Stabilizability of systems with signals in $\ell_2(\mathbb{Z})$

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Abstract

In this paper a system is considered as a (possibly unbounded) linear operator from $\ell_2(\mathbb{Z})$ to $\ell_2(\mathbb{Z})$. Georgiou and Smith [3] concluded that there are intrinsic difficulties in using $\ell_2(\mathbb{Z})$ as underlying signal space, since even a simple causal convolution system is not stabilizable is the usual sense of the term. We discuss this problem, and we "solve" the problem by adapting the definition of stabilizability. Finally, we compare the obtained controller with the one we obtain by restricting our system to $\ell_2(\mathbb{N}_0)$.

1 Introduction

We consider an operator theoretical approach towards discrete-time systems over the signal space $\ell_2(\mathbb{Z})$, that is, a linear, discrete-time system is considered as a (possibly unbounded) operator from $\ell_2(\mathbb{Z})$ to $\ell_2(\mathbb{Z})$. This is an approach towards linear systems often used in modern control theory, see for example [1], [12]. Georgiou and Smith [3] studied the simple example

$$(Pu)(t) = \sum_{n=-\infty}^{t} 2^{t-n} u(n), \quad u \in D(P), \tag{1.1}$$

where the domain of P, denoted by $D(P)$, consists of all sequences $u \in \ell_2(\mathbb{Z})$ such that $Pu \in \ell_2(\mathbb{Z})$. They were led to the counter-intuitive conclusion that the system could not be stabilized in the usual sense of the term, since its graph is not closed; moreover, taking the closure of its graph leads one to an extended definition of the system which is non-causal, and this is physically unreasonable. This contrasts with the situation described in their earlier work [2], where use of the signal space $L_2(\mathbb{R}_+)$ gives no such difficulties. We call the phenomena that a causal system can have a non-causal closure the Georgiou-Smith paradox.

Jacob and Partington [6, 7] studied the problem of closability of graphs of linear systems with one-dimensional inputs and outputs and its connections with causality. The graph of a closed system is described by an element of $L_\infty(\mathbb{T})^2$, called the *symbol* of P, and notions such as causality and causal closability are characterized by means of equivalent conditions in terms of the symbol.

In this paper, we characterize stable feedback systems in terms of the symbol of the involved systems. In contrast to systems over the signal space $\ell_2(\mathbb{N}_0)$ a stable feedback system is not necessarily causal, and so stable and causal feedback systems are characterized as well. While working with stable and causal feedback systems the algebra $H_\infty(\mathbb{D})$ becomes important. For example it is shown that a feedback system can only be stable and causal if the systems defining the feedback system possess symbols which are left-invertible over $H_\infty(\mathbb{D})$. Another problem which we address is stabilizability. Since even some simple LTI-system of practical relevance are closable, but not closed, we do not use the standard definition of (feedback) stabilizability. Instead we say a LTI-system is stabilizable if an extended definition of the system, the closure, is stabilizable in the usual sense of the term by a causal controller and the feedback system is causal as well. Finally, the Georgiou-Smith paradox is solved, and we show that system (1.1) is stabilizable. Note, that the results of this paper can also be found in the author's habilitation thesis [5].

We proceed as follows. In Section 2 we introduce some notion and we review LTI-systems. The main results of this paper are given Section 3 and 4, where we study the notion of stable and causal feedback systems (Section 3), and stabilizability (Section 4). Finally, in Section 5 we solve the Georgiou-Smith paradox and in Section 6 we compare the obtained controllers with the one of the on $\ell_2(\mathbb{N}_0)$ restricted system.

2 Preliminaries and review of LTI-systems

We introduce the following notation. We define $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$, and $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$. $H_\infty(\mathbb{D})$ denotes the Hardy space of bounded holomorphic function f on $\mathbb{D}$, and by $H_2(\mathbb{D})$ we denote the Hardy space of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ satisfying

$$\sup_{r\in(0,1)} \left(\int_0^{2\pi} |f(re^{it})|^2 \, dt \right)^{1/2} < \infty.$$

