On the convergence of a feedback control strategy for multilevel quantum systems

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Abstract

We present in this paper a class of feedback strategies that solve the steering problem for finite dimensional quantum systems. The control is designed to let a suitable distance between the state and the target decrease. Sufficient conditions are given to ensure convergence of this process.

1 Introduction

Recent theoretical and technical advances in the field of laser technology and microelectronic devices have further motivated the study of coherent control of quantum-mechanical systems [4, 9, 6]. In particular, the field of quantum computation [11] is very attracting from a control theoretic point of view. Actually, the information (encoded in the so-called qubits) is carried by the finite-dimensional complex state of the system and any logic operation is performed by steering the state of the system to a specific target.

The difficulty of this task lies mainly in three facts. First of all, the mathematical theory which is involved is not simple. Indeed, the model of the system, based on the well-known Schrödinger equation, is nonlinear in the control. Moreover, the physical state of the system does not live in an ordinary euclidean but in a projective complex space. States are equivalence classes: we will deal with vectors which represents the state but not in a canonical way, since they are not uniquely determined. More details about these facts are given in the following section.

Second, classical control theory cannot be employed in the usual way because of the 'collapse of the wave function' that occurs in the measurement process. Every measurable quantity of a quantum system, called *observable*, is associated with an operator whose eigenvalues are the possible outcomes of the measure. The state, that evolves continuously following the Schrödinger equation, contains only information on the a priori probability on the possible outcomes of every measurement. According to the orthodox interpretation of quantum mechanics, when the value of some measure is known, the state of the system becomes equal to the eigenvector corresponding to the measured eigenvalue (it 'collapses' on the eigenspace). Therefore the role of feedback, which is fundamental for control purposes, has to be considered with much more attention, since the observation process modifies the state of the system.

In the end, every quantum object tends to interact with its surroundings. This phenomenon is called 'entanglement' and is the basis of many quantum computing algorithms, when it affects the mutual interaction between different systems (each giving rise to one qubit). On the other hand, when the system interacts with the ambient, it loses the information it contains and becomes, in some sense, a classical object. This process, called decoherence, occurs after an amount of time that depends on the system. Many methods have been proposed to maintain coherence of a qubit [14] but this cannot be done during the transformation that performs a logic operation. Therefore the control of the system has to be done within a critical time. Note that the increase of energy, which would permit a quicker steering of the state, is dangerous too, since it could affect not only the state of the designed system, but also of other systems in its neighborhood, i.e. it could change the value of other qubits.

In this paper we present a strategy for state steering that is based on a feedback, in general not linear, from the state. The feedback is constructed in order to cause the distance between the state and the target to decrease. Even if the implementation of continuous feedback is not an impossible task (see, e.g. [7, 8, 12]), we will not be concerned with practical issues. Instead, the presented control strategy permits to compute a control that can be employed successively in open loop. Note that a discrete feedback scheme could be developed as is explained in [13]: intermediate suitable measures could be done to let the state collapse to the value predicted by the off-line simulation.

As regards the optimality of the proposed methodology, it has to be further investigated, since it highly depends on the exact form of the feedback, that is here only supposed to belong to a vast class of functions.

Before we enter the details about the control strategy and show sufficient conditions for its convergence, we need to recall some necessary notations and concepts from quantum mechanics and to give a more precise meaning to the notion of distance between states. This is done in the following two sections.

2 Finite dimensional quantum systems

Even if usually only one vector is taken to represent it, the state of a quantum system is associated with a ray of vectors, i.e. with a one dimensional vector subspace, of a complex Hilbert space $\mathcal{H}$. Therefore we will talk of (physical) states and of state vector.

