LQG-balancing in infinite dimensions

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Abstract

In this paper, we study the existence of LQG-balanced realizations (i.e., realizations for which the solutions of the control and filter Riccati equations are equal) for discrete time infinite-dimensional systems.

1 Introduction

Balanced realizations of a linear system are those for which the observability and controllability gramians are both equal and diagonal. In [6] Ober and Montgomery-Smith showed that they exist for a large class of transfer functions. For the subset of stable irrational transfer functions with an impulse response $h \in \mathbf{L}^1 \cap \mathbf{L}^2(0, \infty; \mathbb{C}^{p\times m})$ the problem of approximation by the truncations of the balanced realization was studied in Glover *et al.* [3]. The main result was that if the Hankel operator is nuclear (the sum of the Hankel singular values is finite), then the $\mathbf{L}^\infty$-error of the k-th approximation is bounded above by twice the sum of the tail of the Hankel singular values starting with the $(k+1)$-th one. This has interesting implications for controller design based on reduced order models (see Curtain [1]). Of course, balanced realizations only exist for stable transfer functions and in [1] it was proposed that LQG-balanced realizations could prove a useful alternative. LQG-balanced realizations of a system are those for which the solutions Q and P of the associated control and filter Riccati equations are equal. For finite-dimensional systems the eigenvalues of PQ are invariants of the system and the truncations of the LQG-balanced realizations are good approximants of the original system. The complete theory for the finite-dimensional case can be found in Mustafa and Glover [5], but little has been developed in infinite dimensions. An outline of results for a special class of systems was sketched in [1]. The first step is to prove the existence of LQG-balanced realizations for infinite-dimensional systems, but since for continuous-time systems even the existence of solutions to Riccati equations is complicated, in this paper we concentrate on this problem for discrete-time infinite-dimensional systems. This will prove to be an important step in establishing the existence of LQG-balanced realizations for continuous time infinite-dimensional linear systems.

In section 2 we define infinite-dimensional discrete-time linear systems and various notions of stability. In section 3 we collect some facts about Lyapunov and Riccati equations that we need. In section 4 we study the relation between a discrete-time system and an associated stable factor system, in particular, the relation between the Riccati equations of the discrete-time system and the Lyapunov equations of the associated stable factor system. We use this relation in section 5 to prove our main result: the existence of LQG-balanced realizations for infinite-dimensional discrete-time systems under very general assumptions.

2 Discrete-time systems

In this article H, X, U, Y denote separable Hilbert spaces, $\mathbf{H}^2(H)$ denotes the Hilbert space of holomorphic functions on the unit disc with values in H that are square integrable over the unit circle and $\mathbf{H}^\infty(U, Y)$ denotes the Banach space of bounded holomorphic functions on the unit disc with values in $\mathcal{L}(U, Y)$, the space of bounded operators from U to Y. We call a quadruple of operators $(A, B, C, D) \in \mathcal{L}(X) \times \mathcal{L}(U, X) \times \mathcal{L}(X, Y) \times \mathcal{L}(U, Y)$ a discrete-time linear system. We call $G(z) = D + Cz(I - zA)^{-1}B$ its transfer function. In general we call any function $G(z)$ that is holomorphic and bounded on some disc centered at the origin a transfer function. (A, B, C, D) is a stable discrete-time system if

- it is input stable, i.e., $B^*(I - zA^*)^{-1}x \in \mathbf{H}^2(U)$ for all $x \in X$;
- it is output stable, i.e., $C(I - zA)^{-1}x \in \mathbf{H}^2(Y)$ for all $x \in X$;
- it is input-output stable, i.e., $D + Cz(I - zA)^{-1}B \in \mathbf{H}^\infty(U, Y)$.

This notion of stability is a translation to discrete time of a continuous time stability notion introduced by Staffans in [9] (see also Staffans [8] and Mikkola [4]).

3 Lyapunov and Riccati equations

In this section we collect some elementary facts about Lyapunov and Riccati equations that we shall need.

