Coprimeness Conditions For Pseudorational Transfer Functions

Yutaka Yamamoto
Department of Applied Analysis and Complex Dynamical Systems
Graduate School of Informatics
Kyoto University
Kyoto 606-8501
JAPAN
yy@i.kyoto-u.ac.jp

Abstract

Coprimeness conditions play important roles in various aspects of system/control theory: realization, controllability, stabilization, just to name a few. While the issue is now well understood for finite-dimensional systems, it is far from being settled for infinite-dimensional systems. This is due to a wide variety of situations in which this issue occurs, and several variants of coprimeness notions, which are equivalent in the finite-dimensional context, turn out to be non-equivalent. This paper studies the notions of spectral, approximate and exact coprimeness for pseudorational transfer functions. A condition is given under which these notions coincide.

1 Introduction

Coprimeness conditions play important roles in various aspects of system/control theory: realization, controllability, stabilization, just to name a few.

The class of pseudorational transfer functions (impulse responses) [7, 8, 9] is known to be quite effective in dealing with delay systems, and some infinite-dimensional servomechanism control, known as repetitive control; for details, see [10]. The crux of this framework is that they allow a natural class of fractional representations as the ratio of distributions with compact support [7]. In the Laplace domain, they are also the ratio of entire functions of exponential type—the simplest extension of polynomials.

It is not suprising that the coprimeness of such a fractional representation is intimately related to the notion of reachability/controllability. Reflecting the infinite-dimensional nature, however, there arise several distinct notions of reachability, and accordingly those of coprimeness. The interrelationships among them are thus of interest from both system theoretic and control points of view. We mainly consider the three notions: spectral coprimeness, approximate coprimeness and exact coprimeness (Bezout condition). In particular, we give a partial answer to the problem of finding a condition under which the exact coprimeness holds, posed as an open problem in [11].

2 Pseudorational Impulse Responses and Their Realizations

We start by defining the notions of time-invariant linear systems, pseudorational impulse resposes, and their realizations. We confine ourselves, without losing much generality, to the single-input single-output case. Generalization to the multivariable case can be suitably obtained by introducing left factorizations.

Let $\mathcal{E}'(\mathbb{R}_-)$ denote the space of distributions having compact support contained in the negative half line $(-\infty, 0]$. Distributions such as Dirac's delta δ_a placed at $a \leq 0$, its derivative δ'_a are examples of elements in $\mathcal{E}'(\mathbb{R}_-)$. An impulse response function W ($\operatorname{supp} W \subset [0, \infty)$) is said to be pseudorational if it satisfies the following two conditions:

1. $W = q^{-1} * p$ for some $q, p \in \mathcal{E}'(\mathbb{R}_-)$, where the inverse is taken with respect to convolution;

2. $\operatorname{ord} q^{-1} = -\operatorname{ord} q$, where $\operatorname{ord} q$ denotes the order of a distribution q [4].

Let $\Omega := \underrightarrow{\lim} L^2[-n, 0]$ denote the inductive limit of the spaces $\{L^2[-n, 0]\}_{n>0}$; it is the union of all these spaces endowed with the finest topology that makes all injections $j_n : L^2[-n, 0] \to \Omega$ continuous; see, e.g., [6]. Dually, $\Gamma := L^2_{loc}[0, \infty)$ is the space of all locally Lebesgue square integrable functions with obvious family of seminorms:

$$\|\phi\|_n := \left\{ \int_0^n |\phi(t)|^2 dt \right\}^{1/2}, \quad n = 1, 2, \ldots .$$

This is the projective limit of spaces $\{L^2[0, n]\}_{n>0}$. Ω is the space of past inputs, and Γ is the space of future outputs, with the understanding that the present time is 0. These spaces are equipped with the following natural left shift semigroups:

$$(\sigma_t \omega)(s) \;:=\; \begin{cases} \omega(s+t), & s \leq -t, \\ 0, & -t < s \leq 0, \quad \omega \in \Omega, t \geq 0, s \leq 0. \end{cases} \tag{2.1}$$

$$(\sigma_t \gamma)(s) := \gamma(s+t), \quad \gamma \in \Gamma, t \geq 0, s \geq 0. \tag{2.2}$$

