A Hilbert space approach to self-similar systems

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Abstract

This paper investigates the structural properties of linear self-similar systems, using an invariant subspace approach. The self-similar property is interpreted in terms of invariance of the corresponding transfer function space to a given transformation in a Hilbert space, in a same way as the time invariance property for linear systems is related to the shift-invariance of the Hardy spaces. The transformation in question is exactly that defining the de Branges homogeneous spaces. The explicit form of the corresponding impulse response, which is shown to be described by a hyperbolic partial differential equation, is given.

1 Introduction

Take a linear time-invariant (LTI) system f, with finite energy (*i.e.* $f \in L_2(\mathbb{R}, dt)$). Denote by g the image of f under the logarithmic distorsion of the time axis: $g(t) = f(\ln t)\ t > 0$. With this distorsion, the time-invariance property of f,

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |f(t+\tau)|^2 dt < \infty, \quad \tau \in \mathbb{R}, \tag{1.1}$$

reverberates into g as:

$$\int_{0}^{\infty} |g(t)|^2 d\ln t = \int_{0}^{\infty} |g(\lambda t)|^2 d\ln t < \infty, \tag{1.2}$$

where $\ln \lambda = \tau \in \mathbb{R}$. A linear system g will be termed scale invariant and finite energy (LSI) if (1.2) holds for all positive λ. Let now f be a LSI system and set $g(t) = t^{\nu/2} f(t)$ for some real number ν. Again, reverberating the scale invariance property of f into g yields:

$$\int_{0}^{\infty} t^{\nu} |g(t)|^2 d\ln t = \lambda^{\nu} \int_{0}^{\infty} t^{\nu} |g(\lambda t)|^2 d\ln t < \infty \tag{1.3}$$

which shows that the energy of g is proportional to the scale factor, under a scale transformation. This introduces the

Definition 1.1 (Self-similar). *A continuous time linear system g is said to be self-similar (LSS) with parameter ν, and finite energy if (1.3) holds for any $\lambda > 0$.*

Given a linear system $\mathcal{S}$ with input $u(t)$, we write $y(t) = \mathcal{S}\{u(t)\}$ for the corresponding output. From an input-output point of view, we shall say that a linear system $\mathcal{S}_\nu\{\cdot\}$ is self-similar with parameter ν if, for all $\alpha > 0$,

$$\mathcal{S}_\nu\{u(t)\} = y(t) \Longrightarrow \mathcal{S}_\nu\{u(\alpha t)\} = \alpha^{\nu/2} y(\alpha t). \tag{1.4}$$

with an arbitrary self-similar input $u(t)$ and output $y(t)$

These observations was made by Yazıcı and Kashyap in [1] and the authors proposed the following input-output model for LSS systems:

$$y(t) = t^{\nu/2} \int_0^\infty h\left(\frac{t}{\lambda}\right) u(\lambda) d\ln\lambda = t^{\nu/2} \int_0^\infty h(\lambda) u\left(\frac{t}{\lambda}\right) d\ln\lambda \quad t > 0. \tag{1.5}$$

In this expression, the function $h(\cdot)$ is defined in [1] as the *pseudo*-impulse response associated to the system.

Let $\tilde{y}(t)$ be defined by the integral appearing the above exppression, *i.e.* $\tilde{y}(t) = t^{-\nu/2} y(t)$.

2 Fractional Hardy space

We proceed to introduce a mathematical framework in which LSS systems act naturally.

Definition 2.1. *Let $\nu > -1$ be a real number. We define by $\mathcal{H}_2^\nu$ the set of functions $F(s)$, analytic in the right half plane and of the form*

$$F(s) = \int_0^\infty t^{\nu/2} e^{-st} f(t) dt, \tag{2.6}$$

where $f(t) \in L_2(0,\infty)$.

