Zeros of SISO Infinite-Dimensional Systems

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Abstract

We give a definition of the zeros of an infinite-dimensional system with bounded control and observation operators B and C respectively. The zeros are defined in terms of the spectrum of an operator on an invariant subspace. These zeros are shown to be exactly the invariant zeros of the system. For the case of SISO systems, where also the range of B is not in the kernel of C, we show that this subspace exists and it is the entire kernel of C. We calculate the operator K such that the spectrum of $A + BK$ on $\ker(C)$ is the system zeros, and show that $A + BK$ generates a C_o-semigroup on $\ker(C)$. If the range of B is not in the kernel of C, a number of situations may occur, depending on the nature of B and C.

1 Introduction

The importance of the poles of a transfer function to system dynamics are well-known. The zeros are also important in controller design. For instance, the presence of right-hand-plane zeros restricts the achievable sensitivity reduction. The presence of right-hand-plane zeros also makes the use of an adaptive controller impractical in most situations. Furthermore, if a zero is close to a pole, the system is close to being non-minimal and hence uncontrollable and/or unobservable *e.g.* [6, sect. 2.5]. It is also difficult to obtain robust stability for these systems [*ibid*,sect. 4.6].

For linear finite-dimensional systems

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t),$$

where $x(t) \in R^n$ and A, B and C are matrices, the zero dynamics can be defined in a number of equivalent ways. One of the most fundamental is in terms of an invariant subspace. A subspace V is invariant for a linear finite-dimensional system if for all initial conditions in V there exists a control that keeps the state in V for all times. If this is the case, the control can be a constant state feedback. *e.g.* [9]. The zero dynamics can be obtained by considering the system on the largest feedback-invariant subspace Z contained in the kernel of the observation operator. The system zeros are then analyzed by considering the spectrum of the operator $A + BK$ on Z where K is an operator such that Z is $A + BK$-invariant. The dynamics are independent of the choice of K for which Z is $A + BK$-invariant. The eigenvalues of $A + BK$ on Z are identical to the invariant zeros.

This is not the case for infinite-dimensional systems. In fact, we do not even have a standard definition of zero dynamics valid for general infinite-dimensional systems. Generally, the transmission zeros for infinite-dimensional systems are approximated by calculation of the transmission zeros of a finite-dimensional approximation to the system. These calculated zeros are often very different from the exact zeros, even when the system poles are approximated with good accuracy *e.g.* [3, 5]. Thus, it is useful to have a rigorous definition of the zeros of infinite-dimensional systems based on the state-space realization. A large difficulty in studying zeros of infinite-dimensional systems however is that the largest feedback-invariant subspace in the kernel of C might not exist in the sense mentioned above.

In this paper we will only consider single-input single-output systems with bounded control $b \in D(A)$ and bounded observation c. For control and observation b and c respectively where $\langle b, c\rangle \neq 0$, we show that a largest feedback-invariant subspace exists and it is the entire subspace $c^\perp := \{x \in X \mid \langle x, c\rangle = 0\}$. We give an explicit representation of a feedback K for which $c^\perp$ is $A + bK$-invariant. The spectrum of $A + bK$ is identical to the invariant zeros of the system. If $\langle b, c\rangle = 0$, then the theory is quite different. If $c^\perp$ is A-invariant, then the transfer function is identically 0. In the more interesting case where the transfer function is non-trivial, a number of situations may occur, depending on the nature of b and c. We obtain a characterization of the largest invariant subspace in a number of cases, and show that it is contained in a proper subset of $c^\perp$. In other cases a largest feedback-invariant subspace may not exist. This is illustrated by an example of a delay system.

2 Definition of Zero Dynamics

We consider single-input single-output infinite-dimensional systems with bounded control and observation. Let X be a Hilbert space with inner product $\langle \cdot, \cdot\rangle$ and norm $\|\cdot\|$. Let A be the infinitesimal generator of a C_0-semigroup $T(t)$ on X. Let b and c be elements of X. Let $U = Y = \mathbb{C}$ and $u(t) \in U$. We consider the following single-input single-output system (A, b, c) in X:

$$\dot{x}(t) = Ax(t) + bu(t) \tag{2.1}$$

with the observation

$$y(t) = Cx(t) := \langle x(t), c\rangle. \tag{2.2}$$

This paper extends the work in Byrnes *et. al.* [1], where zero dynamics are defined for

(A, b, c) under the assumptions that

$$b \in D(A), \qquad c \in D(A^*), \qquad \langle b, c\rangle \neq 0. \tag{2.3}$$

In this paper we will remove the restriction $c \in D(A^*)$, and also examine the case where $\langle b, c\rangle = 0$. Our standing assumptions throughout the paper are:

(A1) A is an infinitesimal generator of a C_0-semigroup on the Hilbert space X.