An input/output or a Hankel operator associated with an impulse response function W is defined to be the continuous linear mapping $\mathcal{H}_W : \Omega \to \Gamma$ defined by

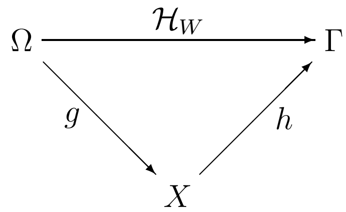
$$\mathcal{H}_W(\omega)(t) := \int_{-\infty}^{0} W(t-\tau)\omega(\tau) d\tau.$$

Let us now introduce the notion of a (linear, time-invariant) system.

Definition 2.1. A (linear, time-invariant) system Σ is a quadruple (X, Φ, g, h) such that

- X is a Banach space, and $\Phi(t)$ is a strongly continuous semigroup defined on it;

- $g : \Omega \to X$ is a continuous linear mapping such that $g\sigma_t = \Phi(t)g$ for all $t \geq 0$;
- $h : X \to \Gamma$ is also a continuous linear map satisfying $h\Phi(t) = \sigma_t h$ for all $t \geq 0$.



The mappings g and h are called reachability map and observability map, respectively. Σ is said to be approximately reachable if g has dense image, and observable if h is one to one. It is topologically observable if h gives a topological homomorphism (i.e., continuously invertible when its codomain is restricted to $\operatorname{im} h$). Σ is weakly canonical if it is approximately reachable and observable; it is canonical if it is further topologically observable. Σ is said to be a realization of an impulse response W if $\mathcal{H}_W = hg$.

The definition above looks a little abstract and appears to have little information needed to analyze linear systems. However, when there are certain "smoothness hypotheses" satisfied, then it is immediate to write down a differential equation description in the following form [7]:

$$\begin{aligned} \frac{dx}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

where A is the infinitesimal generator of $\Phi(t)$, and

$$\begin{aligned} g(\omega)(t) &= \int_{-\infty}^{0} \exp(-At)B\omega(t)dt \\ h(x)(t) &= C\exp(At)x. \end{aligned}$$

These properties justify the terms reachability and observability maps.

For a pseudorational impulse response $W = q^{-1} * p$, one can always associate with it a topologically observable realization $\Sigma^{q,p}$ as follows [7]:

Define X^q as follows:

$$X^q := \{x \in \Gamma \mid \pi_+(q * x) = 0\}$$

where π_+ is the truncation to $(0, \infty)$. It is easy to check X^q is a σ_t-invariant closed subspace of Γ. To define $\Sigma^{q,p}$, take this X^q as the state space with σ_t (restricted to X^q) as its semigroup. Then define $g : \Omega \to X^q$ and $h : X^q \to \Gamma$ as follows.

$$\begin{aligned} g(\omega) &:= \pi_+(q^{-1} * p * \omega) \\ h(x) &= x \text{ (injection)}. \end{aligned}$$

Since h is clearly a topological homomorphism, $\Sigma^{q,p}$ is topologically observable.

3 Coprimenss Conditions

We now introduce various coprimeness conditions.

Definition 3.1. Let $W = q^{-1} * p$ be pseudorational. The pair (q, p) is spectrally coprime if their Laplace transforms $\hat{q}, \hat{p}$ have no common zeros. It is called approximately coprime if there exists a sequence $x_n, y_n \in \mathcal{E}'(\mathbb{R}_-)$ such that

$$q * x_n + p * y_n \to 0$$

in $\mathcal{E}'(\mathbb{R}_-)$. It is exactly coprime or said to satisfy the Bezout condition if there exist $x, y \in \mathcal{E}'(\mathbb{R}_-)$ such that

$$q * x + p * y = \delta.$$

Clearly exact coprimeness is stronger than approximate coprimeness, and the latter is yet stronger than spectral coprimeness. That each implication cannot be reversed can be proven rather easily.