We consider $\ell_2(\mathbb{N}_0)$ as a subset of $\ell_2(\mathbb{Z})$ by extending $x \in \ell_2(\mathbb{N}_0)$ to $\ell_2(\mathbb{Z})$ by defining the sequence to be zero outside $\mathbb{N}_0$. Moreover, $x \in \ell_2(\mathbb{Z})$ is an element of $\ell_2(\mathbb{N}_0)$ if $x(j) = 0$ for $j < 0$. By S we denote the *right shift* on $\ell_2(\mathbb{Z})$, on which is given by

$$(Sx)(j) := x(j-1), \quad j \in \mathbb{Z}.$$

By e_k, $k \in \mathbb{Z}$, we denote the kth unit vector of $\ell_2(\mathbb{Z})$, namely, $e_k(j) := \delta_{k,j}$. By $\hat{\cdot}$ we denote the *z-transform* which is given by

$$\hat{u}(z) := \sum_{j\in\mathbb{Z}} u(j) z^j, \qquad u \in \ell_2(\mathbb{Z}).$$

The z-transform is an isometric isomorphism from $\ell_2(\mathbb{N}_0)$ onto $H_2(\mathbb{D})$, and from $\ell_2(\mathbb{Z})$ onto $L_2(\mathbb{T})$. Following [4] we define a linear time-invariant system as follows.

Definition 2.1. *An operator $P : D(P) \subset \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$, is called a* linear time-invariant system *with input space $\ell_2(\mathbb{Z})$ and output space $\ell_2(\mathbb{Z})$ (or* LTI-system *for short) if P is* linear, *i.e. $G(P)$ is a linear subspace of $\ell_2(\mathbb{Z})$, if P is* shift-invariant, *i.e. $SG(P) = G(P)$, and if P is densely defined, i.e. $\overline{D(P)} = \ell_2(\mathbb{Z})$.*

Here $D(P)$ denotes the *domain* of P and $G(P)$ the *graph* of P, that is,

$$G(P) := \left\{ \begin{pmatrix} x \\ Px \end{pmatrix} \mid x \in D(P) \right\}.$$

Definition 2.2. *We say a LTI-system P is* closed*, if the operator P is closed, that is, if $G(P)$ is a closed subspace of $\ell_2(\mathbb{Z})$, and we say a LTI-system P is* closable*, if the operator P is closable, i.e. if for every sequence $\{u_n\} \subseteq D(P)$ which tends to 0 and for which Pu_n tends to a function $y \in \ell_2(\mathbb{Z})$ we have $y = 0$.*

Equivalently, a LTI-system P is closable if and only if $\overline{G(P)}$ is the graph of an operator. Closability means that there exists a closed LTI-system $T : D(T) \subset \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ such that $D(P) \subset D(T)$ and $Tu = Pu$ for every $u \in D(P)$. If P is closable, then the closure $\overline{P}$ of P is the smallest closed LTI-system, which extends P.

Definition 2.3. *Let P be a LTI-system. Then P is called* causal*, if $u \in \ell_2(\mathbb{N}_0) \cap D(P)$ implies $Pu \in \ell_2(\mathbb{N}_0)$.*

We stated the definition of a LTI-system in the time-domain. Using the z-transform and the fact that the z-transform is an isometric isomorphism, we can interchangingly use the description of P in the frequency domain. The *frequency domain description* of a LTI-system P is given by

$$\hat{P} : D(\hat{P}) \subset L_2(\mathbb{T}) \to L_2(\mathbb{T}),$$
$$\hat{P}\hat{u} := \widehat{Pu}, \quad u \in D(P).$$

$D(\hat{P})$ is the z-transform of $D(P)$ and $G(\hat{P})$ is the z-transform of $G(P)$. Since P is a LTI-system, we have that $\hat{P}$ is linear, that $G(\hat{P})$ is a linear subspace of $L_2(\mathbb{T})$, and that $\hat{P}$ is shift-invariant, that is, $SG(\hat{P}) = G(\hat{P})$. Further, P is causal if and only if $x \in H_2(\mathbb{D})$ implies $\hat{P}x \in H_2(\mathbb{D})$, and P is closable if and only if the operator $\hat{P}$ is closable. In the following we use interchangingly the time-domain and the frequency domain description of a LTI-system. Next we consider an example showing the Georgiou-Smith paradox, that is, a causal LTI-system can have a non-causal closure. The continuous-time version of this example can be found in Georgiou and Smith [2].