(A2) $c \in X$.

(A3) $b \in D(A)$.

The input/output map is

$$y(t) = < \int_0^t T(t-\sigma)bu(\sigma)\, d\sigma, c > . \tag{2.4}$$

Defining $g(t) = \langle T(t)b, c\rangle$, the output is simply the convolution of $g(t)$ and $u(t)$. Taking the Laplace transform on both sides of equation (2.4) gives

$$\hat{y}(s) = G(s)\hat{u}(s), \tag{2.5}$$

where $G(s) = \langle (sI - A)^{-1}b, c\rangle$ is the system transfer function.

Definition 2.1. *The* transmission zeros *of (2.1), (2.2) is the set of all z such that $G(z) = 0$.*

Definition 2.2. *The* invariant zeros *of (2.1), (2.2) are the set of all λ such that*

$$\begin{bmatrix} \lambda I - A & b \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has a solution for $u \in U$ and non-zero $x \in D(A)$.

It is not difficult to show that every transmission zero is an invariant zero. Also, any invariant zero in $\rho(A)$ is a transmission zero. Invariant zeros have the same relation to transmission zeros as the system eigenvalues have to the transfer function poles.

Following are the definitions of invariance that we use here to define zero dynamics. Note that we only consider closed invariant subspaces. Even though the input and output for this system are bounded, for a complete theory we need to consider unbounded feedbacks. We say that a feedback operator K is A-bounded if $K \in \mathcal{B}(D(A), U)$, where $D(A)$ is the domain of A. We point out here that if K is A-bounded, there is no guarantee that $A + bK$ is the generator of a C_0-semigroup, where $D(A + bK) = D(A)$ and $(A + bK)x = Ax + b(Kx) \in X$.

Definition 2.3. *A closed subspace Z of X is feedback invariant if there exists an A-bounded feedback K such that $(A + bK)(Z \cap D(A)) \subset Z$.*

Theorem 2.1. *[11, Thm.II.26] A closed subspace Z is feedback-invariant if and only if it is (A, b)-invariant, that is,*

$$A(Z \cap D(A)) \subset Z + \text{span}\{b\}.$$

Definition 2.4. *A closed subspace Z of X is closed-loop invariant if there exists an A-bounded feedback K such that $(A + bK)(Z \cap D(A)) \subset Z$, and $A + bK$ generates a C_0 semigroup T_K on Z.*

Definition 2.5. *A closed subspace Z of X is open loop invariant if for every $x(0) \in Z$, there exists a $u(\cdot) \in L_2([0, \infty), U)$, such that the solution $x(t) \in Z$ for all $t \geq 0$.*

Closed-loop invariance implies feedback invariance, but the converse statement is not always true [11]. Under certain conditions, some types of invariance are equivalent. For instance, if the feedback is in fact bounded, then open-loop (with the control restricted to continuous functions) and closed-loop invariance are equivalent [11, Thm. II.27], as is feedback invariance.

We now define zero dynamics.

Definition 2.6. *Suppose Z is the largest closed feedback-invariant subspace contained in $c^\perp$, and let K be such that Z is $A + bK$-invariant. If such a Z and K exist, the zero dynamics of (A, b, c) is $A + bK|_Z$. We write this as $(Z, A + bK)$.*

The operator K is not specified as unique in the above theorem. However, if $b \notin Z$, and there are two operators K_1 and K_2 that are both (A, b)-invariant on Z, then $b(K_1x - K_2x) \in Z$ and so $K_1x = K_2x$ for all $x \in Z$.

Theorem 2.2. *If $Z \subseteq c^\perp$ is feedback-invariant and $b \in Z$ then the system transfer function is identically zero.*

Proof: Since Z is feedback-invariant,

$$A(Z \cap D(A)) \subset Z + \mathcal{R}(b) \subset Z.$$

This implies that A is Z-invariant. This fact together with $b \in Z$ implies that every $z \in Z$ can be written $z = (sI - A)\xi(s)$ where $\xi(s) \in D(A) \cap Z$ [11, Lem. II.18], and $s \in [r, \infty)$ for some $r \in \mathbb{R}$. We obtain that $(sI - A)^{-1}b \in Z$ for all $s \in [r, \infty)$. Since $Z \subset c^\perp$, the system transfer function $G(s)$ is identically zero. □

Thus, we may assume that if Z is a feedback-invariant subspace in $c^\perp$, then $b \notin Z$ and also that the feedback K is unique on Z.

