A Riccati equation approach to the standard infinite-dimensional H^∞ problem

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Abstract

We solve the standard (four-block) H^∞ problem for regular well-posed linear systems (in the sense of George Weiss). The state space H, the disturbance space W, the control space U, the regulated output space Z, and the measurement output space Y are all Hilbert spaces of finite or infinite dimension. Our main result is an infinite-dimensional version of the following standard result: there exist a dynamic controller which feeds the measured output y into the control input u, makes the closed loop system exponentially stable, and also makes the norm of the mapping from the external disturbance w to the regulated output z less than a predefined constant $\gamma > 0$ if and only if two algebraic Riccati equations have exponentially stabilizing solutions P_X and P_Y, respectively, and the spectral radius of $P_X P_Y$ is less that γ^2. Another equivalent condition which is given in terms of two nested Riccati equations is available as well. Finally, we establish a generalized version of the standard parameterization of all stabilizing solutions. The exact formulation varies depending on the regularity assumptions that we make, but our assumptions allow for roughly twice as much unboundedness of the control and observation operators as the Pritchard–Salamon class does, and they permit a countable number of pure delays in the input/output responses. Analogous discrete time results are valid as well.

1 Introduction

In this paper we study *well-posed linear systems* in the sense of Salamon [5] and Smuljan [6] which are *weakly regular in the sense of Weiss* [12]. Roughly speaking, for sufficiently smooth and ‘compatible’ data the dynamics of the system is described by the system of equations

$$\begin{cases} x'(t) = Ax(t) + B_1u(t) + B_2w(t) & \text{in } H_{-1}, \\ z(t) = (C_1)_{\mathrm{w}}x(t) + D_{11}u(t) + D_{12}w(t) & \text{in } Z, \\ y(t) = (C_2)_{\mathrm{w}}x(t) + D_{21}u(t) + D_{22}w(t) & \text{in } Y, \\ x(0) = x_0. \end{cases} \tag{1}$$

Here x_0, $x(t) \in H$, $u(t) \in U$, $w(t) \in W$, $z(t) \in Z$, and $y(t) \in Y$, where all these spaces are Hilbert spaces of arbitrary dimensions (finite or infinite). The operator A is the generator of a C_0 semigroup $\mathfrak{A}$ on H. We define $H_1 = \mathcal{D}(A)$ and $H_1^d = \mathcal{D}(A^*)$ (with the graph norms) and let $H_{-1} = (H_1^d)^*$ and $H_{-1}^d = (H_1)^*$ (where we identify the dual of H with H itself). Then $A \in \mathcal{L}(H_1;H)$, and A has a natural extension to an operator in $\mathcal{L}(H;H_{-1})$ (that we still denote by the same letter A). The operators appearing in (1) are bounded linear operators between the appropriate spaces: $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} \in \mathcal{L}\left(\left[\begin{smallmatrix} W \\ U \end{smallmatrix}\right]; H_{-1}\right)$, $C = \left[\begin{smallmatrix} C_1 \\ C_2 \end{smallmatrix}\right] \in \mathcal{L}\left(H_1; \left[\begin{smallmatrix} H \\ Z \end{smallmatrix}\right]\right)$, and $D = \left[\begin{smallmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{smallmatrix}\right] \in \mathcal{L}\left(\left[\begin{smallmatrix} U \\ W \end{smallmatrix}\right]; \left[\begin{smallmatrix} H \\ Z \end{smallmatrix}\right]\right)$. Finally, $C_{\mathrm{w}} = \left[\begin{smallmatrix} (C_1)_{\mathrm{w}} \\ (C_2)_{\mathrm{w}} \end{smallmatrix}\right] := \mathrm{w}-\lim_{\mathsf{a}\to+\bullet} \mathsf{a}(\mathsf{a} - A)^{-1}C$ is the weak Yoshida–Weiss extension of the observation operator C. The *well-posedness* assumption means that, for some $t > 0$ (hence for all $t > 0$), there is a constant $K(t)$ such that for all sufficiently smooth and compatible data, the solution of (1) (exists for all $t \geq 0$ and) satisfies

$$\|x(t)\|_H^2 + \|z\|_{L^2(0,t;Y)}^2 + \|y\|_{L^2(0,t;Z)}^2 \leq K(t)\left(\|x_0\|^2 + \|u\|_{L^2(0,t;W)}^2 + \|w\|_{L^2(0,t;W)}^2\right).$$