The significance of these notions may be obvious from the following:

Facts 3.1. ([7, 8])

1. $\Sigma^{q,p}$ is approximately reachable if and only if (q, p) is approximately coprime. In this case $\Sigma^{q,p}$ gives the canonical realization of $W = q^{-1} * p$.

2. The spectrum of the infinitesimal generator A^q of system $\Sigma^{q,p}$ is given by

$$\sigma(A^q) = \{\lambda \mid \hat{q}(\lambda) = 0\}. \tag{3.3}$$

   Furthermore, every point in $\sigma(A^q)$ is an eigenvalue with finite multiplicity. The resolvent set $\rho(A^q)$ is its complement.

3. For each $\lambda \in \sigma(A^q)$, the generalized eigenfunctions are of the form $\{e^{\lambda t}, te^{\lambda t}, \ldots, t^{n-1}e^{\lambda t}\}$, where n is the geometric multiplicity.

4. The state space X^q is eigenfunction complete if and only if $r(q) = \sup\{t : t \in \operatorname{supp} q\} = 0$.

5. The pair (q, p) is exactly coprime if every element of X^q can be made reachable if we extend the inputs to $\mathcal{E}'(\mathbb{R}_-)$.

The last property easily yields the following:

Proposition 3.1. Approximate coprimeness does not imply exact coprimeness.

Proof Take $q := \delta_{-1}$ and take p to be a C^∞ function with compact support contained in $[-1, 0]$, which is also not identically zero in a neighborhood of the origin. Since $\hat{q} = e^s$ does not admit any zero, the pair is spectrally coprime. It is futher approximately coprime due to the main theorem of [8]. On the other hand, the pair cannot be exactly reachable because $q^{-1} * p$ is a C^∞ function and hence reachable elements in $\Sigma^{q,p}$ are also C^∞ functions, but, on the other hand $X^q = L^2[0, 1]$. □

This example, however, gives the impression that the converse may hold for a more restricted class. One such case would be delay-differential operators.

Although we cannot answer this question in a full generality, we give the following result for a special case.

Theorem 3.1. Let $W = q^{-1} * p$ be pseudorational. Suppose further that p is a polynomial in δ'. Then the pair (q, p) is exactly coprime if and only if it is spectrally coprime.

Note that there is no restriction on q. The proof makes use of a spectral mapping theorem. Suitably introducing the operator action of $p(A^q)$, it can be seen that the pair is exactly coprime if and only if $p(A^q)$ is boundedly invertible. By proving a spectral mapping theorem, this is characterized by $0 \notin p(\sigma(A^q))$.

4 Spectral Mapping Theorem

To prove our main result Theorem 3.1, we need some spectral results. We start by defining some canonical projections.

Let $W = q^{-1} * p$ be pseudorational, and X^q be as above. Let π_+ and π_- be defined by

$$\pi_+\psi := \psi|_{(0,\infty)}, \tag{4.4}$$

$$\pi_-\psi := \psi - \pi_+\psi = (I - \pi_+)\psi. \tag{4.5}$$

These are well defined at least for continuous ψ.

We then define the canonical projection $\pi^q : \Gamma \to X^q$ as follows:

Definition 4.1. Take any $x \in \Gamma \cap C^\infty$. Define

$$\pi^q x := \pi_+(q^{-1} * \pi_-(q * x)). \tag{4.6}$$

Since $x \in C^\infty$, it is easy to see that $\pi^q x$ belongs to Γ. We further have the following lemma:

Lemma 4.1. Let x be a C^∞ function. Then

$$\pi_+(q * \pi_+x) = \pi_+(q * x) \tag{4.7}$$

and

$$\pi_+(q * \pi^q x) = 0. \tag{4.8}$$

That is, $\pi^q x$ belongs to X^q.