Example 2.1. We consider the operator $P : D(P) \subseteq \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ given by

$$\begin{aligned} (Pu)(t) &:= \sum_{n=-\infty}^{t} 2^{t-n} u(n), \qquad u \in D(P), \qquad (2.2) \\ D(P) &:= \{u \in \ell_2(\mathbb{Z}) \mid y(t) := \sum_{n=-\infty}^{t} 2^{t-n} u(n) \text{ converges} \forall\ t \in \mathbb{Z}, \text{ and } y \in \ell_2(\mathbb{Z})\}. \end{aligned}$$

In Jacob [4] it has been proved that P is a causal, closable LTI-system, and that $\overline{P}$ is a non-causal, bounded linear operator from $\ell_2(\mathbb{Z})$ to $\ell_2(\mathbb{Z})$, given by

$$(\overline{P}u)(t) = -\sum_{n=t+1}^{\infty} u(n) 2^{t-n}, \quad t \in \mathbb{Z}, u = \sum_{n=-\infty}^{\infty} u(n) e_n.$$

Further, we have that $\hat{\overline{P}}$ is a bounded, linear operator from $L_2(\mathbb{T})$ to $L_2(\mathbb{T})$, having the form

$$(\hat{\overline{P}}u)(z) = \frac{1}{1-2z}u(z), \quad u \in L_2(\mathbb{T}), z \in \mathbb{T}.$$

Next we introduce the notion of a symbol of a LTI-system.

Definition 2.4. *Let P be a LTI-system. We call a matrix $G \in L_\infty(\mathbb{T})^2$ a* symbol *of P if $G(\hat{P}) = GL_2(\mathbb{T})$ and $G^*G = I$.*

In [4] it is shown that every closed LTI-system possesses a symbol and that the symbol is unique up to an unitary element of $L_\infty(\mathbb{T})$.

3 Stable feedback systems

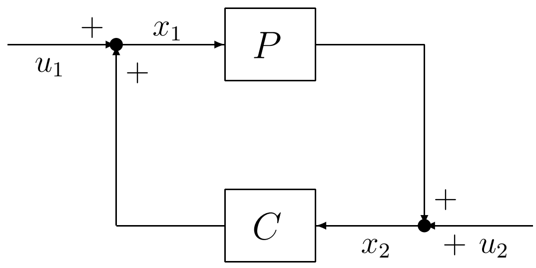

Figure 1: Standard feedback configuration

We consider feedback systems as given in Figure 1, which is the standard feedback configuration used in system and control theory, see Vidyasagar [13] for more details. We say that such a feedback system is stable if all the paths in the loop in Figure 1 are stable, or more precisely

Definition 3.1. *Let P and C be LTI-systems. We say that the feedback system $[P,C]$, as given in Figure 1, is* stable*, if*

$$F_{[P,C]} := \begin{pmatrix} I & C \\ P & I \end{pmatrix} : D(P) \times D(C) \to \ell_2(\mathbb{Z})^2 : \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} \to \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}$$

has a bounded inverse, that is, the operators $u_i \to x_j$, $i,j = 1,2$, are well defined and bounded. If $[P,C]$ is stable, then we denote the inverse of $F_{[P,C]}$ by $H_{[P,C]}$.

In Figure 1 we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 & C \\ P & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

If the feedback system is stable, it is easy to see that the operators $I - PC : D(C) \to \ell_2(\mathbb{Z})$ and $I - CP : D(P) \to \ell_2(\mathbb{Z})$ are boundedly invertible, and that the inverse of $F_{[P,C]}$ is given by

$$\begin{aligned} H_{[P,C]}\begin{pmatrix} u_1 \\ -u_2 \end{pmatrix} &= \begin{pmatrix} (I-CP)^{-1} & -(I-CP)^{-1}C \\ -(I-PC)^{-1}P & (I-PC)^{-1} \end{pmatrix}\begin{pmatrix} u_1 \\ -u_2 \end{pmatrix} \\ &= \begin{pmatrix} (I-CP)^{-1} & -(I-CP)^{-1}C \\ -P(I-CP)^{-1} & I+P(I-CP)^{-1}C \end{pmatrix}\begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}. \end{aligned} \tag{3.3}$$

We have the following necessary conditions for stability of feedback systems. The proof follows Georgiou and Smith [2].

Proposition 3.1. *Let P and C be LTI-systems. If $[P,C]$ is stable, then P and C are closed.*

Proof. Let $\{v_n\}_n \subset D(P)$ be a sequence which converges to v in $\ell_2(\mathbb{Z})$ and $\{Pv_n\}_n$ converges to y in $\ell_2(\mathbb{Z})$. Taking the limit n to ∞ in $\binom{v_n}{0} = H_{[P,C]}\binom{v_n}{Pv_n}$ we get $\binom{v}{0} = H_{[P,C]}\binom{v}{y}$, which implies $v \in D(P)$ and $y = Pv$. Thus P is closed. Similarly, it can be proved that C is closed. $\square$

Next we characterize stable feedback systems $[P,C]$ by means of equivalent conditions. The proof follows Georgiou and Smith [2], see also Vidyasagar [13], and the proof is based on this known result.