We are going to use the Dirac's notation. A state vector is called 'ket' and written $|\psi\rangle \in \mathcal{H}$ while a vector in the dual space is called 'bra' and written $\langle\psi| \in \mathcal{H}^*$ (note that $\mathcal{H} \simeq \mathcal{H}^*$). With this notation, the Schrödinger equation, governing the evolution of the state vector $|\psi\rangle$ is

$$i\hbar|\dot{\psi}\rangle = H|\psi\rangle, \tag{2.1}$$

where $\hbar$ is the Planck constant and $H : \mathcal{H} \to \mathcal{H}$ is an Hamiltonian operator, i.e. an operator such that $H = H^*$. Therefore its eigenvalues are real valued. In the following, the system is scaled in such a way that $\hbar = 1$.

It is easy to see that if $|\psi(t)\rangle$ satisfies equation (2.1), also $|\tilde{\psi}(t)\rangle = e^{i\theta}|\psi(t)\rangle$ does. The Schrödinger equation is therefore invariant with respect to a change of phase. Nevertheless, when θ is a complex valued function of time, $|\tilde{\psi}(t)\rangle$, though representing the same state as $|\psi(t)\rangle$, is not anymore a solution of (2.1). To solve this problem two ways are possible. The most correct one is based on complex projective geometry [2, 3]: in this context it is also possible to give a *projective* formulation of equation (2.1). However, the formalism is rather difficult. We may therefore follow an easier approach: we choose a state vector $|\psi(t)\rangle$ with unitary norm, i.e. assume that $\langle\psi(t)|\psi(t)\rangle \equiv 1$, and require that all the results we derive are independent of the phase.

We are going to treat only the finite dimensional case, i.e. $\mathcal{H}$ is a finite dimensional complex Hilbert space, thus isomorphic to $\mathbb{C}^n$ for some n. We will identify operators with (hermitian) matrices and vectors in $\mathcal{H}$ with vectors in $\mathbb{C}^n$. Therefore $\langle\psi|$ is the conjugate transpose of $|\psi\rangle$ so that $\langle\psi_1|\psi_2\rangle$ is the standard inner product in $\mathbb{C}^n$, that gives rise to the euclidean norm $||\,|\psi\rangle|| = \sqrt{\langle\psi|\psi\rangle}$.

As we said, each operator H represents a quantity that can be measured, and every measured value λ has to be one of the eigenvalues of H. The role of the state is to give the probability of the outcomes. Without entering too much into the details, if the state collapses from $|\psi\rangle$ into $|\psi'\rangle$ after a measure , the *a priori* probability of this event is given by $|\langle\psi'|\psi\rangle|^2$.

This process causes the so-called collapse: after the measurement, the state $|\psi\rangle$ assumes the value of its projection onto the eigenspace associated with λ.

3 Distance between states

Our aim is to let the system reach a final state $|\psi_f\rangle$. To do this we construct a suitable state feedback, that can ensure the asymptotic stability of the closed loop control system by letting the distance between the actual and the final state decrease.

The distance between two states should be computed using formulas from complex projective geometry [3]. However, it can be proved [1] that the Fubini-Study distance d_{FS}, which is the distance measured along the geodesic connecting the states represented by $|\psi_1\rangle$ and $|\psi_2\rangle$, can be computed by the equation

$$\cos d_{\text{FS}}(|\psi_1\rangle, |\psi_2\rangle) = 2|\langle\psi_1|\psi_2\rangle|^2 - 1.$$

Note that the value $|\langle\psi_1|\psi_2\rangle|^2$ is the probability to collapse from state $|\psi_1\rangle$ into $|\psi_2\rangle$ after a measurement and therefore called transition probability.

There are other notions of distance that, as is shown in [15], depend in various ways on the transition probability. The most intuitive one, since it is the euclidean distance of two equivalence classes, is the Bures distance

$$d_{\mathrm{Bures}}(|\psi_1\rangle, |\psi_2\rangle) = \min_{\theta} ||\, |\psi_1\rangle - e^{i\theta}|\psi_2\rangle\, ||.$$

It is not difficult to check that $d^2_{\mathrm{Bures}}(|\psi_1\rangle, |\psi_2\rangle) = 2(1 - |\langle\psi_1|\psi_2\rangle|)$.