Definition 3.1. *The control Lyapunov equation of a discrete-time linear system* (A, B, C, D) *is:*

$$ALA^* - L + BB^* = 0.$$

The observer Lyapunov equation of a discrete-time linear system (A, B, C, D) *is:*

$$A^*LA - L + C^*C = 0.$$

Lemma 3.1. *If the system is input stable, the controllability gramian is the smallest non-negative solution of the control Lyapunov equation. If the system is output stable, the observability gramian is the smallest nonnegative solution of the observer Lyapunov equation.*

Definition 3.2. *The control Riccati equation of a discrete-time linear system* (A, B, C, D) *is:*

$$A_Q^*(I + QBS^{-1}B^*)QA_Q - Q + C^*R^{-1}C = 0 \tag{1}$$

where

$$A_Q := A - B(S + B^*QB)^{-1}(D^*C + B^*QA) \quad R := I + DD^* \quad S := I + D^*D.$$

The filter Riccati equation of a discrete-time linear system (A, B, C, D) *is:*

$$A_P P(I + C^*R^{-1}CP)A_P^* - P + BS^{-1}B^* = 0 \tag{2}$$

where

$$A_P := A - (BD^* + APC^*)(R + CPC^*)^{-1}C.$$

Many different forms of these Riccati equations are used in the literature. In the next lemma we present some alternative forms that are sometimes more convenient.

Lemma 3.2. *For a discrete-time linear system* (A, B, C, D) *the set of nonnegative solutions of the following equations are identical.*

- *The control Riccati equation (1);*
- $\bar{A}^*Q(I + BS^{-1}B^*Q)^{-1}\bar{A} - Q + C^*R^{-1}C = 0$ *where* $\bar{A} = A - BS^{-1}D^*C$*;*
- $A^*QA - Q + C^*C = (C^*D + A^*QB)(S + B^*QB)^{-1}(B^*QA + D^*C)$.

This lemma has an obvious dual version concerning the filter Riccati equation (2). To give sufficient conditions on the system under which the Riccati equations have nonnegative solutions we define the following concepts.

Definition 3.3. *A discrete-time linear system* (A, B, C, D) *is called output stabilizable if there exists an operator* $F \in \mathcal{L}(X, U)$ *such that* $(A + BF, B, [F; C + DF], D)$ *is output stable. A discrete-time linear system* (A, B, C, D) *is called input stabilizable if there exists an operator* $L \in \mathcal{L}(Y, X)$ *such that* $(A + LC, [L, B + LD], C, D)$ *is output stable.*

Remark 3.1. *In the previous definition and in the rest of the article* $[a, b]$ *stands for the row vector of operators with as first component* a *and as second component* b *and* $[a; b]$ *stands for the column vector of operators with as first component* a *and as second component* b*.*

Lemma 3.3. *If a discrete-time linear system is output stabilizable, then its control Riccati equation has a nonnegative solution. If a discrete-time linear system is input stabilizable, then its filter Riccati equation has a nonnegative solution.*

Remark 3.2. *Note that the previous lemma is about existence, not uniqueness. There may be more then one nonnegative solution of the Riccati equation.*

We shall need the following well-known and easy to prove result on the behaviour of the solutions of the Riccati equations under similarity transformations in the state space.

Lemma 3.4. *Let (A, B, C, D) be a discrete-time linear system and suppose that its control Riccati equation has a nonnegative solution Q and its filter Riccati equation has a nonnegative solution P. Let Y be a boundedly invertible operator on the state space. Then $\bar{Q} := Y^{-*}QY^{-1}$ and $\bar{P} := YPY^*$ are solutions of the control, respectively, filter Riccati equation of the discrete-time linear system $(YAY^{-1}, YB, CY^{-1}, D)$.*

Proofs of the lemmas in this section can be found in Opmeer [7].

4 Factor systems

In this section we study the relation between a discrete-time system and an associated stable factor system that we now define via a factorization of its transfer function.

Definition 4.1. *A transfer function G has a right factorization if there exist $[M; N] \in \mathbf{H}^\infty(U, U \oplus Y)$ such that M has an inverse such that $M^{-1}(r\cdot) \in \mathbf{H}^\infty(U, U)$ for some $r > 0$ and $G(z) = N(z)M(z)^{-1}$ on some disc centered at the origin. This function $[M; N]$ is called a right factor of G.*

Two important properties that a factor may or may not have are a normalization property and coprimeness.