Theorem 3.1. *Let P and C be closed LTI-systems with symbols $\binom{m}{n}$ and $\binom{s}{t}$. Then $[P,C]$ is stable if and only if $\begin{pmatrix} m & t \\ n & s \end{pmatrix}$ is invertible over $L_\infty(\mathbb{T})$.*

Proof. The *inverse graph* of a LTI-system P is defined by $G^I(P) := \binom{P}{I}D(P)$. Stability of $[P,C]$ implies $G(P) \cap G^I(C) = \{0\}$ and $G(P) + G^I(C) = \ell_2(\mathbb{Z})^2$. Thus we get

$$L_2(\mathbb{T})^2 = G(\hat{P}) + G^I(\hat{C}) = \begin{pmatrix} m \\ n \end{pmatrix} L_2(\mathbb{T}) + \begin{pmatrix} t \\ s \end{pmatrix} L_2(\mathbb{T}),$$

and thus Corollary 3.10 of [4] shows that the matrix $\begin{pmatrix} m & t \\ n & s \end{pmatrix}$ is invertible over $L_\infty(\mathbb{T})$. On the other hand, the invertibility of $\begin{pmatrix} m & t \\ n & s \end{pmatrix}$ over $L_\infty(\mathbb{T})$ implies that $G(P) \cap G^I(C) = \{0\}$ and $G(P) + G^I(C) = \ell_2(\mathbb{Z})^2$. This shows that $F_{[P,C]}$ is injective and surjective. Thus the inverse of $F_{[P,C]}$ exists, and it only remains to show that the inverse is bounded. Using the Closed Graph Theorem this holds if $F_{[P,C]}$ has a closed graph. The graph of $F_{[P,C]}$ is given by

$$\left\{ \begin{pmatrix} x_1 \\ -x_2 \\ x_1 - Cx_2 \\ Px_1 - x_2 \end{pmatrix} \mid x_1 \in D(P), x_2 \in D(C) \right\},$$

and it is closed since P and C are closed. $\square$

Beside stability another important property of a feedback system is causality, which is defined as follows.

Definition 3.2. *Let P and C be LTI-systems, such that the feedback system $[P, C]$ is stable. We say the feedback system $[P, C]$ is* causal *if $H_{[P,C]}$, as given in (3.3), is causal, that is, $u \in \ell_2(\mathbb{N}_0)^2$ implies $H_{[P,C]}u \in \ell_2(\mathbb{N}_0)^2$.*

Considering systems over $\ell_2(\mathbb{N}_0)$, see Georgiou and Smith [2], we see that every stable feedback system is automatically causal. Unfortunately, this is not the case for systems over $\ell_2(\mathbb{Z})$, as the following example shows. Note that in the example both systems P and C are causal, whereas the feedback system is not causal.

Example 3.1. We consider a feedback system $[P, C]$, which is given by

$$\begin{aligned} Pu &:= u, \quad u \in \ell_2(\mathbb{Z}), \\ (Cu)(t) &:= u(t) - u(t-1), \quad u \in \ell_2(\mathbb{Z}), t \in \mathbb{Z}. \end{aligned}$$

Clearly, P and C are stable, causal LTI-systems, and $G(\hat{P}) = \binom{1}{1} L_2(\mathbb{T})$ and $G(\hat{C}) = \binom{1}{1-z} L_2(\mathbb{T})$. Since $D := 1 \cdot 1 - (1-z) \cdot 1 = z$ is invertible over $L_\infty(\mathbb{T})$, Theorem 3.1 shows that the feedback system $[P, C]$ is stable. In order to show that the feedback system is not causal, we choose the inputs $u_1 := (\dots, 0, 1, 0, \dots)$, where the 1 stands at position 0, and $u_2 := 0$. This choice implies $x_1 = (\dots, 0, 1, 0, \dots)$, where the 1 stands at position -1, and thus the feedback system is not causal.

Next we give a necessary condition for a feedback system to be causal.