We can therefore (by continuity and by the density of the class of smooth and compatible data) *extend the notion of a solution of* (1) to arbitrary initial states $x_0 \in H$, disturbances $w \in L^2_{\mathrm{loc}}(\mathbf{R}^+;W)$, and controls $u \in L^2_{\mathrm{loc}}(\mathbf{R}^+;U)$, and still get a continuous state trajectory $x(t)$ in H, a regulated output $z \in L^2_{\mathrm{loc}}(\mathbf{R}^+;Z)$, and a measurement output $y \in L^2_{\mathrm{loc}}(\mathbf{R}^+;Y)$. *Weak regularity* of the system means that, for some (hence all) $\mathsf{a} \in \mathsf{r}(A)$, the range of the operator $(\mathsf{a} - A)^{-1}B$ is contained in the domain of the operator C_{w}. In this case the equations in (1) remain valid almost everywhere for arbitrary initial states $x_0 \in H$, disturbances $w \in L^2_{\mathrm{loc}}(\mathbf{R}^+;W)$, and controls $u \in L^2_{\mathrm{loc}}(\mathbf{R}^+;U)$. See [9], [10] or [12] for details.

Throughout this article we assume that S is a weakly regular well-posed linear system of the type described above.

2 The standard $H^\bullet$ problem

Let $\mathsf{g} > 0$. The standard $H^\bullet$ problem amounts to finding another (weakly regular and well-posed) system S_c, called a *(exponentially) stabilizing suboptimal controller*, such that if we feed the measurement output y of S into the controller, and feed the controller output into the control input u of S, then the closed loop system becomes exponentially stable[1] and the norm of the map from

[1] For simplicity we here restrict ourselves to the case where the closed loop system is *exponentially stable*. See [4] for a number of other cases.

$w \in L^2(\mathbf{R}^+;W)$ to $z \in L^2(\mathbf{R}^+;Z)$ is less that γ. It turns out that in some cases this class of controllers is not sufficiently large, and it is often more appropriate to allow the controllers to be non-well-posed (when disconnected from the system). The 'correct' class of controllers was introduced by Curtain, Weiss, and Weiss in [2], and it is known under the name *controllers with an internal loop*. Unless otherwise specified, we shall allow our stabilizing suboptimal controllers to have an internal loop. Under sufficient regularity assumptions these controllers will actually be well-posed, and a reader who is not familiar with this class of controllers may think about them as being weakly regular and well-posed. We shall made some further comments on this point below.

As the following theorem shows, under standard coercivity assumptions and certain regularity assumptions, the existence of a stabilizing suboptimal controller is equivalent to the two standard algebraic Riccati equations with their standard signature and coupling conditions.

Theorem 2.1 (H^∞ 4BP $\Leftrightarrow$ CAREs).

(A1) **(Regularity)** *Assume that at least one of (I)–(V) holds, where*

- *(I)* **(Parabolic case)** *A generates an analytic semigroup on H. We let H_β, $\beta \in \mathbf{R}$, be the standard interpolation spaces of fractional order induced by A, and suppose that $B_1 \in \mathcal{L}(U, H_{\beta_1})$, $B_2 \in \mathcal{L}(W, H_{\beta_2})$, $C_1 \in \mathcal{L}(H_{\gamma_1}, Z)$, $C_2 \in \mathcal{L}(H_{\gamma_2}, Y)$, $D \in \mathcal{L}(\left[\begin{smallmatrix} U \\ W \end{smallmatrix}\right], \left[\begin{smallmatrix} Z \\ Y \end{smallmatrix}\right])$, where the parameters β_k, $\gamma_k \in (-1/2, 1/2)$ ($k = 1,2$) satisfy the following additional restrictions: $\gamma_1 < \max\{1/4, 1/2 + \min\{\beta_1, \beta_2\}\}$ and $\beta_2 > \min\{-1/4, \max\{\gamma_1, \gamma_2\} - 1/2\}$;*
- *(II)* ***B* is bounded**, *i.e., $B \in \mathcal{L}(\left[\begin{smallmatrix} U \\ W \end{smallmatrix}\right], H)$, and $C_{\mathrm{w}}\mathfrak{A} \in L^1_{\mathrm{loc}}(\mathbf{R}^+; \mathcal{L}(H, \left[\begin{smallmatrix} Z \\ Y \end{smallmatrix}\right]))$;*[2]
- *(III)* ***C* is bounded**, *i.e., $C \in \mathcal{L}(H, \left[\begin{smallmatrix} Z \\ Y \end{smallmatrix}\right])$, and $\mathfrak{A}B \in L^1_{\mathrm{loc}}(\mathbf{R}^+; \mathcal{L}(\left[\begin{smallmatrix} U \\ W \end{smallmatrix}\right], H))$;*
- *(IV) $\mathfrak{A}B\left[\begin{smallmatrix} u_0 \\ w_0 \end{smallmatrix}\right] \in L^2_{\mathrm{loc}}(\mathbf{R}^+;H)$, $C_{\mathrm{w}}\mathfrak{A}B\left[\begin{smallmatrix} u_0 \\ w_0 \end{smallmatrix}\right] \in L^2_{\mathrm{loc}}(\mathbf{R}^+;\left[\begin{smallmatrix} Z \\ Y \end{smallmatrix}\right])$ and $C_{\mathrm{w}}\mathfrak{A} \in L^2_{\mathrm{loc}}(\mathbf{R}^+;\mathcal{L}(H, \left[\begin{smallmatrix} Z \\ Y \end{smallmatrix}\right]))$ for all $\left[\begin{smallmatrix} u_0 \\ w_0 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} U \\ W \end{smallmatrix}\right]$;*
- *(V) $\mathfrak{A}$ is exponentially stable and both the function $\lambda \mapsto C_{\mathrm{w}}(\lambda - A)^{-1}B$ and its adjoint belong to the strong version of H^2 over some right half-plane $\mathbf{C}^+_\omega = \{\lambda \in \mathbf{C} \mid \Re\lambda > \omega\}$ with $\omega < 0$.*

