Reciprocals of regular linear systems: a survey

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Abstract

This paper proposes a novel approach to studying regular linear systems via their reciprocal systems. Under the generic assumption that A has a bounded inverse, a regular linear system possesses a reciprocal system with four bounded generating operators. Many system theoretic problems for regular linear systems can be translated into equivalent problems for their reciprocal system. Due to the bounded nature of the generators, the problems for the reciprocal system are easier to solve and these solutions can be translated back to solutions for the original regular linear system. Properties of reciprocal systems are reviewed and the success of this approach is illustrated with the LQ control problem, the existence of (pseudo-) coprime factorizations and spectral factorization problems.

1 Stabilizability results for state linear systems

In this section we collect known results for systems with bounded input and output operators. A is the generator of a strongly continuous semigroup on a Hilbert space Z, $B \in \mathcal{L}(U,Z), C \in \mathcal{L}(Z,Y), D \in \mathcal{L}(U,Y)$ with U, Y Hilbert spaces. Following the terminology in Curtain and Zwart [3] we call $\Sigma(A,B,C,D)$ a *state linear systems* and we omit the "D" term if it is not relevant. In Curtain and Oostveen [1] the concepts of a strongly stable system, strong stabilizability and strong detectability were introduced and analyzed for state linear systems. There the focus was on the concept of strong stability of the semigroup as the desirable property for a stable system, but for our present purposes the appropriate definition of a stable state linear system is the following one due to Staffans [10].

Definition 1.1. *The state linear system* $\Sigma(A,B,C,D)$ *is* stable *if*

- *it is input stable, i.e.,* $B^*(sI-A^*)^{-1}z \in \mathbf{H}_2(U)$ *for all* $z \in Z$*;*
- *it is output stable, i.e.,* $C(sI-A)^{-1}z \in \mathbf{H}_2(Y)$ *for all* $z \in Z$*;*
- *it is input-output stable, i.e.,* $D + C(sI-A)^{-1}B \in \mathbf{H}_\infty(\mathcal{L}(U,Y))$.

Definition 1.2. $\Sigma(A,B,C)$ *is* output stabilizable *if there exists an* $F \in \mathcal{L}(Z,U)$ *such that* $\Sigma(A+BF,B,[C:F])$ *is output stable.*
$\Sigma(A,B,C)$ *is* input stabilizable *if there exists an* $L \in \mathcal{L}(Y,Z)$ *such that* $\Sigma(A+LC,[B:L],C)$ *is input stable.*

We summarize the following consequences from Curtain and Oostveen [1].

Theorem 1.1. *If the state linear system* $\Sigma(A,B,C,D)$ *is output stabilizable, then there exists a minimal self-adjoint non-negative definite solution of the control Riccati equation for* $z \in D(A)$

$$A^*Qz + QAz + C^*Cz = (QB + C^*D)S^{-1}(B^*Q + D^*C)z, \tag{1.1}$$

where we denote $S = I + D^*D$ *and* $A_Q = A - BS^{-1}D^*C - BS^{-1}B^*Q$. *Moreover,* $\Sigma(A_Q, B, [C : B^*Q])$ *is output stable and input-output stable and it will be input stable if* $\Sigma(A,B,C)$ *is input stabilizable.*

Unfortunately, input and output stabilizability are not sufficient to guarantee the existence of coprime factorizations over $\mathbf{H}_\infty$, only coprime factorizations over $\mathbf{H}_2$. In fact, it is the pseudo-coprime property introduced in Mikkola [8] that proves to be a key property in the theory of Riccati equations.

Definition 1.3. *We call* $[M : N] \in \mathbf{H}_\infty(\mathcal{L}(U, U \oplus Y)$ right pseudo-coprime *if there exists* $\mu > 0$ *such that for all* $s \in \mathbb{C}_0^+$ *there holds* $M(s)^*M(s) + N(s)^*N(s) \geq \mu I$.

If dim $Y < \infty$, right pseudo-coprimeness is equivalent to the more usual concept of right coprimeness over $\mathbb{C}_0^+$, i.e., $[M : N]$ is right coprime over $\mathbb{C}_0^+$ if there exist $\tilde{X}, \tilde{Y}$ such that $[\tilde{X} : \tilde{Y}]^t \in \mathbf{H}_\infty(\mathcal{L}(U \oplus Y, U))$ and for all $s \in \mathbb{C}_0^+$ there holds $\tilde{X}M - \tilde{Y}N = I$.

Definition 1.4. *The transfer function* $\mathbf{G}$ *of the state linear system* $\Sigma(A,B,C,D)$ *has a* right (pseudo-) coprime factorization *if there exist* $[M : N] \in \mathbf{H}_\infty(\mathcal{L}(U, U \oplus Y)$ *that are right (pseudo-) coprime,* M *has an inverse* $M^{-1}(\cdot + \omega) \in \mathbf{H}_\infty(\mathcal{L}(U))$ *for some* ω *and* $\mathbf{G}(s) = N(s)M(s)^{-1}$ *on some right half-plane.*

Analogous definitions hold for the left versions. We quote a recent result that is a generalization of an earlier result in Curtain and Oostveen [2] for finite-dimensional U, Y.