However, our choice is based on the so-called Hilbert-Schmidt (or, equivalently, on the trace) distance, since it permits easier calculations. In particular we are going to study the function

$$V(|\psi\rangle) = \frac{1}{2}d^2_{\mathrm{HS}}(|\psi\rangle, |\psi_f\rangle) = \frac{1}{4}d^2_{\mathrm{trace}}(|\psi\rangle, |\psi_f\rangle) = 1 - |\langle\psi_f|\psi\rangle|^2, \tag{3.2}$$

which will be used as a Lyapunov function to prove stability.

4 Model and feedback control strategy

The model of the system we are concerned with is a particular form of the rescaled Schrödinger equation:

$$i|\dot{\psi}(t)\rangle = H(t)|\psi(t)\rangle, \quad H(t) = H_0 + H_c(t), \quad H_c(t) = \sum_{l=1}^{r} H_l u_l(t), \tag{4.3}$$

where H_0 is the unperturbed Hamiltonian and $H_c(t)$ is the interaction Hamiltonian. The latter represents the effect of the control on the system. Each scalar control $u_l(t)$ acts on the system in the way specified by the hermitian matrix H_l.

The problem we wish to solve is then the following: given an initial state of the system $|\psi_0\rangle$, find controls in feedback form $u_l(|\psi\rangle)$ that move the state to a desired final state $|\psi_f\rangle$. Before we go on, we state two important assumptions about the final state $|\psi_f\rangle$.

H1 The final state is an eigenstate of the unperturbed system, i.e.

$$H_0|\psi_f\rangle = \lambda_0|\psi_f\rangle. \tag{4.4}$$

H2 The final state is not an eigenvalue of all the operators H_l, i.e.

$$\exists l \in \{1, \ldots, r\} \text{ such that } H_l|\psi_f\rangle \neq \lambda|\psi_f\rangle\ \forall\lambda \in \mathbb{R}.$$

Remark 4.1. *These are not technical hypotheses: condition (4.4) of **H1** ensures that once the final state is reached, no control is necessary to keep the system in that state. Indeed, if $|\psi_f\rangle$ is an eigenvector of H_0, the solution of equation (4.3) with no controls and starting from $|\psi_f\rangle$ is*

$$i|\dot{\psi}(t)\rangle = H(t)|\psi(t)\rangle = H_0|\psi(t)\rangle \ \Rightarrow\ |\psi(t)\rangle = e^{-iH_0t}|\psi_f\rangle = e^{-i\lambda_0 t}|\psi_f\rangle,$$

and therefore, even if the state vector $|\psi(t)\rangle$ is not stationary, the physical state is and coincides with the state $|\psi_f\rangle$.

*As for hypothesis **H2**, note that any notion of distance between two states always depends on the modulus of their inner product. If the final state were an eigenvector of H_l for any $l = 0, \dots, r$ with eigenvalue λ_l, then we would have*

$$i\langle\psi_f|\dot{\psi}(t)\rangle = \langle\psi_f|H(t)|\psi(t)\rangle = \left(\lambda_0 + \sum_{l=1}^{r} \lambda_l u_l(t)\right)\langle\psi_f|\psi(t)\rangle = u(t)\langle\psi_f|\psi(t)\rangle, \tag{4.5}$$

where we introduced the real-valued function $u(t)$ just to simplify the notation. Consider now that the derivative of $|\langle\psi_f|\psi(t)\rangle|^2$ is

$$\frac{d}{dt}|\langle\psi_f|\psi(t)\rangle|^2 = \langle\dot{\psi}(t)|\psi_f\rangle\langle\psi_f|\psi(t)\rangle + \langle\psi(t)|\psi_f\rangle\langle\psi_f|\dot{\psi}(t)\rangle = 2\,\mathrm{Re}[\langle\psi(t)|\psi_f\rangle\langle\psi_f|\dot{\psi}(t)\rangle], \tag{4.6}$$

and so, by replacing the result found in (4.5), we obtain that we would have

$$\frac{d}{dt}|\langle\psi_f|\psi(t)\rangle|^2 = 2\,\mathrm{Re}[-iu(t)|\langle\psi_f|\psi(t)\rangle|^2] = 0.$$

*In other words, if hypothesis **H2** were not satisfied, we could never bring $|\psi(t)\rangle$ near to $|\psi_f\rangle$.*

The control strategy that we propose is explained by the following statement.