(A2) **(Nonsingularity)** *Assume that $D_{11}^* D_{11} \gg 0$, $D_{22} D_{22}^* \gg 0$,[3] and that for some $\epsilon > 0$,*[4]

$$
\begin{aligned}
(ir - A)x_0 = B_1 u_0 &\Longrightarrow \|(C_1)_{\mathrm{w}} x_0 + D_{11} u_0\|_Z \ge \epsilon \|x_0\|_H, \quad \text{and} \\
(ir - A^*)x_0 = C_2^* y_0 &\Longrightarrow \|(B_2^*)_{\mathrm{w}} x_0 + D_{22}^* y_0\|_W \ge \epsilon \|x_0\|_H,
\end{aligned}
$$

for all $x_0 \in H$, $u_0 \in U$, $y_0 \in Y$, and $r \in \mathbf{R}$.[5]

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[2]Recall that $\mathfrak{A}$ is the semigroup generated by A.

[3]The notation $E \gg 0$ or $E \ll 0$ means that E is *strictly* positive or negative definite, i.e., $E \ge \epsilon$ or $E \le -\epsilon$ for some $\epsilon > 0$.

[4]Here $B^*_{\mathrm{w}} = \left[\begin{smallmatrix} (B_1^*)_{\mathrm{w}} \\ (B_2^*)_{\mathrm{w}} \end{smallmatrix}\right] := \mathrm{w}-\lim_{\alpha \to +\infty} \alpha(\alpha - A^*)^{-1}B^*$ is the weak Yoshida–Weiss extensions of B^*.

[5]The first of the two equations above implies that $x_0 \in \mathcal{D}(C_{\mathrm{w}})$, and the second equation implies that $x_0 \in \mathcal{D}(B^*_{\mathrm{w}})$.

Under the above assumptions (A1) and (A2), there is a suboptimal exponentially stabilizing controller for S *(possibly with internal loop) if and only if conditions (1.)–(3.) below hold:*

(1.) **(P_X-CARE)** $D_{12}^*D_{12} - D_{12}^*D_{11}(D_{11}^*D_{11})^{-1}D_{11}^*D_{12} \ll g^2$*, and the algebraic Riccati equation*

$$\begin{cases} K_X^* S_X K_X = A^* P_X + P_X A + C_1^* C_1, \\ S_X = \begin{bmatrix} D_{11}^*D_{11} & D_{11}^*D_{12} \\ D_{12}^*D_{11} & D_{12}^*D_{12} - g^2 \end{bmatrix}, \\ K_X = -S_X^{-1}\left(\begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1 + B_{\mathrm{w}}^* P_X \right), \end{cases} \tag{1}$$

has a solution triple $P_X \in L(H; D(B_{\mathrm{w}}^*))$*,* $S_X \in L(\left[\begin{smallmatrix} U \\ W \end{smallmatrix}\right])$*, and* $K_X \in L(H_1, \left[\begin{smallmatrix} U \\ W \end{smallmatrix}\right])$*, with* $P_X \geq 0$*, such that the semigroup generated by* $A + BK_X$ *is exponentially stable (in particular, this implies that* S *is exponentially stabilizable).*[6]