Definition 4.2. *A right factor $[M; N] \in \mathbf{H}^\infty(U, U \oplus Y)$ is called normalized if for almost all z on the unit circle $M(z)^*M(z) + N(z)^*N(z) = I$.*

Definition 4.3. *A right factor $[M; N] \in \mathbf{H}^\infty(U, U \oplus Y)$ is called right coprime if there exist $[\tilde{X}, \tilde{Y}] \in \mathbf{H}^\infty(U \oplus Y, U)$ such that for all z in the unit disc $\tilde{X}(z)M(z) - \tilde{Y}(z)N(z) = I$.*

We remark that since a factor is in $\mathbf{H}^\infty$, it always has a realisation as a discrete-time linear system (see Young [11]). For any given realization of a right factor of a transfer function G we can construct a realization of G as a discrete-time linear system.

Lemma 4.1. *Let $(\check{A}, \check{B}, [\check{C}_1; \check{C}_2], [\check{D}_1; \check{D}_2])$ be a realization of a right factor of a transfer function G. Then $\check{D}_1$ is boundedly invertible and the system*

$$A := \check{A} - \check{B}\check{D}_1^{-1}\check{C}_1 \quad B := \check{B}\check{D}_1^{-1} \quad C := \check{C}_2 - \check{D}_2\check{D}_1^{-1} \quad D := \check{D}_2\check{D}_1^{-1} \tag{3}$$

is a realization of G.

And if we are given a realization of G and a nonnegative solution of the control Riccati equation of this realization we can construct a realization of a right factor.

Lemma 4.2. *Let (A, B, C, D) be a discrete-time linear system and suppose that its control Riccati equation has a nonnegative solution Q. Define the discrete-time system $(\breve{A}, \breve{B}, \breve{C}, \breve{D})$ by*

$$\breve{A} := A + BF \quad \breve{B} := BW^{-1/2} \quad \breve{C} := [F; C + DF] \quad \breve{D} := [I; D]W^{-1/2} \tag{4}$$

where

$$W := S + B^*QB \quad S := I + D^*D \quad F := -W^{-1}(D^*C + B^*QA).$$

This system is a realization of a right factor $[M; N]$ of the transfer function of (A, B, C, D). Moreover, for all $z \in \rho(\breve{A})$ $M(z)^*M(z) + N(z)^*N(z) = I$ and there exist $\tilde{X}, \tilde{Y}$ such that $\tilde{X}(\cdot)u \in \mathbf{H}^2(U)$ for all $u \in U$, $\tilde{Y}(\cdot)y \in \mathbf{H}^2(U)$ for all $y \in Y$ and $\tilde{X}(z)M(z) - \tilde{Y}(z)N(z) = I$ for all z in the unit disc.

Remark 4.1. *Note that the system (A, B, C, D) can be recovered from the system $(\breve{A}, \breve{B}, \breve{C}, \breve{D})$ defined by (4) using formulas (3).*

Remark 4.2. *If X is finite-dimensional, $\sigma(\breve{A})$ is finite and therefore $M(z)^*M(z)+N(z)^*N(z) = I$ for almost all z on the unit circle and the factor $[M, N]$ is thus normalized. If X, U, Y are all finite-dimensional, then $\tilde{X} \in \mathbf{H}^\infty(U, U)$ and $\tilde{Y} \in \mathbf{H}^\infty(Y, U)$ and the factor $[M, N]$ is thus coprime.*

In the the infinite-dimensional case this is in general not true, but we do have the following corollary.

Corollary 4.1. *With the assumptions as in the previous lemma and the extra assumptions that U and Y are finite-dimensional and the spectrum of A has at most countably many points on the unit circle, the system defined by (4) is a realization of a normalized right coprime factor of G.*

We have the following results on the connection between the Riccati equations of a discrete-time linear system and the Lyapunov equations of its factor system.