Proposition 4.1. *Suppose that the initial state $|\psi_0\rangle$ is not orthogonal to the final state $|\psi_f\rangle$, i.e. that*

$$\langle\psi_f|\psi_0\rangle \neq 0. \tag{4.7}$$

Then with the state feedback control

$$u_l(|\psi\rangle) = g_l(\mathrm{Im}[\langle\psi|\psi_f\rangle\langle\psi_f|H_l|\psi\rangle]), \quad \forall l = 1, \dots, r, \tag{4.8}$$

where $g_l(x)$ are functions such that $g_l(0) = 0$ and $x \mapsto x g_l(x)$ is positive definite for $x \in \mathbb{R}$, the closed loop system is stable.

Proof. Consider the Lyapunov function V defined in (3.2). It is clear that it is positive definite, being zero only when $|\psi\rangle$ and $|\psi_f\rangle$ are the same physical state. Let us compute its derivative $\dot{V}(t)$ along the trajectory imposed by (4.3). We can employ equation (4.6) and, since $\mathrm{Re}\, ix = -\,\mathrm{Im}\, x$, obtain that

$$\begin{aligned}\dot{V}(t) &= -\frac{d}{dt}|\langle\psi_f|\psi(t)\rangle|^2 = 2\,\mathrm{Re}[\langle\psi(t)|\psi_f\rangle\langle\psi_f|iH(t)|\psi(t)\rangle] \\ &= -2\,\mathrm{Im}[\langle\psi(t)|\psi_f\rangle\langle\psi_f|H_0|\psi(t)\rangle] - 2\sum_{l=1}^{r} u_l(|\psi(t)\rangle)\,\mathrm{Im}[\langle\psi(t)|\psi_f\rangle\langle\psi_f|H_l|\psi(t)\rangle]\end{aligned}$$

where, by hypothesis (4.4), $\langle\psi(t)|\psi_f\rangle\langle\psi_f|H_0|\psi(t)\rangle = \lambda_0|\langle\psi_f|\psi(t)\rangle|^2 \in \mathbb{R}$ and therefore

$$= -2\sum_{l=1}^{r} g_l(\mathrm{Im}[\langle\psi(t)|\psi_f\rangle\langle\psi_f|H_l|\psi(t)\rangle])\,\mathrm{Im}[\langle\psi(t)|\psi_f\rangle\langle\psi_f|H_l|\psi(t)\rangle] \leq 0, \tag{4.9}$$

with the proposed feedback law (4.8). Therefore, being $\dot{V}(t)$ negative semidefinite, the system is stable. □

As we required before, the control strategy is independent of the phase of both $|\psi\rangle$ and $|\psi_f\rangle$. Actually, if for instance we alter the phase of $|\psi\rangle$ by Δ, then

$$\langle e^{i\Delta}\psi(t)|\psi_f\rangle\langle\psi_f|H_l|e^{i\Delta}\psi(t)\rangle = e^{-i\Delta}\langle\psi(t)|\psi_f\rangle e^{i\Delta}\langle\psi_f|H_l|\psi(t)\rangle = \langle\psi(t)|\psi_f\rangle\langle\psi_f|H_l|\psi(t)\rangle.$$

Moreover, note that when $|\psi\rangle$ and $|\psi_f\rangle$ represent the same physical state (the target has been reached), the feedback controls are $u_l(|\psi\rangle) = 0$ since in (4.8) the argument of Im is a real number.