Proof Write

$$x = \phi + \pi_+ x, \quad \pi_+ \phi = 0.$$

Then

$$q * \pi_+ x = q * x - q * \phi.$$

Since $\pi_+(q * \phi) = 0$, (4.7) readily follows.

It now follows that

$$\pi_+(q * \pi^q x) = \pi_+(q * \pi_+(q^{-1} * \pi_-(q * x))) = \pi_+(q * (q^{-1} * \pi_-(q * x))) = \pi_+(\pi_-(q * x)) = 0.$$

□

It is not difficult to see the continuity of π^q with respect to the (locally) L^2 topology. Hence this mapping can be extended to the whole space Γ, and gives a canonical projection from Γ onto X^q. We also denote this extended mapping by π^q.

Via this projection we now introduce the following action on X^q:

Definition 4.2. Let $p \in \mathcal{E}'(\mathbb{R}_-)$. Define $p(A^q)$ by

$$p(A^q)x := \pi^q(p * x), \tag{4.9}$$

where $x \in X^q$ is sufficiently smooth so that $p * x$ is a locally L^2 function.

Let us first note the following:

Lemma 4.2. Let p be a polynomial in δ':

$$p = a_n \delta^{(n)} + \cdots + a_0 \delta,$$

i.e., its Laplace transform being

$$\hat{p}(s) = a_n s^n + \cdots + a_0.$$

Then $p(A^q) = a_n (A^q)^n + \cdots + a_0 I$.

Proof Consider the case $p = \delta'$. Take any $x \in X^q$ such that $A^q x = dx/dt \in X^q$. By definition, $q * x = \phi \in \mathcal{E}'(\mathbb{R}_-)$. By (4.9),

$$\begin{aligned} p(A^q)x &= \pi^q(\delta' * x) = \pi_+(q^{-1} * \pi_-(q * \delta' * x)) \\ &= \pi_+(q^{-1} * \pi_-(\delta' * \phi)) = \pi_+(q^{-1} * \delta' * \phi) \\ &= dx/dt = A^q x. \end{aligned}$$

The general case is entirely analogous. □

We now have the following spectral mapping theorem:

Theorem 4.1. Let p be a polynomial in δ'. Then $\sigma(p(A^q)) = \hat{p}(\sigma(A^q))$, and hence

$$\sigma(p(A^q)) = \{\hat{p}(\lambda) : \hat{q}(\lambda) = 0\}. \tag{4.10}$$

Proof We check this directly for operators of form $(A^q)^n$. For simplicity of notation, assume $n = 2$. The general case is entirely similar.

It is easy to see that $\sigma(A^q)^2 \subset \sigma((A^q)^2)$. Conversely, take any $\lambda \neq \sigma(A^q)^2$. We show such a λ belongs to the resolvent set of $(A^q)^2$. Let $\lambda = \mu^2$. Then by definition, $\mu, -\mu \notin \sigma(A^q)$, so that $\hat{q}(\mu), \hat{q}(-\mu) \neq 0$. Consider the equation

$$((A^q)^2 - \lambda)y = \left(\frac{d^2}{dt^2} - \lambda\right) y = x.$$

This should be solvable for every $x \in X^q$, and the solution y should depend continuously on x. Taking the Laplace transform of both sides, we obtain $(s^2 - \mu^2)\hat{y} = \hat{x}$, which readily yields

$$\hat{y}(s) = \frac{1}{s^2 - \mu^2}\hat{x} + \frac{y_0}{s - \mu} + \frac{y_1}{s + \mu}$$

for some constants y_0, y_1. Then

$$\hat{q}(s)\hat{y}(s) = \frac{\hat{q}(s)}{s^2 - \mu^2}\left\{\hat{x}(s) + (s + \mu)y_0 + (s - \mu)y_1\right\}.$$