(2.) **(P_Y-CARE)** $D_{12}D_{12}^* - D_{12}D_{22}^*(D_{22}D_{22}^*)^{-1}D_{22}D_{12}^* \ll g^2$*, and the algebraic Riccati equation*

$$\begin{cases} K_Y^* S_Y K_Y = AP_Y + P_Y A^* + B_2 B_2^*, \\ S_Y = \begin{bmatrix} D_{22}D_{22}^* & D_{22}D_{12}^* \\ D_{12}D_{22}^* & D_{12}D_{12}^* - g^2 \end{bmatrix}, \\ K_Y = -S_Y^{-1}\left(\begin{bmatrix} D_{22} \\ D_{12} \end{bmatrix} B_2^* + C_{\mathrm{w}} P_Y \right), \end{cases} \tag{2}$$

has a solution triple $P_Y \in L(H; D(C_{\mathrm{w}}))$*,* $S_Y \in L(\left[\begin{smallmatrix} Y \\ Z \end{smallmatrix}\right])$*, and* $K_X \in L(H_1^d, \left[\begin{smallmatrix} Y \\ Z \end{smallmatrix}\right])$*, with* $P_Y \geq 0$*, such that the semigroup generated by* $A^* + C^* K_Y$ *is exponentially stable (in particular, this implies that* S *is exponentially detectable).*

(3.) **(Coupling condition)** $r\,(P_X P_Y) < g^2$.

The proof of this theorem is given in [4].

Assume that (A1)–(A2) and (1.)–(3.) in Theorem 2.1 are satisfied. Then it is possible to parameterize the set of all exponentially stabilizing suboptimal controllers in the ‘standard’ way. We first choose any invertible $X \in L(\left[\begin{smallmatrix} U \\ W \end{smallmatrix}\right])$ such that[7] $X^* \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] X = S_X$ and $X_{21} = 0$, and define the operator quadruple $\left[\begin{smallmatrix} A_Z & B_Z \\ C_Z & D_Z \end{smallmatrix}\right] \in L\left(\begin{bmatrix} D(A_Z) \\ Y \\ U \end{bmatrix}; \begin{bmatrix} H \\ W \\ U \end{bmatrix} \right)$ by (here $K_X = \left[\begin{smallmatrix} K_{X1} \\ K_{X2} \end{smallmatrix}\right] \in L(H_1, \left[\begin{smallmatrix} U \\ W \end{smallmatrix}\right])$):

$$\left[\begin{array}{c|c} A_Z & B_Z \\ \hline C_Z & D_Z \end{array}\right] = \left[\begin{array}{c|cc} A^* + K_{X2}^*(B_2^*)_{\mathrm{w}} & C_2^* + K_{X2}^* D_{22}^* & -K_{X1}^* \\ \hline X_{22}^{-*}(B_2^*)_{\mathrm{w}} & X_{22}^{-*} D_{22}^* & X_{22}^{-*} X_{12}^* \\ 0 & 0 & 1 \end{array}\right]$$

with $D(A_Z) := \{x \in H \mid A_Z x \in H\} = \{x \in D((B_2^*)_{\mathrm{w}}) \mid A_Z x \in H\}$. Then the algebraic Riccati equation

$$\begin{cases} K_Z^* S_Z K_Z = A_Z^* P_Z + P_Z A_Z + B_2 X_{22}^{-1} X_{22}^{-*} B_2^*, \\ S_Z = D_Z^* \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] D_Z, \\ K_Z = -S_Z^{-1}\left(\begin{bmatrix} D_{22} X_{22}^{-1} \\ X_{12} X_{22}^{-1} \end{bmatrix} X_{22}^{-*} B_2^* + (B_Z^*)_{\mathrm{w}} P_Z \right), \end{cases} \tag{3}$$

[6] All the exponentially stabilizing solutions of the different Riccati equation appearing in this article are unique.
[7] We denote the identity operator in any Hilbert space by 1.

has a unique solution triple $P_Z \in L(H, D((B_Z^*)_w))$, $S_Z \in L(\left[\begin{smallmatrix} Y \\ Z \end{smallmatrix}\right])$ and $K_Z \in L(D(A_Z), \left[\begin{smallmatrix} Y \\ U \end{smallmatrix}\right])$ such that $P_Z \geq 0$, $S_{Z11} \gg 0$, $S_{Z22} - S_{Z21} S_{Z11}^{-1} S_{Z12} \ll 0$ and $A_Z + B_Z K_Z$ is exponentially stable. The solution of this Riccati equation gives us a suitably normalized doubly coprime factorization of the transfer function from the control input u to the measurement output y, and by using this factorization in the Youla parameterization and restricting the Youla parameter to be exponentially stable and have norm less than one we get a parameterization of the set of all the exponentially stabilizing suboptimal controllers.

