Stability and boundedness of continuous- and discrete-time systems

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Abstract

In this paper we investigate the relation between discrete- and continuous-time systems. More precisely, we investigate the stabiity properties of the semigroup generated by A, and the sequence A_d^n, $n \in \mathbb{N}$, where $A_d = (I + A)(I - A)^{-1}$.

1 Introduction

For finite-dimensional spaces it is very easy to see that the eigenvalues of the matrix A are in the right-half plane if and only if the eigenvalues of $A_d := (I + A)(I - A)^{-1}$ are inside the unit circle. Since the location of the eigenvalues determines the stability of the associated system, it is now an easy consequence that the system

$$\dot{x}(t) = Ax(t) \tag{1.1}$$

is stable if and only if the system

$$x(n + 1) = A_d x(n) \tag{1.2}$$

is stable. For infinite-dimensional systems the situation is more complicated. Although the eigenvalues still possess the same property as in the finite dimensional situation, they don't longer rule the stability. Since many years it is an open question whether the stability of (1.1) and (1.2) are equivalent. For Banach spaces it is not hard to find a counter example, and so we concentrate on the situation where the state space is a Hilbert space.

For Hilbert spaces there are some positive results, such as for dissipative operator and for normal operators. The results of Crouzeix et. al. [1] and Palencia [5] imply that if A is the infinitesimal generator of the holomorphic semigroup $T(t)$ which is sectorially bounded, i.e.,

$\|T(t)\| \leq M$ for all t with $|\arg(t)| < \theta$, for some $\theta > 0$, then A_d is power bounded. i.e., $\|A_d^n\| \leq M_d$ for all $n \in \mathbb{N}$. Recently, Guo and Zwart [3] showed that the stability of (1.1) and (1.2) are equivalent if A generates a holomorphic semigroup. We note that the systems (1.1) and (1.2) are naturally related via the Gayley transform, see e.g. Curtain and Oostveen [2]. In this paper we show that the stability of (1.1) implies that of (1.2) under the extra assumption that the system

$$\dot{x}(t) = A^{-1}x(t) \tag{1.3}$$

is stable. Before we can show this result, we need some new stability results. This will be the subject of the next section.

2 Stability results

In this section we relate the stability of (1.1) and (1.2) to the solution of a certain Lyapunov equation. It is well-know that (1.1) is exponentially stable, i.e., the semigroup generated by A satisfies $\|T(t)\| \leq Me^{-\omega t}$ for some $\omega > 0$ if and only if there exists a positive, bounded operator Q satisfying

$$A^*Q + QA = -I.$$

For the uniform boundedness and strong stability of (1.1) we prove similar results. These results are inspired by the results of Shi and Feng [7], and Tomilov [8]. We begin with the discrete-time.

Theorem 2.1. *Let A_d be a bounded operator on the Hilbert space Z, then the following are equivalent*

1. *A_d is power bounded, i.e., $\|A_d^n\| \leq M$ for all $n \in \mathbb{N}$.*
2. *For all $r \in (0,1)$ there exist positive operators $R(r), \tilde{R}(r) \in \mathcal{L}(Z)$ such that*

$$r^2 A_d^* R(r) A_d - R(r) \leq -I \tag{2.4}$$

$$r^2 A_d \tilde{R}(r) A_d^* - \tilde{R}(r) \leq -I. \tag{2.5}$$

Furthermore, they satisfy

$$(1-r)\|R(r)\| \leq M, \qquad (1-r)\|\tilde{R}(r)\| \leq \tilde{M}, \tag{2.6}$$

for certain constants M and $\tilde{M}$ independent of r.

Proof We first show that 1. implies 2. Let A_d be power bounded, then we see that $\| \left(A_d^*\right)^n \| = \|A_d^n\| \leq M$. Now we define for $r \in (0,1)$

$$R(r) = \sum_{n=0}^{\infty} r^n \left(A_d^*\right)^n A_d^n r^n.$$

It is easy to see that $R(r)$ is positive and satisfies (2.4). There is even equality. Furthermore, since A_d and A_d^* are power bounded, we see

$$\|R(r)\| \leq \sum_{n=0}^{\infty} r^{2n} M^2 = \frac{M^2}{1-r^2}$$

Thus we see that $R(r)$ satisfies (2.6). Repeating the argument for A_d^* gives the remaining part of the assertion.