Proposition 3.2. *Let P and C be LTI-systems, such that the feedback system $[P, C]$ is stable and causal. Then there exist symbols $G_P, G_C \in H_\infty(\mathbb{D})^2$ of P and C, respectively, such that G_P and G_C are left-invertible over $H_\infty(\mathbb{D})$.*

Proof. By $\binom{m}{n} \in L_\infty(\mathbb{T})^2$ we denote a symbol of P, and by $\binom{s}{t} \in L_\infty(\mathbb{T})^2$ a symbol of C. We define $u_1 \in H_\infty(\mathbb{D})$ by $u_1 = 0$, and $u_2 \in H_\infty(\mathbb{D})$ by $u_2 = 1$. Thus u_2 is invertible over $H_\infty(\mathbb{D})$. Since the feedback system $[P, C]$ is stable and causal, there exist unique elements $x_1 \in H_2(\mathbb{D})$ and $x_2 \in H_2(\mathbb{D})$ such that

$$\begin{aligned} x_1 &= \hat{C}x_2, \\ x_2 &= u_2 + \hat{P}x_1. \end{aligned}$$

We have $\hat{C}x = ts^{-1}x$ and $\hat{P}u = nm^{-1}u$ for $x \in D(\hat{C})$ and $u \in D(\hat{P})$. Thus we get $x_1 = m(sm - tn)tu_2$ and $x_2 = m(sm - tn)su_2$ which implies $x_1, x_2 \in H_\infty(\mathbb{D})$. Clearly $x_2 \neq 0$. Thus $ts^{-1} \in R(H_\infty(\mathbb{D}))$. In a similar manner it can be proved that $nm^{-1} \in R(H_\infty(\mathbb{D}))$. We now write $ts^{-1} = \tilde{t}\tilde{s}^{-1}$ and $nm^{-1} = \tilde{n}\tilde{m}^{-1}$ with $\tilde{n}, \tilde{m}, \tilde{s}, \tilde{t} \in H_\infty(\mathbb{D})$, $\gcd_{H_\infty(\mathbb{D})}(\tilde{n}, \tilde{m}) = 1$ and $\gcd_{H_\infty(\mathbb{D})}(\tilde{s}, \tilde{t}) = 1$. The stability and causality of the feedback system shows that the functions

$$\tilde{m}(\tilde{s}\tilde{m} - \tilde{t}\tilde{n})^{-1}\tilde{s}, \tilde{m}(\tilde{s}\tilde{m} - \tilde{t}\tilde{n})^{-1}\tilde{t}, \tilde{n}(\tilde{s}\tilde{m} - \tilde{t}\tilde{n})^{-1}\tilde{s}, \text{ and } \tilde{n}(\tilde{s}\tilde{m} - \tilde{t}\tilde{n})^{-1}\tilde{t}$$

are holomorphic and bounded on $\mathbb{D}$. By Lemma 4 of Smith [10] we get that

$$(\tilde{s}\tilde{m} - \tilde{t}\tilde{n})^{-1}\tilde{s}, \quad \text{ and } (\tilde{s}\tilde{m} - \tilde{t}\tilde{n})^{-1}\tilde{t}$$

are holomorphic and bounded on $\mathbb{D}$. Using again Lemma 4 of Smith [10] we see that $(\tilde{s}\tilde{m} - \tilde{t}\tilde{n})^{-1} \in H_\infty(\mathbb{D})$. This shows that $\binom{\tilde{m}}{\tilde{n}}$ and $\binom{\tilde{s}}{\tilde{t}}$ are left-invertible over $H_\infty(\mathbb{D})$. Clearly, $G(\hat{P}) = \binom{\tilde{m}}{\tilde{n}}L_2(\mathbb{T})$ and $G(\hat{C}) = \binom{\tilde{s}}{\tilde{t}}L_2(\mathbb{T})$. However, $\binom{\tilde{m}}{\tilde{n}}$ and $\binom{\tilde{s}}{\tilde{t}}$ are in general not inner.

We define $U \subset H_2(\mathbb{D})^2$ by $U := \binom{m}{n}H_2(\mathbb{D})$. Following the proof of Proposition 5.5 of [4] we get that U is a closed subset of $H_2(\mathbb{D})^2$. Moreover, U is shift-invariant in $H_2(\mathbb{D})^2$. Thus by the Beurling-Lax theorem, see Lax [8], there is a number $r \in \{1, 2\}$ and an inner function $G \in H_\infty(\mathbb{D})^{2\times r}$ such that $U = GH_2(\mathbb{D})^r$. Thus we get

$$GH_2(\mathbb{D})^r = \begin{pmatrix} m \\ n \end{pmatrix} H_2(\mathbb{D}). \tag{3.5}$$