Remark 4.2. *Note that if $\langle\psi_f|\psi(t_0)\rangle \neq 0$ and the feedback laws (4.8) are applied, then*

$$\langle\psi_f|\psi(t)\rangle \neq 0 \text{ for every } t > t_0. \tag{4.10}$$

Indeed, this is a consequence of the fact that, with V the Lyapunov function (3.2), by the proof of Proposition 4.1, $\frac{d}{dt}|\langle\psi_f|\psi(t)\rangle|^2 = -\dot{V}(t) \geq 0$.

The fact that condition (4.10) holds, permits to construct other feedback laws. For example, definitions

$$u_l(|\psi\rangle) = g_l(\mathrm{Im}[e^{-i\angle\langle\psi_f|\psi\rangle}\langle\psi_f|H_l|\psi\rangle]), \tag{4.11}$$

work as well, since $\angle\langle\psi_f|\psi\rangle$ (i.e. the phase of $\langle\psi_f|\psi\rangle$)is always defined. If an arbitrary value is assigned to $\angle 0$ (e.g. $\angle 0 = 0$), then it could be possible to start from a state orthogonal to the final one. Indeed, depending on the values of $\langle\psi_f|H_l|\psi_0\rangle$, one or more controls could be activated. (To be sure that at least one control is different from zero more conditions are needed, as in Theorem 4.1).

It is interesting to point out that the state feedbacks (4.11) can be obtained by direct computation, as in the proof of Proposition 4.1, by choosing a different Lyapunov function $V(|\psi\rangle)$ based on the Fubini-Study or on the Bures distances, defined in Section 3.

To overcome the problems given by an initial state such that $\langle\psi_f|\psi\rangle = 0$, one possibility would be to make a measure of a suitable quantity. This would change (and at the same time furnish) the state of the system. Then the feedback control could be used.

However, the proposed control laws may not provide a closed loop system which is convergent to $|\psi_f\rangle$ but one that may remain on some stationary orbit instead of reaching the desired final state.

A sufficient condition for this controllability question is given by the following theorem.

Theorem 4.1. *Consider the conditions:*

$$\begin{cases} \langle\psi_f|[H_0, H_l]|\psi\rangle = 0 \\ \langle\psi_f|\lambda_l I - H_l|\psi\rangle = 0 \text{ for some } \lambda_l \in \mathbb{R}, \end{cases} \tag{4.12}$$

where $[A, B] = AB - BA$ are the commutator brackets. Suppose that there is no state vector $|\psi\rangle$ that satisfies the above conditions $\forall l = 1, \ldots, r$ and

$$\langle\psi_f|\psi\rangle \neq 0, \quad |\langle\psi_f|\psi\rangle| \neq 1.$$

Then, under the hypotheses of Proposition 4.1, the feedback control strategy (4.8) is also convergent.

Proof. To prove convergence of $|\psi(t)\rangle$ to $|\psi_f\rangle$, i.e., asymptotic stability, we use the Krasovskii criterion with the same Lyapunov function V introduced in Proposition 4.1. In other words we aim to show that the set $\mathcal{V}$ of states such that $\dot{V} = 0$ does not contain trajectories of the system, with the exception of the trajectories corresponding to the final state $|\psi_f\rangle$.

First of all, note that by equation (4.9) and by the definition of functions g_l given in the statement of Proposition 4.1, for all the state vectors $|\psi\rangle \in \mathcal{V}$ the feedback is $u_l(|\psi\rangle) = 0$. So, if we suppose that there exists a trajectory of the system (4.3) entirely contained in $\mathcal{V}$, then it must satisfy the following Schrödinger equation

$$|\dot{\psi}(t)\rangle = -iH_0|\psi(t)\rangle. \tag{4.13}$$

Now we characterize $\mathcal{V}$. Again by equation (4.9), we know that $\dot{V} = 0$ if and only if the state $|\psi(t)\rangle$ satisfies the following system of equations in $|\psi\rangle$

$$\mathrm{Im}[\langle\psi|\psi_f\rangle\langle\psi_f|H_l|\psi\rangle] = 0, \quad \forall l = 1, \dots, r.$$

Notice that $\mathrm{Im}\, a^* b = 0$ if and only if there exist two real numbers η and ν, not both zero, such that $\eta a = \nu b$. Moreover, if $a \neq 0$, necessarily $\nu \neq 0$ and therefore we get the equivalent condition $\lambda a = b$, for some $\lambda \in \mathbb{R}$.