Theorem 2.3. *The set of eigenvalues of the zero dynamics (if they exist) is the set of invariant zeros of the system.*

Proof: Indicate the largest feedback-invariant subspace in $c^\perp$ by Z and let K be a suitable feedback. As shown above, we may assume $b \notin Z$ and that $K|_Z$ is unique. Choose a decomposition for $X = X_1 \oplus X_2$ where $Z = X_1$ and $b \in X_2$. Let P indicate the projection onto X_2. Any $x \in X$ can be written as $x_1 + x_2$, where $x_1 = (I-P)x \in X_1$ and $x_2 = Px \in X_2$. Then, A can be decomposed as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{2.6}$$

where

$$A_{11} = (I-P)A|_{X_1}, \quad A_{12} = (I-P)A|_{X_2}, \quad A_{21} = PA|_{X_1}, \quad A_{22} = PA|_{X_2}. \tag{2.7}$$

Since $Z \subseteq c^{\perp}$, and $b \in X_2$, the system can be written

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t) \\ y(t) &= \langle x_2, c \rangle. \end{aligned}$$

Writing $K = [K_1 \quad K_2]$,

$$u(t) = [K_1 \quad K_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} + bK_1 & A_{22} + bK_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Since X_1 is $A + bK$-invariant,

$$A_{21}x_1 = -bK_1x_1 \tag{2.8}$$

for all $x_1 \in X_1$. Since K_2 does not affect the action of $A + bK$ on Z, we set $K_2 = 0$. Note that $A_{11} = (A + bK)|_Z$.

Suppose that λ is an eigenvalue of $(A + bK)|_Z = A_{11}$ with eigenvector $x_1 \in D(A) \cap Z$. Setting $x = x_1$, $u = -Kx_1$ we see that λ is also an invariant zero.

We now show that every invariant zero is an eigenvalue of $(A + bK)|_Z = A_{11}$. Write

$$\begin{bmatrix} \lambda I - A & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} \lambda I - A - bK & b \\ c & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ K & I \end{bmatrix}. \tag{2.9}$$

Thus, if λ is an invariant zero of (A, b, c) then it is an invariant zero of $(A + bK, b, c)$. Using the decomposition $X = X_1 \oplus X_2$, with this choice of K

$$\begin{bmatrix} \lambda I - A - bK & b \\ c & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \tag{2.10}$$

becomes

$$\begin{bmatrix} \lambda I - A_{11} & -A_{12} & 0 \\ 0 & \lambda I - A_{22} & b \\ 0 & c & 0 \end{bmatrix} \begin{bmatrix} x_{o1} \\ x_{o2} \\ u_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.11}$$

where not both of x_{o1}, x_{o2} are zero. Thus, $x_{o2} \in c^{\perp}$ and

$$A\begin{bmatrix} x_{o1} \\ x_{o2} \end{bmatrix} = \lambda \begin{bmatrix} x_{o1} \\ x_{o2} \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix}(-Kx_o + u_o).$$

This implies that span$[x_{o1}, x_{o2}]$ is in an (A, b)-invariant subspace. Since Z is the largest (A, b) subspace in $c^{\perp}$, $x_{o2} = 0$, and so $x_{o1} \neq 0$. This implies that x_{o1} is an eigenvalue of A_{11} with eigenvector λ. □

3 The case $\langle b, c\rangle \neq 0$

As in [1], for $x \in X$ we define the projection

$$Px = \frac{\langle x, c\rangle}{\langle b, c\rangle} b. \tag{3.1}$$

Theorem 3.1. *If $\langle b, c\rangle \neq 0$ then the zero dynamics for (A, b, c) exist, and is $(c^{\perp}, A + bK)$, where*

$$Kx = -\frac{\langle A(I-P)x, c\rangle}{\langle b, c\rangle}.$$

Furthermore, $A + bK$ generates a C_o-semigroup on $c^{\perp}$ and so $c^{\perp}$ is closed-loop invariant.

Proof: Let $X_1 = c^{\perp}$ and $X_2 = \text{span}\{b\}$. Any $x \in X$ can be written as $x_1 \oplus x_2$, where $x_1 = (I - P)x \in X_1$ and $x_2 = Px \in X_2$. Writing X as $X_1 \oplus X_2$, A can be decomposed as (2.6, 2.7).