Since $\binom{m}{n}$ is left-invertible over $H_\infty(\mathbb{D})$, we get that $r = m$. Let X be a left-inverse of $\binom{m}{n}$. Then (3.5) shows $XGH_2(\mathbb{D}) = H_2(\mathbb{D})$, and hence XG is invertible over $H_\infty(\mathbb{D})$. Thus G is also left-invertible over $H_\infty(\mathbb{D})$. Further, equation (3.5) shows that there are functions $Q_1, Q_2 \in H_2(\mathbb{D})$ such that

$$G = \begin{pmatrix} m \\ n \end{pmatrix} Q_1, \quad \begin{pmatrix} m \\ n \end{pmatrix} = GQ_2.$$

Now the left-invertibility of G and $\binom{m}{n}$ implies that $Q_1, Q_2 \in H_\infty(\mathbb{D})$, and that Q_1, Q_2 are invertible over $H_\infty(\mathbb{D})$ with $Q_1^{-1} = Q_2$. We now split G as $G = \binom{m_1}{n_1}$ with $m_1 \in H_\infty(\mathbb{D})$ and $n_1 \in H_\infty(\mathbb{D})$. Since $m \neq 0$ we get $m_1 \neq 0$, and so $\binom{m_1}{n_1}$ is a symbol of P which has the required properties. $\square$

Remark 3.1. Note that if G is a symbol of a LTI-system P satisfying $G \in H_\infty(\mathbb{D})^2$, then G needs not to be left-invertible over $H_\infty(\mathbb{D})$. For example, $\frac{1}{\sqrt{2}}\binom{z}{z}$ is a symbol of $P := I$, but not left-invertible over $H_\infty(\mathbb{D})$.

Next we formulate equivalent conditions for a stable feedback system to be causal. The result is based on standard results using the coprime factorization approach (see Vidyasagar [13]), see also Georgiou and Smith [2].

Theorem 3.2. *Let P and C be LTI-systems with symbols $\binom{m}{n} \in L_\infty(\mathbb{T})^2$ and $\binom{s}{t} \in L_\infty(\mathbb{T})^2$ such that the feedback system $[P, C]$ is stable. Then $[P, C]$ is causal if and only if $\begin{pmatrix} m & t \\ n & s \end{pmatrix}$ is invertible over $H_\infty(\mathbb{D})$.*

Proof. If $\begin{pmatrix} m & t \\ n & s \end{pmatrix}$ is invertible over $H_\infty(\mathbb{D})$ then it is easy to see that $[P, C]$ is causal. The converse direction follows directly from the proof of Proposition 3.2. $\square$

4 Stabilizability

The system C is called a *controller* of P. Note that a controller always is a closed LTI-system. There are different possibilities to define the notion of stabilizability for a LTI-system P. The simplest one would be to require that there exists a LTI-system C such that the feedback system $[P, C]$ is stable. This is the usual definition used in system and control theory. However, in our situation this definition is not suitable, since it would rule out a huge class of important systems from being stabilizable and this definition would not guarantee that the feedback system is causal. Thus we adapt the defintion as follows.

Definition 4.1. *A LTI-system P is called* stabilizable *if P is closable and if there exists a causal LTI-system C such that the feedback system $[\overline{P}, C]$ is stable and causal.*

We have the following equivalent conditions for stabilizability.

Theorem 4.1. *A closable LTI-system P is stabilizable if and only if the system $\overline{P}$ possesses a symbol $G \in H_\infty(\mathbb{D})^2$ which is left-invertible over $H_\infty(\mathbb{D})$.*

Proof. If P is stabilizable then Proposition 3.2 shows that $\overline{P}$ possesses a symbol $G \in H_\infty(\mathbb{D})^2$ which is left-invertible over $H_\infty(\mathbb{D})$.

Let us now assume that $\overline{P}$ possesses a symbol $\binom{m}{n} \in H_\infty(\mathbb{D})^2$ which is left-invertible over $H_\infty(\mathbb{D})$. Using Tolokonnikov's Lemma (see for example Nikolski [9, page 293]) there exist matrices $s \in H_\infty(\mathbb{D})$ and $t \in H_\infty(\mathbb{D})$ such that $X = \begin{pmatrix} m & t \\ n & s \end{pmatrix}$ is invertible over $H_\infty(\mathbb{D})$. Since $s \neq 0$, we can define C via the graph $G(\hat{C}) = \binom{s}{t} L_2(\mathbb{T})$. The stability of $[P, C]$ is then implied by Theorem 3.1, since X is invertible over $H_\infty(\mathbb{D})$, and hence over $L_\infty(\mathbb{T})$, and the causality of $[P, C]$ is implied by Theorem 3.2. However, it can happen that C is not causal. Clearly, for every $Q \in H_\infty(\mathbb{D})^2$, the matrix $X_Q := \begin{pmatrix} m & t+mQ \\ n & s+nQ \end{pmatrix} = \begin{pmatrix} m & t \\ n & s \end{pmatrix} \begin{pmatrix} I & Q \\ 0 & I \end{pmatrix}$ is invertible over $H_\infty(\mathbb{D})$, and so it is enough to show that there is a matrix $Q \in H_\infty(\mathbb{D})^{m \times p}$ such that $s + nQ$ is invertible over $H_\infty(\mathbb{D})$. In this case C is causal as well. Since $(n \;\; s)$ is right-invertible, we have $\inf_{i \in \mathbb{D}}(\|n(z)\| + \|s(z)\|) > 0$. Now Treil [11] shows the existence of a function $Q \in H_\infty(\mathbb{D})$ such that $s + nQ$ is invertible over $H_\infty(\mathbb{D})$, and thus the theorem is proved. □