Theorem 1.2. *If the state linear system* $\Sigma(A,B,C,D)$ *is input and output stabilizable, and* $\sigma(A) \cap i\mathbb{R}$ *is at most countable, then its transfer function has a normalized doubly pseudo-coprime factorization.*

2 Reciprocal systems of regular linear systems

Here we review the concept of a reciprocal system that was introduced in Curtain [4] for a regular linear system with generators A, B, C, D. A generates a strongly continuous semigroup $T(t)$ on a Hilbert space Z, U, Y are Hilbert spaces, $D \in \mathcal{L}(U,Y)$, $C \in \mathcal{L}(D(A), Y)$, $A^{-1}B \in \mathcal{L}(U, Z)$, and B and C are admissible control and observation operators with respect to $T(t)$, i.e., given $\tau > 0$ there exists a $\gamma > 0$ such that for all

$z \in D(A)$ $\int_0^\tau \|CT(t)z\|^2 \, \mathrm{d}\tau \leq \gamma \|z\|^2$, and for any $\tau > 0$ there exist a $\beta > 0$ such that for all $u \in \mathbf{L_2}(0, \tau; U)$, $\| \int_0^\tau T(\tau - s)Bu(s) \, \mathrm{d}s\|^2 \leq \beta \int_0^\tau \|u(s)\|^2 \, \mathrm{d}s$. If the above definitions can be extended to $\tau = \infty$, then the term *infinite-time admissible* is used. Of course, these are the time domain equivalents of input and output stability in Definition 1.1, which also applies to regular linear systems. In Grabowski [7] it is shown that C is an infinite-time admissible observation operator for $T(t)$ if and only if the observation Lyapunov equation has a self-adjoint non-negative definite solution $L_C \in \mathcal{L}(Z)$

$$A^* L_C z + L_C A z = -C^* C z \quad \text{for all } z \in D(A). \tag{2.2}$$

The transfer function of a regular linear system is given by $\mathbf{G}(s) = D + C_\Lambda (sI - A)^{-1} B$, where C_Λ denotes the Lambda extension of C. For each $u \in U$, $\mathbf{G}u$ has the limit Du as s approaches infinity along the positive real axis. Notice that if zero is in the resolvent of A, then all the generating operators $A^{-1}, A^{-1}B, CA^{-1}$ are bounded. This motivates the following definition.

Definition 2.1. *Suppose that the regular linear system with generating operators A, B, C, D is such that A has a bounded inverse. Its* reciprocal system *is the state linear system $\Sigma(A^{-1}, A^{-1}B, -CA^{-1}, D + \mathbf{G}(0))$.*

The justification for this definition is the nice relationship between the regular linear system and its reciprocal system shown in [4].

Theorem 2.1. *Suppose that A, B, C, D are generating operators of a regular linear system and zero is in the resolvent of A. Then*

1. *$C_\Lambda (sI - A)^{-1} B = -C_\Lambda A^{-1} B - CA^{-1}(\frac{1}{s} I - A^{-1})^{-1} A^{-1} B$ for $s \in \rho(A)$.*

2. *C is an infinite-time admissible observation operator for $T(t)$ if and only if CA^{-1} is one such for $T_-(t) = \exp A^{-1} t$. If they are infinite-time admissible, then their observability gramians are identical.*

3. *B is an infinite-time admissible control operator for $T(t)$ if and only of $A^{-1}B$ is one such for $T_-(t)$. If they are infinite-time admissible, then their controllability gramians are identical.*

4. *The input, output and input-output stability properties of the regular linear system and its reciprocal system are identical.*

The concept of r-output stabilizability plays a crucial role in the theory

Definition 2.2. *The regular linear system with generating operators A, B, C, D is* output stabilizable *if there exists $F \in \mathcal{L}(D(A), U)$ such that with $C^e = \begin{bmatrix} F \\ C \end{bmatrix}$*

- A, B, C^e *are generators of a regular linear system with transfer function* $\mathbf{G}_F$*;*

- $[I\ :\ 0]$ *is an admissible feedback operator for* $\mathbf{G}_F$*, i.e.,* $\mathbf{G}_F(I - [I\ :\ 0]\mathbf{G}_F)^{-1}$ *is a regular linear transfer function and the closed-loop generator* A_F *generates a strongly continuous semigroup* T_F*, where*
  $A_F z = (A + BF_\Lambda)z$ *for* $z \in D(A_F) = \{z \in D(F_\Lambda) | (A - BF_\Lambda)z \in Z\}$.