We now concentrate on the implication 2. to 1. We first show that from (2.4) it follows that

$$R(r) \geq \sum_{n=0}^{\infty} r^n \left(A_d^*\right)^n A_d^n r^n. \tag{2.7}$$

To prove this we note that from (2.4)

$$\begin{aligned} \sum_{n=0}^{N} r^n \left(A_d^*\right)^n A_d^n r^n &= \sum_{n=0}^{N} r^n \left(A_d^*\right)^n I A_d^n r^n \\ &\leq \sum_{n=0}^{N} r^n \left(A_d^*\right)^n \left[R(r) - r A_d^* R(r) A_d r\right] A_d^n r^n \\ &= R(r) - r^{N+1} \left(A_d^*\right)^{N+1} R(r) A_d^{N+1} r^{N+1}. \end{aligned} \tag{2.8}$$

Since $R(r) \geq 0$, this implies that

$$\sum_{n=0}^{N} r^n \left(A_d^*\right)^n A_d^n r^n \leq R(r).$$

Thus $r^n \left(A_d^*\right)^n A_d^n r^n$ is summable, and we get that (2.7) holds. Similarly, we can show that

$$\tilde{R}(r) \geq \sum_{n=0}^{\infty} r^n A_d^n \left(A_d^*\right)^n r^n.$$

Using this and (2.7) we can show the uniform boundedness of A_d^n. We start with the following simple equality.

$$\begin{aligned} |(n+1) r^n \langle y, A_d^n x \rangle| &= \left| \sum_{k=0}^{n} \langle y, r^{n-k} A_d^{n-k} A_d^k r^k x \rangle \right| \\ &= \left| \sum_{k=0}^{n} \langle r^{n-k} \left(A_d^*\right)^{n-k} y, A_d^k r^k x \rangle \right| \\ &\leq \left[\sum_{k=0}^{n} \| r^{n-k} \left(A_d^*\right)^{n-k} y \|^2 \right]^{1/2} \left[\sum_{k=0}^{n} \| A_d^k r^k x \|^2 \right]^{1/2} \\ &= \left[\sum_{k=0}^{n} \| r^k \left(A_d^*\right)^k y \|^2 \right]^{1/2} \left[\sum_{k=0}^{n} \| A_d^k r^k x \|^2 \right]^{1/2}. \end{aligned}$$

Thus by (2.7) we have that

$$|(n+1)r^n\langle y, A_d^n x\rangle| \leq \langle y, \tilde{R}(r)y\rangle^{1/2}\langle x, R(r)x\rangle^{1/2} \tag{2.9}$$

Using the conditions in (2.6) we see that

$$|(n+1)r^n\langle y, A_d^n x\rangle| \leq M_2 \frac{1}{1-r}\|y\|\|x\|.$$

Thus

$$\|A_d^n\| \leq \frac{M_2}{(n+1)r^n(1-r)}.$$

Since this holds for all $r \in (0,1)$, we can take the infimum over all these r's. The miminum over the right-hand side is attained in $r = n/(n+1)$. Thus

$$\|A_d^n\| \leq \frac{M_2}{(n+1)(\frac{n}{n+1})^n(\frac{1}{n+1})} = M_2\left(1+\frac{1}{n}\right)^n.$$

Since the right-hand side can be bounded independently of n, we have that $\|A_d^n\|$ is uniformly bounded. ■

The above result we can use to prove strong stability.

Theorem 2.2. *Let A_d be a bounded operator on the Hilbert space Z, then the following are equivalent*

1. *A_d^n is strongly stable, i.e., $A_d^n x \to 0$ for $n \to \infty$.*
2. *For all $r \in (0,1)$ there exist bounded positive operators $R(r)$ and $\tilde{R}(r)$ which satisfy the Lyapunov inequalities (2.4) and (2.5). Furthermore, $\tilde{R}(r)$ satisfies*

$$(1-r)\|\tilde{R}(r)\| \leq \tilde{M} \tag{2.10}$$

for some $\tilde{M}$, and $R(r)$ satisfies

$$\lim_{r\uparrow 1}(1-r)\langle R(r)x, x\rangle = 0. \tag{2.11}$$

Proof We first prove that 1. implies 2. Define for $r \in (0,1)$

$$R(r) = \sum_{n=0}^{\infty} r^n \left(A_d^*\right)^n A_d^n r^n,$$

$$\tilde{R}(r) = \sum_{n=0}^{\infty} r^n A_d^n \left(A_d^*\right)^n r^n.$$

As we have seen in the proof of the previous theorem, these operators satisfies (2.4), and (2.5), respectively. Furthermore, since the strong stability of A_d implies the uniform boundedness of A_d^n and $(A_d^*)^n$, we see that $\tilde{R}(r)$ satisfies (2.10).