5 Discussion of the Georgiou-Smith paradox

We now study again the convolution system introduced in Example 2.1. In Example 2.1 the LTI-system P was given by (2.2). We showed already that P is causal, closable, but not closed, and the closure of P is not causal.

Georgiou and Smith [3] studied exactly this example in the continuous time situation. It is a belief that P is stabilized by a proportional negative feedback of gain greater than one. This is true if we study P on $\ell_2(\mathbb{N}_0)$, see [2]. Since system (2.2) is not closed, Georgiou and Smith [3] concluded that P is not stabilizable in the usual sense of the term, that is, there does not exists a LTI-system C such that $[P, C]$ is stable. Moreover, they realized already that the closure of P is a stable system which is non-causal, and this is physically unreasonable. This contrasts with the situation described in their earlier work [2], where use of the signal space $L_2(\mathbb{R}_+)$ gives no such difficulties.

How can we overcome this problem? The idea is to identify P with $\overline{P}$, and to stabilize $\overline{P}$ in such a way that the controller and the feedback system are causal. In our example $\overline{P}$ is already stable, but not causal, and so actually we have to find a causal controller which causalizes $\overline{P}$, and we will see that this can be done by proportional negative feedback of gain greater than one.

We saw that the graph of $\overline{P}$ is given by

$$G(\hat{\overline{P}}) = \begin{pmatrix} 1 \\ \frac{1}{1-2z} \end{pmatrix} L_2(\mathbb{T}).$$

However, in order to be able to causalize $\overline{P}$, we have to choose the symbol of $\overline{P}$ in $H_\infty(\mathbb{D})^2$. For example we have

$$G(\hat{\overline{P}}) = \begin{pmatrix} 1-2z \\ 1 \end{pmatrix} L_2(\mathbb{T}).$$

Note that $1-2z$ is invertible over $L_\infty(\mathbb{T})$. Now the matrix

$$\begin{pmatrix} 1-2z & c \\ 1 & 1 \end{pmatrix}$$

is invertible over $H_\infty(\mathbb{D})$ (and hence over $L_\infty(\mathbb{T})$) for every $c < -1$, and so P is stabilized by proportional negative feedback of gain greater than one, and the controller as well as the feedback system is causal.

6 Systems over the signal space $\ell_2(\mathbb{N}_0)$

In the $\ell_2(\mathbb{N}_0)$-setting a LTI-system is a (possibly) unbounded operator $P : D(P) \subset \ell_2(\mathbb{N}_0) \to \ell_2(\mathbb{N}_0)$ which is linear and shift-invariant with respect to the right-shift on $\ell_2(\mathbb{N}_0)$. It was shown by Jacob and Partington [6] that every causal LTI-system over $\ell_2(\mathbb{N}_0)$ has the form

$$(Pu)(t) := \sum_{n=0}^{t} g(t-n)u(n), \quad u \in D(P), \tag{6.6}$$

for some $g(t) \in \mathbb{C}$. Thus every causal LTI-system is closable, and every stable LTI-system is causal. Further, it can be shown (see [6]), that a LTI-system is causal if and only if it is of the form (6.6). Results concerning stable feedback systems and stabilizability are similar to those obtained by Georgiou and Smith [2] for systems over the signal space $L_2(\mathbb{R}_+)$.