Since $y \in X^q$ if and only if $q * y \in \mathcal{E}'(\mathbb{R}_-)$ which in turn is equivalent to $\hat{q}(s)\hat{y}(s)$ being an entire function of exponential type satisfying the Paley-Wiener estimate [7]. This is valid if and only if $(s^2 - \mu^2)|(\hat{x}(s) + (s + \mu)y_0 + (s - \mu)y_1)$, i.e., $(s - \mu)|(\hat{x}(s) + (s + \mu)y_0)$ and $(s + \mu)|(\hat{x}(s) + (s - \mu)y_1)$. This holds if and only if

$$\begin{aligned} \hat{x}(\mu) + 2\mu y_0 &= 0 \\ \hat{x}(\mu) - 2\mu y_1 &= 0. \end{aligned}$$

This is clearly possible so the resolvent equation $((A^q)^2 - \lambda)y = x$ is solvable. Since y_0 and y_1 depend on $\hat{x}(\mu)$ continuously, this correspondence is continuous. The case for $(A^q)^n$ is entirely similar, with n free parameters to determine $\hat{y}$. It is then easy to extend the result to their linear combinations. □

5 Proof of Main Theorem

We start with the following proposition.

Proposition 5.1. Let $W = q^{-1} * p$ be pseudorational. The pair (q, p) is exactly coprime if and only if $p(A^q)$ is invertible in X^q.

Proof Suppose there exist $a, b \in \mathcal{E}'(\mathbb{R}_-)$ such that

$$q * a + p * b = \delta.$$

Then by substituing A^q into these, we obtain

$$q(A^q)a(A^q) + p(A^q)b(A^q) = I.$$

Now note that $q(A^q)x = \pi^q(q * x) = 0$, because $\mathrm{supp}(q * x) \subset (-\infty, 0]$. This implies $p(A^q)b(A^q) = I$, i.e., $p(A^q)$ is invertible.

Conversely, suppose $p(A^q)$ is an invertible operator. Consider the commutative diagram:

$$\begin{array}{ccc} X^q & \xrightarrow{\;p(A^q)^{-1}\;} & X^q \\ \uparrow j & & \uparrow j \\ X^q \cap C^\infty[0,\infty) & \xrightarrow[p(A^q)^{-1}]{} & X^q \cap C^\infty[0,\infty) \end{array}$$

where j is the inclusion. Since $p(A^q)^{-1}$ is shift-invariant, it leaves elements in $C^\infty[0, \infty)$ invariant. Thus we obtain the above commutative diagram. We also see that it is continuous with respect to the C^∞ topology of $X^q \cap C^\infty[0, \infty)$. To this end, suppose $x_n \to x$ and $p(A^q)^{-1}x_n \to y$ in $X^q \cap C^\infty[0, \infty)$. But this also implies that they converge with respect to the topology of X^q (i.e., L^2 topology). Then by the continuity of $p(A^q)^{-1}$ in X^q, $y = p(A^q)^{-1}x$. But this equality holds also in $X^q \cap C^\infty[0, \infty)$. This means that $p(A^q)^{-1}$ restricted to $X^q \cap C^\infty[0, \infty)$ has closed graph, and hence it is continuous.

We now note that $(X^q \cap C^\infty[0, \infty))' = \mathcal{E}'(\mathbb{R}_-)/(q * \mathcal{E}'(\mathbb{R}_-))$. This is because $C^\infty[0, \infty)' = \mathcal{E}'(\mathbb{R}_-)$ with respect to the duality

$$\langle \phi, \psi \rangle := \langle \phi, \check{\psi} \rangle, \quad \phi \in \mathcal{E}'(\mathbb{R}_-), \psi \in C^\infty[0, \infty)$$

where $\check{\psi}(t) := \psi(-t)$. Then the polar of the $(X^q \cap C^\infty[0, \infty))$ is easily seen to be equal to $q * \mathcal{E}'(\mathbb{R}_-)$, and hence $(X^q \cap C^\infty[0, \infty))' = \mathcal{E}'(\mathbb{R}_-)/(q * \mathcal{E}'(\mathbb{R}_-))$ by [3]. This yields the continuity of