Let $\varepsilon > 0$ be given, and choose N such that $\|A_d^n x\| \leq \varepsilon$ for all $n \geq N$. Then we obtain

$$\begin{aligned}
\langle x, R(r)x\rangle &= \sum_{n=0}^{\infty} \|r^n A_d^n x\|^2 \\
&\leq \sum_{n=0}^{N-1} \|r^n A_d^n x\|^2 + \sum_{n=N}^{\infty} r^{2n}\varepsilon^2 \\
&= \sum_{n=0}^{N-1} \|r^n A_d^n x\|^2 + \varepsilon^2 \frac{r^{2N}}{1-r^2}
\end{aligned}$$

Thus

$$\lim_{r\uparrow 1}(1-r)\langle R(r)x, x\rangle \leq \frac{1}{2}\varepsilon^2.$$

Since this holds for any $\varepsilon > 0$, we have shown (2.11).

Now we show that 2. implies 1. From (2.9) and (2.10), one sees that

$$\|A_d^n x\|^2 \leq \frac{\tilde{M}}{r^{2n}(n+1)^2(1-r)}\langle x, R(r)x\rangle$$

For $1-r$ sufficiently close to zero, we get that

$$\|A_d^n x\|^2 \leq \frac{\tilde{M}}{r^{2n}(n+1)^2(1-r)}\frac{M}{1-r}\varepsilon.$$

Now one can proceed as in the first proof, and similarly show that for sufficient large n, $\|A_d^n x\| \leq 2eM\varepsilon$. ∎

There is naturally a similar result in continuous time. Since we shall only use the simple implications, and since the proofs are very similar, we omit the proofs.

Theorem 2.3. *Let A be a closed, densely defined operator on the Hilbert space Z, then the following are equivalent*

1. *A is the infinitesimal generator of an uniformly bounded C_0-semigroup $T(t)$, i.e., $\|T(t)\| \leq M$ for all $t \geq 0$.*

2. *For all $\lambda > 0$ there exist unique positive solutions of the Lyapunov equations*

$$(A-\lambda I)^* Q(\lambda) + Q(\lambda)(A-\lambda I) = -I \tag{2.12}$$

$$(A-\lambda I)\tilde{Q}(\lambda) + \tilde{Q}(\lambda)(A-\lambda I)^* = -I \tag{2.13}$$

which satisfy that there exists a constant M and $\tilde{M}$ such that

$$\lambda\|Q(\lambda)\| \leq M, \qquad \lambda\|\tilde{Q}(\lambda)\| \leq \tilde{M}. \tag{2.14}$$

Theorem 2.4. *Let A be a closed, densely defined operator on the Hilbert space Z, then the following are equivalent*

1. *A is the infinitesimal generator of a strongly stable C_0-semigroup, i.e., $T(t)x \to 0$ for $t \to \infty$.*

2. *For all $\lambda > 0$ there exist unique positive solutions of the Lyapunov equations (2.13) and (2.14). Furthermore, the solution $\tilde{Q}(\lambda)$ of (2.13) satisfies*

$$\lambda \|\tilde{Q}(\lambda)\| \leq \tilde{M} \tag{2.15}$$

for some $\tilde{M}$, and the solution $Q(\lambda)$ of (2.12) satisfies

$$\lim_{\lambda \downarrow 0} \lambda \langle Q(\lambda)x, x \rangle = 0. \tag{2.16}$$

These theorem are very useful in showing the relation between bounded semigroups and bounded co-generator.

Theorem 2.5. *Let A and A^{-1} both be the infinitesimal generator of a bounded C_0-semigroup on the Hilbert space Z. Then the operator $A_d := (I + A)(I - A)^{-1}$ is power bounded.*

Furthermore, if the semigroups generated by A and A^{-1} are strongly stable, then A_d is strongly stable.