- C^e *is an infinite-time admissible observation operator for* T_F*.*

It is output-input-output stabilizable *if, in addition, the regular linear system with generating operators* A_F, B, C^e *is input-output stable.*
It is r-output stabilizable *if it is output stabilizable and* A_F^{-1} *is bounded.*

Note that our definition is independent of D. In particular, a system is r-output stabilizable if F exponentially stabilizes (A, B) in the sense of Rebarber [9]. Our definition of r-stabilizability is different from others in Staffans [10] and in Mikkola [8]. The motivation for yet another new concept is the nice relationship with the concept of output stabilizability of the reciprocal system shown in [4].

Lemma 2.1. *Suppose that* A, B, C, D *are generating operators of a regular linear system with* A^{-1} *bounded. If it is r-output stabilizable, its reciprocal system is output stabilizable.*

There are similar definitions and results for the dual concept of r-input stabilizability. The condition of the invertibility of A_F does not seem natural, so the sufficient conditions for this property from [6] are useful.

Lemma 2.2. *Suppose that the regular linear system with generating operators* A, B, C, D *is input stabilizable and output-input-output stabilizable with closed loop generator* A_F*. If* A *is boundedly invertible, then so is* A_F *and the system is r-output stabilizable.*

3 Results for regular linear systems

In this section we report on a number of results for regular linear systems that can be proven by translating the analagous results for their reciprocal systems. Note that typical assumptions on the regular linear systems are in terms of r-input and r-output stabilizability. The proofs are relatively simple, due to the fact that the reciprocal systems have bounded generators. The first results represent the first steps in developing a simpler theory of Riccati equations for regular linear systems. The first theorem states an equivalence between the Riccati equation for a regular linear system with *bounded* B operator and a corresponding Riccati equation for its reciprocal system.

Theorem 3.1. *Let A, B, C, D be the generating operators of a regular linear system with A^{-1} and B bounded operators. Then its control Riccati equation (1.1) has a self-adjoint non-negative solution if and only if the control Riccati equation for its reciprocal system*

$$A^{-*}Q + QA^{-1} + A^{-*}C^*CA^{-1} = L^*S_-^{-1}L, \tag{3.3}$$

with $L = B^*A^{-*}Q - D_-^*CA^{-1}$, $S_- = I + D_-^*D_-$ has a self-adjoint non-negative solution. Moreover, $A_-^Q = A^{-1} - A^{-1}BS_-^{-1}B^*A^{-*}Q + A^{-1}BS_-^{-1}D_-^*CA^{-1}$ has as its inverse the A-bounded operator $A_Q = A - BS^{-1}D^*C - BS^{-1}B^*Q$, which is an infinitesimal generator of a strongly continuous semigroup.

This leads to a partial converse to Lemma 2.1.

Corollary 3.1. *Let A, B, C be the generating operators of a regular linear system such that A has a bounded inverse. If B is bounded, this system is r-ouput stabilizable if and only if its reciprocal system is output stabilizable.*

In [5] control problems for stable regular linear systems were considered. These problems correspond to a more general class of Riccati equations than (1.1) and they are amenable to a spectral factorization approach (see Weiss and Weiss [12] and [11]). Under the assumption that A^{-1} is bounded, it is shown that the control problems corresponding to the Riccati equations (1.1) and (3.3) have the same solution and the minimal cost is given by $< Qz, z >$, where Q is the minimal solution of (3.3). As was already known, (1.1) need not always be well-defined, but (3.3) is. In the case that both are well-defined, they have identical solutions. The unstable case with unbounded B can now be solved by introducing a stabilizing feedback in the usual way (work in progress). The main advantage is that if the underlying optimal control problem has a solution, there is a corresponding Riccati equation that is well-defined and with all operators bounded. This has potential for numerical approximations.
In some applications the Riccati equation is used merely as a tool to deduce other results and in these cases one can use the reciprocal Riccati to good advantage. An example of such an application is the following new result on the existence of pseudo-coprime factorizations in [6] that was deduced from the corresponding result Theorem 1.2 applied to the reciprocal system. The proof of Theorem 1.2 used Theorem 1.1 on the Riccati equation of the reciprocal system, but the Riccati equation for the original regular system was not needed.

Theorem 3.2. *Let A, B, C, D be the generating operators of a regular linear system with transfer function $\mathbf{G}$. If the system is r-input stabilizable and r-output stabilizable, A is boundededly invertible and the intersection of its spectrum with the imaginary axis is at most countable, then $\mathbf{G}$ has a normalized doubly pseudo-coprime factorization. If the dimensions of U and Y are finite, then it is coprime.*

Probably the most fruitful application of reciprocal systems is to obtain explicit formulas for solutions to spectral factorization problems in terms of the original operators A, B, C, D.

We refer the reader to the paper by Curtain and Sasane in these proceedings. The reciprocal approach could also prove useful for other problems for regular linear systems, for example, H-infinity control, sampling and the robust stability radius. Finally, we remark that similar conclusions hold if we replace the assumption that A^{-1} is bounded by the assumption that $(i\omega I - A)^{-1}$ is bounded for some real ω.

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