$\mathcal{H}_2^\nu$ is a *fractional* Hardy space which, for $\nu = 0$, reduces to the classical Hardy space $\mathcal{H}_2$ of the right half-plane. In that particular case, the integral transform (2.6) reduces to the Laplace transform. In fact, this *fractional* Hardy space is shown [2] (see also [3]) to have the following

Property 2.1. *The set $\mathcal{H}_2^\nu$ given in definition 2.1 is a reproducing kernel Hilbert space,*

1. *with reproducing kernel*
$$\mathcal{K}(\xi, s) = \Gamma(1+\nu)(\bar{\xi} + s)^{-1-\nu} \tag{2.7}$$

2. *and norm*
$$\|F(s)\|_\nu^2 = \int_0^\infty |f(t)|^2 dt \tag{2.8}$$

If $\tilde{y}(t)$ is in $L_2(0, +\infty)$ then $y(t)$ admits a Laplace transform,

$$Y(s) = \int_0^\infty t^{\nu/2} e^{-st} \tilde{y}(t) dt, \tag{2.9}$$

which then belongs to $\mathcal{H}_2^\nu$. Now a key feature of $\mathcal{H}_2^\nu$ is given by the

Property 2.2. *Let $\kappa > -1$ be a real number and set $\nu = 1 + 2\kappa$ which also satisfies obviously $\nu > -1$. If $F(s)$ is in $\mathcal{H}_2^\nu$ then, for any $\alpha > 0$, $\alpha^{1+\kappa}F(\alpha s)$ also belongs to $\mathcal{H}_2^\nu$ and this expression has the same norm as $F(s)$.*

By this last property, it appears that the mathematical framework of fractional Hardy spaces allows the concept of self similarity to be expressed, in the frequency domain, in terms similar than that in the time domain. In particular, we show in [2] that the class of ARFIMA systems finds expression, naturally, within this framework. Indeed, this appears clearly from the form of the reproducing kernel $\mathcal{K}(\xi, s)$ in (2.7).

An example of a linear self-similar model constructed from $\mathcal{H}_2^\nu$ is given in [2] along with a numeric computation.

3 de Branges homogeneous spaces

The interpretation of self-similarity in terms of the invariance of the corresponding "transfer function" space to the transformation $F(s) \to \alpha^{1+\kappa}F(\alpha s)$ appeals to further developement. Note that this invariance property is typically that which defines homogeneous de Branges spaces of entire functions [4]. We recall that a Hilbert space of entire functions is said to be homogeneous of order ν, with $\nu > -1$ if it is isometrically invariant to the transformation $F(s) \to \alpha^{1+\nu}F(\alpha s)$ for all $0 < \alpha \leq 1$. Therefore, and now on, we shall define

Definition 3.1. *A linear ν-self-similar system as a linear system whose transfer function belongs to a homogeneous space of order ν.*

The structure of homogeneous spaces admits a complete characterization due to de Branges [4, Theorem 50]. Let us consider the definitions

$$A(\lambda, z) = 2^\nu\,\Gamma(1+\nu)\,(\lambda z)^{-\nu}\mathcal{J}_\nu(\lambda z) \tag{3.10a}$$

$$B(\lambda, z) = 2^\nu\,\Gamma(1+\nu)\,\lambda^{2\nu+1}\,(\lambda z)^{-\nu}\mathcal{J}_{\nu+1}(\lambda z) \tag{3.10b}$$

where $\mathcal{J}_\eta(\cdot)$ is the Bessel function of the first kind, of order η. We also introduce the function $\mathcal{E}(\lambda, z)$ defined by

$$\mathcal{E}(\lambda, z) = \lambda^{\nu+\frac{1}{2}}A(\lambda, z) - i\lambda^{-\nu-\frac{1}{2}}B(\lambda, z). \tag{3.11}$$

A consequence of the above mentioned de Branges's theorem is the following one (see [2]) which mimicks verbatim the Paley-Wiener theorem on entire functions of exponential type and square integrable on the real line.