In this decomposition $b = [0, b]^T$. Letting $K_1 = K(I - P)$ and $K_2 = KP$,

$$A + bK = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} + bK_1 & A_{22} + bK_2 \end{bmatrix}.$$

Noting that

$$A_{21}x_1 = b\frac{\langle Ax_1, c\rangle}{\langle b, c\rangle},$$

we choose

$$Kx_1 = K_1x_1 = -\frac{\langle Ax_1, c\rangle}{\langle b, c\rangle} = -\frac{\langle A(I-P)x, c\rangle}{\langle b, c\rangle}$$

with domain $D(K) = D(A)$. The choice of K_2 is arbitrary, so we let $K_2 = 0$. Then

$$A + bK = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{3.2}$$

and it is clear that $(A + bK)(X_1 \cap D(A)) \subset X_1$. Also, for $x_1 \in X_1$, $(A + bK)x_1 = A_{11}x_1$. It remains to show that $(A + bK)$ generates a C_0 semigroup on X_1.

Since $b \in D(A)$ and K is rank one and A-bounded $A + bK$ generates a C_0 semigroup on X [8, Thm. 1(c)], $S(t)$.

We need to show that X_1 is invariant under $S(t)$. This does not follow immediately from the invariance of X_1 under $A + bK$. If we can show that for any $\lambda \in \rho(A + bK)$ the image of X_1 under $(\lambda - (A + bK))$ is all of X_1, then X_1 is $e^{(A+bK)t}$ invariant [11, Lem. I.4]. We can write $(\lambda - (A + bK))x$ as

$$\begin{bmatrix} (\lambda - A_{11})x_1 - A_{12}x_2 \\ (\lambda - A_{22})x_2 \end{bmatrix}.$$

Since the image of X_2 under A_{12} is $\text{span}\{(I - P)Ab\}$, and the range of $(\lambda - (A + bK))$ is all of X, we see that the image of X_1 under $\lambda - (A + bK)$ contains $X_1 \ominus \text{span}\{(I - P)Ab\}$. To show that the image of X_1 under $\lambda - (A + bK)$ also contains $\text{span}\{(I - P)Ab\}$, note that there must be x_1 and x_2 which solve

$$\begin{bmatrix} (\lambda - A_{11})x_1 - A_{12}x_2 \\ (\lambda - A_{22})x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \left(\lambda - \frac{\langle Ab,c\rangle}{\langle b,c\rangle}\right) b \end{bmatrix}$$

Writing x_2 as αb for some scalar α, we see that the bottom row of this matrix equation is

$$\alpha \left[\lambda - \frac{\langle Ab, c\rangle}{\langle b, c\rangle}\right] b = \left[\lambda - \frac{\langle Ab, c\rangle}{\langle b, c\rangle}\right] b,$$

which implies that $x_2 = b$. Plugging this into the first row of the matrix equation, we obtain that

$$(\lambda - A_{11})x_1 - (I - P)Ab = 0.$$

This shows that the image of X_1 under $\lambda - (A + bK)$ contains $\text{span}\{(I - P)Ab\}$. Hence, $S(t)$ is a C_0 semigroup on X_1. □

4 The case $\langle b, c\rangle = 0$

In addition to $\langle b, c\rangle = 0$, we assume that $c \in D(A^*)$ and $\langle b, A^*c\rangle \neq 0$. If $A^*c \in \text{span}\{c\}$ then A is $c^\perp$-invariant. Since $b \in c^\perp$, the same argument in the proof of Theorem 2.2 implies that the transfer function is identically zero. We therefore also assume that $A^*c \notin \text{span}\{c\}$.

Theorem 4.1. *Suppose that $\langle b, c\rangle = 0$, $c \in D(A^*)$, $A^*c \notin \text{span}\{c\}$ and $\langle b, A^*c\rangle \neq 0$. Then the zero dynamics for (A, b, c) exist. The zero dynamics are $((c)^\perp \cap (A^*c)^\perp, A + bK)$ with*

$$Kx = -\frac{\langle A(I - Q)(I - P)x, (I - P)A^*c\rangle}{\langle b, A^*c\rangle},$$