$$\mathcal{E}'(\mathbb{R}_-)/(q * \mathcal{E}'(\mathbb{R}_-)) \xrightarrow{p*} \mathcal{E}'(\mathbb{R}_-)/(q * \mathcal{E}'(\mathbb{R}_-)),$$

and also its invertibility. Then it follows that $[p*b] = [\delta]$ in $\mathcal{E}'(\mathbb{R}_-)/(q*\mathcal{E}'(\mathbb{R}_-))$, i.e., $p*b \equiv \delta \mod q$, i.e., $p * b - \delta = q * a$ for some $a \in \mathcal{E}'(\mathbb{R}_-)$. This implies $q * (-a) + p * b = \delta$. That is, (p, q) is exactly coprime. □

Remark 5.1. For a corresponding (and more detailed) treatment in the finite-dimensional context, see [1].

We are now ready to prove our main theorem 3.1.

Proof of Theorem 3.1 Suppose (p, q) are spectrally coprime. This means $\hat{p}(\lambda) \neq 0$ for every λ such that $\hat{q}(\lambda) = 0$. In other words, for every $\lambda \in \sigma(A^q)$, $\hat{p}(\lambda) \neq 0$. Since $\sigma(p(A^q)) = \hat{p}(\sigma(A^q))$ by Theorem 4.1, this means $0 \neq \sigma(p(A^q))$, i.e., $p(A^q)$ is invertible. By Proposition 5.1, (p, q) is exactly coprime. □

6 Further Generalizations

6.1 Relationship with a Dunford-type Integral

The action (4.9) is given in terms of the canonical projection operator π^q. On the other hand, it is also possible to define such actions using the so-called Dunford integral.

We will show that such a formula can be obtained for $\Sigma^{q,p}$ that is eigenfunction complete, i.e., property 4 of Facts 3.1 is satisfied.

The following theorem holds:

Theorem 6.1. Let $\Sigma^{q,p}$ be as above, and suppose that

1. $r(q) = 0$,
2. there exists $c \in \mathbb{R}$ such that $\sigma(A^q) \subset \{\lambda : \mathrm{Re}\,\lambda < c\}$, and
3. the resolvent $(\lambda I - A^q)^{-1}$ decays with the order of $1/\lambda$ as $\lambda \to \infty$.

Then, for a polynomial p and $x \in X^q$ that is sufficiently smooth,

$$p(A^q)x = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} p(\lambda) \cdot (\lambda I - A^q)^{-1} x d\lambda, \tag{6.11}$$

where $p(\lambda) \cdot x$ is understood to be the one obtained by extracting the analytic part when expanded in powers of λ, i.e., its strictly proper part [1].

Proof Let γ be the path that goes from $c - j\omega$ up to $c + j\omega$ and along a large semicircle on the left-hand side of the complex plane to return $c - j\omega$. As $\omega \to \infty$, this path encircles all points in the spectrum $\sigma(A^q)$. Also, the path integral

$$\int p(\lambda) \cdot (\lambda I - A^q)^{-1} x d\lambda$$

along the large semicircle approaches zero as $\omega \to \infty$, since the integrand decays sufficiently fast due to the regularity of x. To prove (6.11) it is thus enough to show that $p(A^q)x$ equals the path integral along this closed semicircle. To this end, we first take a vector that belongs to the generalized eigenspace corresponding to an eigenvalue μ. For simplicity, assume $x = e^{\mu t}$. Then

$$(\lambda I - A^q)^{-1} x = e^{\lambda t} \hat{x}(\lambda) - e^{\lambda t} * x = e^{\lambda t} \frac{1}{\lambda - \mu} - e^{\lambda t} * e^{\mu t} = \frac{e^{\mu t}}{\lambda - \mu}.$$

It follows that

$$\frac{\lambda e^{\mu t}}{\lambda - \mu} = e^{\mu t} + \frac{\mu e^{\mu t}}{\lambda - \mu},$$

[1] This is possible since every element of $(\lambda I - A^q)^{-1} x$ is meromorphic in λ [7].