Proof So we have to show that there exist solutions of (2.4) and (2.5). We shall only show (2.4) since the proof of the other inequality goes very similar. Let $Q(\lambda)$ be the solution of (2.12), and let $S(\lambda)$ be the solution of

$$(A^{-1} - \lambda)^* S(\lambda) + S(\lambda)(A^{-1} - \lambda) = -I. \tag{2.17}$$

Multiplying both sides by A^* from the left and A from the right, it is easy to see that this equation is equivalently formulated as

$$-2\lambda A^* S(\lambda) A + S(\lambda) A + A^* S(\lambda) = -A^* A. \tag{2.18}$$

Now take an $r \in (0, 1)$, and consider the left hand side of (2.4).

$$\begin{aligned} & r^2 A_d^* R A_d - R \\ & \quad = (I - A)^{-*} \left[r^2 (I + A)^* R (I + A) - (I - A)^* R (I - A) \right] (I - A)^{-1} \\ & \quad = (r^2 + 1) \cdot \\ & \qquad (I - A)^{-*} \left[\frac{r^2 - 1}{r^2 + 1} R + \frac{r^2 - 1}{r^2 + 1} A^* R A + A^* R + R A \right] (I - A)^{-1}. \end{aligned}$$

Now we choose $-2\lambda = \frac{r^2-1}{r^2+1}$ and $R(r) = Q(\lambda) + S(\lambda)$.

$$\begin{aligned}
& r^2 A_d^* R(r) A_d - R(r) \\
& \quad = (r^2+1)(I-A)^{-*} \\
& \quad\quad [-2\lambda S(\lambda) - 2\lambda A^* S(\lambda) A + A^* S(\lambda) + S(\lambda) A] (I-A)^{-1} + \\
& \quad\quad (r^2+1)(I-A)^{-*} \\
& \quad\quad [-2\lambda Q(\lambda) - 2\lambda A^* Q(\lambda) A + A^* Q(\lambda) + Q(\lambda) A] (I-A)^{-1} \\
& \quad = (r^2+1)(I-A)^{-*} [-2\lambda S(\lambda) - A^* A] (I-A)^{-1} + \\
& \quad\quad (r^2+1)(I-A)^{-*} [-I - 2\lambda A^* Q(\lambda) A] (I-A)^{-1} \\
& \quad \leq (r^2+1)(I-A)^{-*} [-A^* A - I] (I-A)^{-1} \leq -\gamma I.
\end{aligned}$$

From it follows that $\frac{1}{\gamma} R(r)$ satisfies (2.4). Since the behavior of R at one is like Q and S at zero, we obtain the result. ■

Note that since we don't know whether the inverse of a bounded A generates a bounded semigroup, we are still missing the proof for the case that A is a bounded operator. This will be the result of the next corollary.

Corollary 2.1. *Let $A \in \mathcal{L}(Z)$ be the infinitesimal generator of a bounded C_0-semigroup. Then the operator $A_d := (I+A)(I-A)^{-1}$ is power bounded.*

Furthermore, if e^{At} is strongly stable, then A_d^n is strongly stable.

Proof For $r \in (0,1)$ we define λ as in the proof of Theorem 2.5, i.e.,

$$\lambda = \frac{1-r^2}{2(1+r^2)}.$$

Furthermore, we choose

$$R(r) = Q(\lambda)$$

With this choice, we obtain similar as in previous proof that

$$\begin{aligned}
r^2 A_d^* R(r) A_d - R(r) &= (r^2+1)(I-A)^{-*} [-I - 2\lambda A^* Q(\lambda) A] (I-A)^{-1} \\
&\leq -(r^2+1)(I-A)^{-*}(I-A)^{-1} \leq -\gamma I,
\end{aligned}$$

where the last inequality follows since A is a bounded operator. Thus we see that A_d is power bounded. ■

It is well-known that A generates an analytic semigroup which is sectorially bounded, i.e., $\|T(t)\| \leq M$ for all t such that $|\arg(t)| < \theta$ for some positive θ if and only if $\|(sI-A)^{-1}\| \leq m/|s|$ for all complex s with $|\arg(s)| < \pi/2 + \theta$, see e.g. [6, Theorem 2.5.2]. From this it is easy to see that A generates an analytic, sectorially bounded semigroup if and only if A^{-1} generates an analytic, sectorially bounded semigroup. Thus from Theorem 2.5 we conclude the following.

Corollary 2.2. *Let A be the infinitesimal generator of an analytic, sectorially bounded C_0-semigroup on the Hilbert space Z, and let A^{-1} exist as a closed operator. Then the operator $A_d := (I + A)(I - A)^{-1}$ is power bounded.*

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