Let P be a closed LTI-system over $\ell_2(\mathbb{N}_0)$. Then there exist $M \in H_\infty(\mathbb{D})$, and $N \in H_\infty(\mathbb{D})$ such that

$$G(\hat{P}) = \begin{pmatrix} m \\ n \end{pmatrix} H_2(\mathbb{D}) = GH_2(\mathbb{D}) \tag{6.7}$$

and G is inner. As for LTI-system over $\ell_2(\mathbb{Z})$, a matrix G satisfying (6.7) is called a *symbol* of P. Moreover, P is stabilizable (as system over $\ell_2(\mathbb{N}_0)$) if and only if G is left-invertible over $H_\infty(\mathbb{D})$.

Every LTI-system over $\ell_2(\mathbb{Z})$ can be *restricted* to a LTI-system over $\ell_2(\mathbb{N}_0)$ in the following way. Let P be a LTI-system over $\ell_2(\mathbb{Z})$. Then we define $P_\mathbb{N} : D(P_\mathbb{N}) \subset \ell_2(\mathbb{N}_0) \to \ell_2(\mathbb{N}_0)$ by

$$\begin{aligned} P_\mathbb{N} u &:= Pu, \quad u \in D(P_\mathbb{N}), \\ D(P_\mathbb{N}) &:= \{u \in D(P) \cap \ell_2(\mathbb{N}_0) \mid Pu \in \ell_2(\mathbb{N}_0)\}, \end{aligned}$$

and we get that $P_\mathbb{N}$ is a LTI-system over $\ell_2(\mathbb{N}_0)$. If P is closable, then $P_\mathbb{N}$ is closable, and an interesting question is the relation between the symbols of $\overline{P}$ and $\overline{P}_\mathbb{N}$. We show that if $\overline{P}_\mathbb{N}$ is stabilizable, then P is stabilizable, and that these systems can be described by the same symbol and they are stabilized by the same set of controllers.

Theorem 6.1. *Let P be a closable LTI-system over $\ell_2(\mathbb{Z})$. If $\overline{P}_\mathbb{N}$ is stabilizable, then P is stabilizable. Moreover, every symbol of $\overline{P}_\mathbb{N}$ is a symbol of $\overline{P}$, and the sets of controllers stabilizing $\overline{P}_\mathbb{N}$ equals the set of controllers stabilizing P.*

Proof. Let $\binom{m_\mathbb{N}}{n_\mathbb{N}}$ be an arbitrary symbol of $\overline{P}_\mathbb{N}$, and let $\binom{m}{n}$ be a symbol of $\overline{P}$. Since $\overline{P}_\mathbb{N}$ is stabilizable, we have that $\binom{m_\mathbb{N}}{n_\mathbb{N}} \in H_\infty(\mathbb{D})^2$ and that $\binom{m_\mathbb{N}}{n_\mathbb{N}}$ is left-invertible over $H_\infty(\mathbb{D})$. Since

$$\begin{pmatrix} m_\mathbb{N} \\ n_\mathbb{N} \end{pmatrix} H_2(\mathbb{D}) = G(\overline{P}_\mathbb{N}) \subset G(\overline{P}) = \begin{pmatrix} m \\ n \end{pmatrix} L_2(\mathbb{T}),$$

there exists a function $Q \in L_2(\mathbb{T})$ such that

$$\begin{pmatrix} m_\mathbb{N} \\ n_\mathbb{N} \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix} Q.$$

$\binom{m}{n}^* \binom{m}{n} = 1$ implies $Q \in L_\infty(\mathbb{T})$. Since $\binom{m_\mathbb{N}}{n_\mathbb{N}}$ is left-invertible over $H_\infty(\mathbb{D})$, there is a $Q_L \in L_\infty(\mathbb{T})$ such that $Q_L Q = I$, which implies $Q_L Q = 1$. Thus Q is invertible over $L_\infty(\mathbb{T})$, and $\binom{m_\mathbb{N}}{n_\mathbb{N}}$ is a symbol of $\overline{P}$. This proves that P is stabilizable and that every symbol of $\overline{P}_\mathbb{N}$ is a symbol of $\overline{P}$. That the sets of controllers coincide follows from Theorem 3.2 and Georgiou and Smith [2, Lemma 1]. ☐

Theorem 6.2. *Let P be a closed LTI-system over $\ell_2(\mathbb{Z})$. Then P is stabilizable (as a system on $\ell_2(\mathbb{Z})$) if and only if $P_\mathbb{N}$ is stabilizable as a system on $\ell_2(\mathbb{N}_0)$). Moreover, every symbol of $P_\mathbb{N}$ is a symbol of P, and the set of controllers stabilizing $P_\mathbb{N}$ equals the set of controllers stabilizing P.*

Proof. The proof is similar to the proof of the previous theorem. ☐

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