so that

$$\lambda \cdot (\lambda I - A^q)^{-1} x = \frac{\mu e^{\mu t}}{\lambda - \mu}.$$

Since this function is analytic except at $\lambda = \mu$, the above integral easily reduces to

$$\frac{1}{2\pi j} \oint_{|\lambda - \mu| = \epsilon} \frac{\mu e^{\mu t}}{\lambda - \mu} d\lambda = \mu e^{\mu t} = A^q(e^{\mu t})$$

by Cauchy's formula. The same argument easily carries over to the higher-order generalized eigenvector, and in turn, to their linear combinations with different eigenvalues.

Since the set of all such elements is dense in X^q by the eigenfunction completeness, (6.11) holds by continuity. □

It is possible to extend this formula for functions analytic and bounded in such a domain, e.g., $e^{t\lambda}$, but the spectral properties of thus defined operators are more delicate to be discussed here.

6.2 Actions via More General Class of Functions

Theorem 3.1 can be proven if the spectral mapping property carries over. An immediate conjecture one may be led to is that it also holds for p which is a polynomial in two variables δ' and δ_{-T}, i.e., differentiation and delays. However, this case presents a considerable difficulty. In fact, it is known in general that the spectral mapping theorem does not hold in this case [2].

However, a slightly different generalization is still possible.

Let f be a complex-valued function such that

1) its domain $D(f)$ is an open set in $\mathbb{C}$ that contains $\sigma(A^q)$;

2) the complement of $D(f)$ is compact;

3) f is differentiable in $D(f)$ and is bounded as $|\lambda| \to \infty$.

Then it is known that the spectral mapping theorem holds for such f, i.e., $\sigma(f(A^q)) = f(\sigma(A^q))$. Hence if $\hat{p}(s)$ satisfies this condition, then spectral coprimeness still implies exact coprimeness. Note however that the interesting case $p(s) = e^{-Ts}$ does not satisfy this condition since exponentials are not bounded at infinity.

Conclusion

We have given a condition under which spectrally coprimeness implies exact coprimeness for the class of pseudorational transfer functions. This is based upon the functional calculus, and in this connection, a relationship with the Dunford integral is investigated. Generalizations to a more general context will require a more elaborate theory in this direction.

Acknowledgment The author wishes to thank Paul Fuhrmann for very fruitful discussions.

References

[1] P. A. Fuhrmann, A Polynomila Approach To Linear Algebra, Springer-Verlag, New York, 1996.

[2] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.

[3] H. H. Schaefer, Topological Vector Spaces, Springer-Verlag, New York, 1971.

[4] L. Schwartz, Théorie des Distributions, IIme Edition, Hermann, Paris, 1966.

[5] A. E. Taylor, Introduction to Functional Analysis, John-Wiley & Sons, New York, 1958.

[6] F. Treves, Topological Vector Spaces, Distributions and Kernels, Academic Press, New York, 1967.

[7] Y. Yamamoto, "Pseudo-rational input/output maps and their realizations: a fractional representation approach to infinite-dimensional systems," SIAM J. Control & Optimiz., 26: 1415-1430, 1988.

[8] Y. Yamamoto, "Reachability of a class of infinite-dimensional linear systems: an external approach with applications to general neutral systems," SIAM J. Control & Optimiz., 27: 217-234, 1989.

[9] Y. Yamamoto, "Equivalence of internal and external stability for a class of distributed systems," MCSS, 4: 391-409, 1991.

[10] Y. Yamamoto, "Learning control and related problems in infinite-dimensional systems," Essays on Control: Perspectives in the Theory and its Applications, Ed. by H. L. Trentelman and J. C. Willems, pp. 191–222, Birkhäuser, 1993.

[11] Y. Yamamoto, "Coprimeness of factorizations over a ring of distributions," in Open Problems in Mathmatical Systems and Control Theory, pp. 281–284, V. D. Blondel et al. Eds., Springer, 1998.