Lemma 4.3. *Suppose that $(\breve{A}, \breve{B}, \breve{C}, \breve{D})$ is a stable realization of the normalized right factor $[M; N]$ with controllability gramian $L_B > 0$ and observability gramian L_C. Then*

$$\breve{B}^* L_C \breve{A} + \breve{D}^* \breve{C} = 0$$

$$\breve{B}^* L_C \breve{B} + \breve{D}^* \breve{D} = I$$

and L_C is a solution of the control Riccati equation of the discrete-time linear system (A, B, C, D) defined by (3). Assume further that $I - L_B L_C$ is boundedly invertible. Then $P := L_B(I - L_B L_C)^{-1}$ is a solution of the filter Riccati equation of the discrete-time linear system (A, B, C, D) defined by (3).

Remark 4.3. *The assumption that* $I - L_B L_C$ *is boundedly invertible is not a very nice one since it depends on the realization. Sufficient conditions for* $I - L_B L_C$ *to be boundedly invertible are:*

- M *and* N *are right coprime;*

or

- $[M; N]$ *has a compact Hankel operator* Γ *and* $I - \Gamma^*\Gamma$ *is boundedly invertible.*

These two conditions do not depend on the realization, but only on the transfer function.

The following lemma is the discrete-time version of a continuous-time result (lemma 9.4.10) in Curtain and Zwart [2].

Lemma 4.4. *Let* (A, B, C, D) *be a discrete-time linear system and suppose that its control Riccati equation has a nonnegative solution* Q *and its filter Riccati equation has a nonnegative solution* P*. Then* Q *is a solution of the observer Lyapunov equation and* $P(I + QP)^{-1}$ *is a solution of the control Lyapunov equation of the system* $(\check{A}, \check{B}, \check{C}, \check{D})$ *defined by (4).*

Proofs of these lemmas can be found in [7].

5 LQG-balancing

In this section we give conditions for the existence of a LQG-balanced realization.

Definition 5.1. *The discrete-time linear system* (A, B, C, D) *is called a LQG-balanced realization if its control and filter Riccati equations (1) (2), respectively, have nonnegative solutions that are equal.*

We use the previous lemmas and the following result by Young on the existence of balanced realizations [11] to prove our main theorem.

Lemma 5.1 (Young). *A function in* $\mathbf{H}^\infty$ *has a stable discrete-time realization such that the controllability and observability gramians are equal and positive.*

Lemma 5.2. *Let* G *be a transfer function such that either of the following conditions hold:*

1. G *has a normalized right coprime factor;*

2. G *has a normalized right factor with compact Hankel* Γ *such that* $I - \Gamma^*\Gamma$ *is boundedly invertible;*

then G *has a LQG-balanced realization.*

Proof. According to remark 4.3 and lemma 5.1 the assumptions of lemma 4.3 are satisfied and we can take $L_B = L_C$ in this lemma. We then obtain a realization of G with a solution of the control Riccati equation L_C and a solution of the filter Riccati equation $L_C(I - L_C^2)^{-1}$. We use lemma 3.4 with $Y := (I - L_C^2)^{-1/4}$ to obtain another realization of G for which $L_C(I - L_C^2)^{-1/2}$ is a solution of both the control and the filter Riccati equation. □

There is of course a dual lemma for left factors. Thus we have:

Theorem 5.1. *Let G be a transfer function such that one of the following conditions hold:*

1. *G has a normalized left coprime factor;*
2. *G has a normalized right coprime factor;*
3. *G has a normalized left factor with compact Hankel Γ such that $I - \Gamma\Gamma^*$ is boundedly invertible;*
4. *G has a normalized right factor with compact Hankel Γ such that $I - \Gamma^*\Gamma$ is boundedly invertible;*

then G has a LQG-balanced realization.

We have the following corollary.

Corollary 5.1. *Let (A, B, C, D) be a discrete-time linear system with transfer function G and suppose that its control Riccati equation has a nonnegative solution. Let $[M; N]$ be the transfer function of the factor system defined by (4) and let Γ be its Hankel operator. Assume that $[M; N]$ is normalized and that in addition either of the following conditions hold:*

1. *$[M; N]$ is right coprime;*
2. *Γ is compact and $I - \Gamma^*\Gamma$ is boundedly invertible;*

then G has a LQG-balanced realization.

There is of course a dual statement which involves the filter Riccati equation. Note that we assume the existence of a solution of one of the Riccati equations of a certain realization of G and with some additional assumptions we get that the other Riccati equation (of a different realization) also has a solution.

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