This is exactly the case, since $\langle\psi_f|\psi\rangle \neq 0$ for any value $|\psi\rangle$ of the state, by Remark 4.2. Therefore $|\psi\rangle$ belongs to $\mathcal{V}$ if and only if it satisfies the system of r equations $\lambda_l\langle\psi_f|\psi\rangle = \langle\psi_f|H_l|\psi\rangle = 0$, i.e.,

$$|\psi\rangle \in \mathcal{V} \;\Leftrightarrow\; \forall l = 1, \dots, r \;\exists \lambda_l \text{ such that } \langle\psi_f|\lambda_l I - H_l|\psi\rangle = 0.$$

We only need to analyze one single equation $F(\lambda, |\psi\rangle) = 0$, where

$$F : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C},\; (\lambda, |\psi\rangle) \mapsto \langle\psi_f|\lambda I - H_l|\psi\rangle,$$

and the index l is not determined for now, but will be fixed later. Before giving the details, we briefly explain this fact.

By considering also the equation $\dot{\lambda} = 0$, we extend equation (4.13) to define a dynamical system on the manyfold $\mathcal{M} = \mathbb{R} \times \mathbb{C}^n$. This dynamical system is associated with the vector field

$$X(\lambda, |\psi\rangle) = (0, -iH_0|\psi\rangle) \tag{4.14}$$

(see [10] for the notions of differential geometry that are used). The projection $\pi(\lambda, |\psi\rangle) = |\psi\rangle$ permits to go back to the original system. In particular, $\mathcal{V} = \pi(F^{-1}(0))$.

Suppose that we are able to show that, for some l, $F^{-1}(0)$ is a submanyfold of $\mathcal{M}$. Hence it admits a tangent space at the point $(\lambda, |\psi\rangle) \in F^{-1}(0)$. So, if it does not contain the tangent vector $X(\lambda, |\psi\rangle)$, it follows that the integral curve of X through p is not contained in $F^{-1}(0)$, i.e., that the trajectory of system (4.13) is not contained in $\mathcal{V}$.

For $F^{-1}(0)$ to be a submanifold of $\mathcal{M}$ it suffices to prove that the differential of F at $(\lambda, |\psi\rangle)$ is surjective for every $(\lambda, |\psi\rangle) \in F^{-1}(0)$. Indeed,

$$\begin{aligned} dF_{(\lambda,|\psi\rangle)}(\mu, |\phi\rangle) &= \frac{d}{ds}F(\lambda + s\mu, |\psi\rangle + s|\phi\rangle)\Big|_{s=0} = \frac{d}{ds}\langle\psi_f|(\lambda + s\mu)I - H_l|\psi + s\phi\rangle\Big|_{s=0} \\ &= \langle\psi_f|\lambda I - H_l|\phi\rangle + \mu\langle\psi_f|\psi\rangle. \end{aligned} \tag{4.15}$$

Let us fix the index l, which exists by hypothesis **H2**, such that $\lambda I - H|\psi_f\rangle \neq 0$ for every $\lambda \in \mathbb{R}$. This choice clearly permits to conclude that (4.15) is surjective.