Under the standard normalizing conditions

$$D_{12} = 0, \qquad D_{11}^* \begin{bmatrix} C_1 & D_{11} \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

condition (1.) can be written in the form

$$((B_1^*)_w P_X)^* (B_1^*)_w P_X - \gamma^{-2} ((B_2^*)_w P_X)^* (B_2^*)_w P_X = A^* P_X + P_X A + C_1^* C_1, \tag{4}$$

with the added requirements that $P_X \in L(H, D(B_w^*))$, $P_X \geq 0$, and that $A + (\gamma^{-2} B_2 (B_2^*)_w - B_1 (B_1^*)_w) P_X$ is exponentially stable. (Note that now $S_X = J_\gamma := \left[\begin{smallmatrix} 1 & 0 \\ 0 & -\gamma^2 \end{smallmatrix}\right]$ and $K_X = \left[\begin{smallmatrix} -(B_1^*)_w P_X \\ \gamma^{-2} (B_2^*)_w P_X \end{smallmatrix}\right] \in L(H, \left[\begin{smallmatrix} U \\ W \end{smallmatrix}\right])$.) If B is bounded, then (4) takes the classical form

$$P_X (B_1 B_1^* - \gamma^{-2} B_2 B_2^*) P_X = A^* P_X + P_X A + C_1^* C_1. \tag{5}$$

Analogous remarks apply to (2.) and (4.). In this case the suboptimal controllers will even be well-posed (i.e., no internal loops are needed). We thus observe that the classical H^∞ algebraic Riccati equations become special cases of (1.)–(4.).

Above we have given one necessary and sufficient condition for the existence of a stabilizing suboptimal controller which involves the two independent algebraic Riccati equations (1) for P_X and (2) for P_Y, plus the spectral radius condition $\rho(P_X P_Y) < \gamma^2$. This solution is symmetric with respect to the original system and its dual. Another non-symmetric equivalent description is also available. This solution is based on the two nested Riccati equations (1) for P_X and (3) for P_Z, and it does not contain any further coupling conditions.

3 Extensions

The setting that we have described above is one of the most restrictive ones treated in [4]. Analogous results are true under weaker assumptions, but the weaker the assumptions become, the more complicated the statements and the conclusions of the theorems become. One major class of results found in [4] relaxes the exponential stabilizability assumption to other versions of stability, one of which is 'output stability in the energy sense' (every initial state $x_0 \in H$ and input $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in L^2(\mathbf{R}^+; \left[\begin{smallmatrix} U \\ W \end{smallmatrix}\right])$ of the closed loop system produces an output $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right] \in L^2(\mathbf{R}^+; \left[\begin{smallmatrix} Z \\ Y \end{smallmatrix}\right])$). Furthermore, it is possible to allow even more unbounded control and observation operators and less smooth transfer functions than those appearing in Theorem 2.1. One major feature which complicates the theory is the following: if we allow pure delays in the input/output responses (such delays appear naturally in transmission lines and other systems with a hyperbolic behaviour), then the formulas for the input, output, and mixed input/output cost operators S_X, S_Y, respectively S_Z change, and they are no longer determined exclusively by the feedthrough operator D. For example, the formula for S_X in (1) should be replaced by

$$S_X = \begin{bmatrix} D_{11}^* D_{11} & D_{11}^* D_{12} \\ D_{12}^* D_{11} & D_{12}^* D_{12} - \mathrm{g}^2 \end{bmatrix} + \lim_{\mathsf{a} \to +\bullet} B_{\mathrm{w}}^* P_X (\mathsf{a} - A)^{-1} B,$$

and analogous changes are needed in the formulas defining S_Y and S_Z. In particular, the standard normalization conditions $D_{12} = 0$ and $D_{21} = 0$ no longer simplify the theory significantly, since these conditions no longer lead to the corresponding simplifications of S_X, S_Y, and S_Z.

The early history of the problem is explained on [3]. The theory was extended to the class of smooth Pritchard–Salamon systems by van Keulen in [11]. The stable full information $H^\bullet$ problem in the well-posed linear setting has been discussed in [7] and [8]. A frequency domain solution (under more restrictive assumptions) is given in [1].

We refer the reader to the references cited above and to [4, Chapter 12] for further details and for discussions of the remaining literature. Analogous discrete time results are also given in [4].

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