mathtt{m}}\end{bmatrix}=\Lambda(i\omega)^*\begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & -I_{\mathtt{m}}\end{bmatrix}\Lambda(i\omega)$$

for $\omega \in \mathbb{R}$ and since $G_-(s) = G\left(\frac{1}{s}\right)$, we obtain a $J-$spectral factorization over H_2 for G:

$$\begin{bmatrix} I_{\mathtt{p}} & 0 \\ G(i\omega)^* & I_{\mathtt{m}} \end{bmatrix} \begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & -\sigma^2 I_{\mathtt{m}} \end{bmatrix} \begin{bmatrix} I_{\mathtt{p}} & G(i\omega) \\ 0 & I_{\mathtt{m}} \end{bmatrix} = \Lambda(\frac{1}{i\omega})^* \begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & -I_{\mathtt{m}} \end{bmatrix} \Lambda(\frac{1}{i\omega})$$

for $\omega \in \mathbb{R}$. The final solution to our original suboptimal Nehari problem is

$$K(-s) = -G(0) + K_-\left(-\frac{1}{s}\right) = -G(0) + R_1\left(-\frac{1}{s}\right) R_2^{-1}\left(-\frac{1}{s}\right),$$

where

$$\begin{bmatrix} R_1\left(-\frac{1}{s}\right) \\ R_2\left(-\frac{1}{s}\right) \end{bmatrix} = \Lambda\left(-\frac{1}{s}\right)^{-1} \begin{bmatrix} Q_0(-s) \\ I_{\mathtt{m}} \end{bmatrix},$$

and $Q_0(-\cdot) \in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ satisfies $\|Q_0\| \leq 1$. From Curtain and Oostveen [4], we have the following formula.

$$\Lambda\left(-\frac{1}{s}\right)^{-1} = \begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & \frac{1}{\sigma} I_{\mathtt{m}} \end{bmatrix} + \frac{1}{\sigma^2} \begin{bmatrix} CA^{-1}L_B \\ \sigma\left(A^{-1}B\right)^* \end{bmatrix} \left(\frac{1}{s}I - (A^{-1})^*\right)^{-1} N_\sigma^* \begin{bmatrix} (-CA^{-1})^* & \frac{1}{\sigma}L_C A^{-1}B \end{bmatrix},$$

for all $s \in \mathbb{C}_0^+$. While it is tempting to try to write this in terms of its reciprocal, we know that this will not in general be well-defined (see Staffans [13]). So we leave the explicit solution as it stands.

A similar approach to the optimal Hankel norm problem for well-posed linear systems has been taken in Curtain and Sasane [3]. For other applications of reciprocal system see Curtain [2].

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