The tangent space at $(\lambda, |\psi\rangle) \in F^{-1}(0)$ is equal to $\ker dF_{(\lambda,|\psi\rangle)}$. Hence, we have to check if $dF_{(\lambda,|\psi\rangle)}(X(\lambda, |\psi\rangle)) = 0$. This, by (4.15) and (4.14), is equivalent to

$$dF_{(\lambda,|\psi\rangle)}(X(\lambda, |\psi\rangle)) = dF_{(\lambda,|\psi\rangle)}(0, -iH_0|\psi\rangle) = -i\langle\psi_f|(\lambda I - H_l)H_0|\psi\rangle = 0. \tag{4.16}$$

Note, finally, that since $F(\lambda, |\psi\rangle) = 0$ we have

$$\langle\psi_f|\lambda H_0|\psi\rangle = \lambda_0\lambda\langle\psi_f|\psi\rangle = \lambda_0\langle\psi_f|H_l|\psi\rangle = \langle\psi_f|H_0H_l|\psi\rangle,$$

where we also used relation (4.4) of hypothesis **H1**. Condition (4.16) becomes then

$$\langle\psi_f|(\lambda I - H_l)H_0|\psi\rangle = \langle\psi_f|H_0H_l - H_lH_0|\psi\rangle = \langle\psi_f|[H_0, H_l]|\psi\rangle = 0.$$

The theorem is proved since, by hypothesis, this equation does not admit non-trivial solutions. □

5 The two-level spin system — an example

In this section we show that the standard model of a two-level spin system allows to perform a simple logic operation with one single feedback control.

A two-level spin system, e.g. the spin of an electron, is the simplest finite dimensional quantum system. The spin is a quantity that can be measured along every direction and the result is $\pm s$ for some constant s. In this paper, we let $s = 1$ for the sake of simplicity. Once we consider one direction, which traditionally is the z axis, the two eigenvectors associated with the eigenvalues -1 and 1 are simply denoted by $|0\rangle$ and $|1\rangle$. This shows that the spin system furnishes a 'quantum hardware' that can handle one bit of information, i.e., it is a qubit.

To be more detailed, consider the following set of matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which satisfy the relations $\sigma_j^2 = I$, $j = x, y, z$ and $\sigma_x\sigma_y\sigma_z = iI$. Their linear combinations, with real coefficients, generate the whole set of matrices with eigenvalues ± 1, i.e., of spin operators. In particular, it can be proved that, given any versor $a = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}$, the operator $\sigma_a = \sum_j a_j \sigma_j$ is the spin operator along the direction a. Note also that the set $\{I, \sigma_x, \sigma_y, \sigma_z\}$ generates every hermitian operator.

With this choice, the states of the qubit are

$$|0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The model usually employed defines $H_0 = \sigma_z$, thus ensuring that the states of the qubit $|0\rangle$ and $|1\rangle$, (which coincide with the eigenvectors of H_0) are stable with no control. Then one control acts by means of the matrix $H_1 = \sigma_y$. To perform a NOT, the simplest logic operation, we have to steer the state from $|\psi_0\rangle = |1\rangle$ to $|\psi_f\rangle = |0\rangle$ (the opposite transformation is symmetrical).

By Theorem 4.1, we have convergence. Indeed, note that

$$[H_0, H_1] = \sigma_z \sigma_y - \sigma_y \sigma_z = -2i\sigma_x.$$

Since σ_x is exactly the operator that performs the NOT operation, $\sigma_x|\psi_f\rangle = \sigma_x|0\rangle = |1\rangle$. Therefore the first of conditions (4.12) can be written as

$$\langle\psi_f|[H_0, H_1]|\psi\rangle = 0 \Leftrightarrow \langle 0|\sigma_x|\psi\rangle = 0 \Leftrightarrow \langle 1|\psi\rangle = 0 \Leftrightarrow |\psi\rangle = |0\rangle = |\psi_f\rangle,$$

and admits only the trivial solution.

It is not difficult to show that the simple open loop bang-bang control presented, for instance, in [5] can be achieved by the feedback control (4.8) where

$$g_1(x) = \operatorname{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0. \\ -1 & x < 0 \end{cases}$$

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