Theorem 3.1. *If $\varphi(x) \in L_2(-\infty, \infty)$ and vanishes outside the interval $[-\lambda, \lambda]$ then the eigentransform $F(\lambda, z)$ defined by the integral*

$$F(\lambda, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{E}(x, z)\varphi(x)dx \tag{3.12}$$

is, for each fixed λ, an entire function which belongs to the de Branges's homogeneous space of order ν. Moreover, if $\nu \geq -\frac{1}{2}$ then, $F(\lambda, z)$ is, for each fixed λ, an entire function of exponential type at most λ and square integrable on the real line.

Conversely, any element $G(z)$ of a homogeneous space of order ν, is equal to an $F(\lambda, z)$ as given in (3.12) for some λ and some function φ in $L_2(-\infty, \infty)$ which vanishes outside the interval $[-\lambda, \lambda]$.

This theorem shows that any function $F(\lambda, z)$ obtained as in (3.12), with $\nu \geq -\frac{1}{2}$, belongs to the Paley-Wiener space of entire function of exponential type (at most λ) and square summable on the real axis. Therefore any such function can also be written as the Fourier-Laplace transform of an element of L_2.

4 A class of linear self-similar systems

Based on Theorem 3.1 one may see each function of any de Branges's homogeneous space of order $\nu \geq -\frac{1}{2}$ as the transfer function of some linear self-similar system. Hence, the usual concept of transfer function does apply to these systems, within the framework of our definition 3.1.

In the following, we exhibit the structure of any linear self-similar system, directly in the signal space.

To begin, note that one may readily check from (3.11) and the definitions (3.10a), (3.10b), that $\mathcal{E}(\lambda, z)$ satisfies the self-similar property

$$\mathcal{E}(\lambda, \alpha z) = \alpha^{-\nu-\frac{1}{2}} \mathcal{E}(\alpha\lambda, z), \text{ for all } \alpha > 0 \tag{4.13}$$

and, as a function of λ, it also satisfies the differential equation

$$\frac{d}{d\lambda}\mathcal{E}(\lambda, z) = \frac{\nu + \frac{1}{2}}{\lambda}\, \mathcal{E}^*(\lambda, z) - iz\mathcal{E}(\lambda, z), \tag{4.14}$$

where

$$\mathcal{E}^*(\lambda, z) \triangleq \overline{\mathcal{E}(\lambda, \bar{z})} = \mathcal{E}(-\lambda, z). \tag{4.15}$$

This last relation is easily obtained upon noting that $A(\lambda, z)$ and $B(\lambda, z)$ are real for z real. For fixed λ, let $\psi(\lambda, t)$ be the inverse Fourier-Laplace transform of $\mathcal{E}(\lambda, z)$ as in

$$\mathcal{E}(\lambda, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izt} \psi(\lambda, t) dt. \tag{4.16}$$

Then, the self-similar property of $\mathcal{E}(\lambda, z)$ (4.13) reverberates on $\psi(\lambda, t)$ as:

$$\psi(\lambda, t) = \alpha^{-\nu+\frac{1}{2}}\, \psi(\alpha\lambda, \alpha t), \text{ for all } \alpha > 0. \tag{4.17}$$

Now, using Mikusiǹski's operational calculus [5], it follows that the ordinary differential equation (4.14) is transformed in the time domain into the partial differential equation

$$\frac{\partial\psi(\lambda, t)}{\partial\lambda} = \frac{\nu + \frac{1}{2}}{\lambda} \psi(-\lambda, t) - \frac{\partial\psi(\lambda, t)}{\partial t}, \quad a.e. \tag{4.18}$$

with the initial condition

$$\psi(\lambda, 0) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\nu+1)}{\Gamma(\nu+\frac{1}{2})}\lambda^{\nu-\frac{1}{2}} - i\lambda^{\nu-\frac{1}{2}} \tag{4.19}$$

and where $\psi(-\lambda, t)$ verifies, by virtue of (4.15), the relations

$$\psi(-\lambda, t) = \overline{\psi(\lambda, t)} \tag{4.20a}$$

$$\psi(\lambda, -t) = \psi(\lambda, t). \tag{4.20b}$$

A solution of this equation is obtained in [2] as:

$$\psi(\lambda, t) = \begin{cases} \frac{\sqrt{\pi}\,\Gamma(\nu+1)}{\Gamma(\nu+\frac{1}{2})}\dfrac{\lambda^{\nu+\frac{1}{2}}}{\lambda - t} - i\lambda^{-\nu-\frac{1}{2}}\,(\lambda - t)^{2\nu}, & \text{for } |t| < \lambda \\ 0 & \text{for } |t| \geq \lambda \end{cases} \tag{4.21}$$

We can now summarize the preceding developments by the

Theorem 4.1. *Let $\nu \geq -\frac{1}{2}$ be a real number and let $F(\lambda, z)$, where λ is some fixed positive real number, be an element of de Branges's homogeneous space of order ν. Then $F(\lambda, z)$ is expressible in the form*

$$F(\lambda, z) = \frac{1}{2\pi}\int_{-\lambda}^{\lambda}\varphi(x)\int_{-\infty}^{\infty}\psi(x, t)e^{-itz}dt\,dx, \tag{4.22}$$

for some $\varphi \in L_2$, with support in $[-\lambda, \lambda]$, and where $\psi(x, t)$ is the function defined by (4.21).

Finally, we have:

Corollary 4.1. *A linear self-similar system with parameter $\nu \geq -\frac{1}{2}$ has, at the scale $\lambda > 0$, an impulse response $h(\lambda, t)$ of the form*

$$\begin{aligned} h(\lambda, t) = & \frac{\sqrt{\pi}\,\Gamma(\nu+1)}{\Gamma(\nu+\frac{1}{2})}\int_t^{\lambda}\frac{x^{\nu+\frac{1}{2}}[\varphi(x)+\varphi(-x)]}{x-t}\,dx \\ & - i\int_t^{\lambda} x^{-\nu-\frac{1}{2}}[\varphi(x)-\varphi(-x)](x-t)^{2\nu}\,dx \end{aligned} \tag{4.23}$$

At each fixed scale λ, the transfer function of the system belongs to the de Branges homogeneous space of order ν.

The input-output relation of such a system, with input $u(t)$ and ouput $y(\lambda, t)$ at time t and scale λ, is therefore given by

$$y(\lambda, t) = h(\lambda, t) \star u(t) = \int_{-\lambda}^{\lambda}\varphi(x)\psi(x, t) \star u(t)dx \tag{4.24}$$

Note however that we have from the relation (4.20b),

$$h(\lambda, -t) = h(\lambda, t),$$

which shows that the impulse response $h(\lambda, t)$ does not vanish for $t < 0$. The corresponding system is therefore not causal. Nonetheless, its self-similar behaviour appears through the invariance

$$h_{\varphi(\cdot)}(\lambda, \alpha t) = \alpha^{\nu}\, h_{\sqrt{\alpha}\,\varphi(\alpha\cdot)}(\tfrac{\lambda}{\alpha}, t) \text{ for all } \alpha > 0, \tag{4.25}$$

where the dependance of $h(\lambda, \cdot)$ on $\varphi \in L_2([-\lambda, \lambda])$ has been make explicit by the notation $h_{\varphi}(\lambda, \cdot)$ and where the factor $\sqrt{\alpha}$ in $h_{\sqrt{\alpha}\varphi(\alpha\cdot)}(\frac{\lambda}{\alpha}, \cdot)$ is to keep unchanged the norm of the associated function.

5 Conclusion

We interpret the self-similar property in the following terms: *A linear self-similar system with parameter $\nu \geq -\frac{1}{2}$ is a linear system whose transfer function belongs to a de Branges's homogeneous space of order ν.* This interpretation has allowed us to introduce a linear time-invariant model which presents a self-similar behaviour when observed through different scales. The model thus depends on two parameters: a time parameter and a scale parameter.

References

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