where for $x \in X$ and $x_1 \in (c)^\perp$,

$$Px = \frac{\langle x, c\rangle}{\|c\|^2} c$$

and

$$Qx_1 = \frac{\langle x_1, (I-P)A^*c\rangle}{\langle b, A^*c\rangle} b.$$

Proof: Let $X_1 = c^\perp$ and $X_2 = \text{span}\{c\}$. Any $x \in X$ can be written as $x_1 + x_2$, where $x_1 = (I-P)x \in X_1$ and $x_2 = Px \in X_2$. Writing X as $X_1 \oplus X_2$, A can be written as (2.6, 2.7). In this decomposition $b = [b, 0]^T$, since $b \in X_1$. Letting $K_1 = K(I-P)$ and $K_2 = KP$,

$$A + bK = \begin{bmatrix} A_{11} + bK_1 & A_{12} + bK_2 \\ A_{21} & A_{22} \end{bmatrix}.$$

If a subspace $Z \in X_1$ is to satisfy $(A+bK)(Z \cap D(A)) \subset Z$, we see that $A_{21}z = 0$ for any $z \in D(A) \cap Z$. This condition is equivalent to $\langle Az, c\rangle = 0$. Since $c \in D(A^*)$, any $(A+bK)$-invariant subspace $Z \subseteq c^\perp \cap (A^*c)^\perp$. For $z \in Z \cap D(A)$, $(A+bK)z = (A_{11} + bK_1)z$. Define

$$Z = c^\perp \cap (A^*c)^\perp \tag{4.1}$$

and decompose $c^\perp$ into $\tilde{X}_1 \oplus \tilde{X}_2$ where $\tilde{X}_1 = (I-Q)c^\perp$ and $\tilde{X}_2 = Qc^\perp$. In particular, $\tilde{X}_1 = Z$ and $b \in \tilde{X}_2$. Since $A^*c \notin \text{span}\{c\}$, we can apply Theorem 3.1 on $c^\perp$ with c replaced by $(I-P)A^*c$, A replaced by A_{11} and P replaced by Q. The feedback used to obtain the zero dynamics is

$$\begin{aligned} Kx &= -\frac{\langle A_{11}(I-Q)x_1, (I-P)A^*c\rangle}{\langle b, A^*c\rangle} \\ &= -\frac{\langle A(I-Q)(I-P)x, (I-P)A^*c\rangle}{\langle b, A^*c\rangle}. \end{aligned}$$

The arguments in Theorem 3.1 can be used to show that the zero dynamics for (A, b, c) in this case are $(Z, A+bK)$. □

The following example illustrates that if $\langle b, c\rangle = 0$ the zero dynamics as defined in Defn. 2.6 might not exist.

Example: The following example of a controlled delay equation first appeared in [7].

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) - x_2(t-1) \\ \dot{x}_2(t) &= u(t) \\ y(t) &= x_1(t). \end{aligned} \tag{4.2}$$

The system of equations (4.2) can be written in the standard state-space form (2.1, 2.2) as follows. Choose the state-space

$$X = R^2 \times L_2(-1,0) \times L_2(-1,0).$$

and define the closed operator A on X

$$A(r_1, r_2, \phi_1, \phi_2) = \begin{bmatrix} \phi_2(0) - \phi_2(-1) \\ 0 \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$$

with domain

$$D(A) = \left\{ (r_1, r_2, \phi_1, \phi_2), \phi_1(0) = r_1, \phi_2(0) = r_2, \phi_1 \in H^1(-1,0), \phi_2 \in H^1(-1,0) \right\}.$$

Also define

$$b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad c = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

In this example we have that $< b, c >= 0$, $b \notin D(A)$ and $c \notin D(A^*)$.

We will show that there does not exist a largest feedback-invariant subspace $Z \subset c^\perp$. Define the set of elements

$$e_k = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \exp(2\pi ikt) \end{bmatrix} \in D(A) \cap c^\perp.$$

The subspace span$\{e_k\}$ is (A, b)-invariant and hence feedback invariant. Define

$$V_n = \text{span}_{-n \leq k \leq n} e_k.$$

Each subspace V_n is feedback invariant. Define also the union of all finite linear combinations of e_k,

$$V = \bigcup V_n.$$

The space V is not closed. If there is a largest feedback-invariant subspace Z in $c^\perp$, then $Z \supset \bar{V}$. The important point now is that although $b \notin V$, $b \in \bar{V}$.

Since b cannot be contained in any feedback invariant subspace (Thm. 2.2), $\bar{V}$ is not feedback-invariant. Hence no largest feedback-invariant subspace exists.

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