H_∞-Control of Acoustic Noise in a Duct with a Feedforward Configuration

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Abstract

A mathematical model of sound propagation in a duct derived from physical principles is described. Experimental results validate the model. Theoretical limits of noise reduction are examined using this model. In many feedforward configurations, the optimal controller can reduce the noise at a point to almost zero. However, no noise reduction is obtained at points below the performance point, and at many such points the noise level is increased.

1 Introduction

A mathematical model of sound propagation in a duct derived from physical principles is described. This physics-based model has non-constant acoustic load impedance at the open end, and a coupled disturbance loudspeaker model at the other end. The final model consists of a system of coupled partial and ordinary differential equations. Experimental results validate the model.

We then formulate the control problem of reducing the noise levels as an $\mathcal{H}_\infty$-optimization problem. The solution of the problem depends not only on sensor and actuator locations, but on the point x at which noise is to be reduced. The noise reduction level achievable with a stable closed loop is calculated for various choices of controller location x_c, measurement location x_m and performance point x. This is an extension of [4], where a similar control problem was analyzed using a simpler mathematical model. For many configurations, including all feedback configurations $x_c < x_m$, the achievable noise reduction achievable with a stable closed loop is limited by time delays. The feedforward case, $x_c > x_m$ is examined in detail. It is shown that any point $x > x_c$, the noise level can be arbitrarily reduced with a stable closed loop. The noise level at points other than that used for performance in controller design is examined. Points above the design point exhibit noise reduction. The noise level at points closer to the disturbance source is in general increased by control.

2 Model

We consider control of acoustic noise in a duct of length L. The disturbance source is a speaker mounted at one end, $x = 0$. The far end, $x = L$, is open. A canceller speaker is mounted partway along the duct at some point x_c, $0 < x_c < L$. This speaker is used to attenuate the effect of a disturbance signal. Here we will briefly describe the mathematical model. Details can be found in [6, 8].

The duct is considered to be a hard-walled structure, with sound dissipation only at the ends. We assume that air pressure $p(x,t)$ and velocity $v(x,t)$ varies only with distance along the duct x and time t.

The open end of the duct ($x = L$) results in a partially reflective and partially absorptive boundary condition. Let $\hat{p}(L,s)$ indicate the Laplace transform of $p(L,t)$, and define $\hat{v}(L,s)$ similarly. The specific acoustic impedance of the open end of the duct is

$$Z_L(s) = \frac{\hat{p}(L,s)}{\hat{v}(L,s)}.$$

An analytical value of Z_L was derived in [2] and [5, p. 1529]. Here a rational approximation will be used [1]. The differential equations for this approximation are as follows. Here $P(t)$ and $V(t)$ are equivalent to the acoustic pressure $p(L,t)$ and velocity $v(L,t)$ respectively and $P_c(t), V_m(t)$ are intermediate variables. Parameter values are in Table 1.

$$\frac{dP_c}{dt} = P_c(-\frac{1}{C})(\frac{1}{R_1} + \frac{1}{R_2}) - \frac{1}{CR_2}P, \tag{2.1}$$

$$\frac{dV_m}{dt} = \frac{P(t)}{M}, \tag{2.2}$$

$$V(t) = \frac{1}{R_2}P(t) + \frac{1}{R_2}P_c(t) + V_m(t). \tag{2.3}$$

It can be shown that $\frac{\hat{P}(L,s)}{\hat{V}(s)} = Z_L(s)$ where

$$Z_L(s) = \pi a^2 \frac{(R_1 + R_2)Ms + R_1 R_2 M C s^2}{(R_1 + R_2) + (M + R_1 R_2 C)s + R_1 M C s^2}. \tag{2.4}$$

A loudspeaker is mounted at the disturbance end of the duct ($x = 0$), acting as a source of noise. Letting $x_D(t)$ indicate the loudspeaker driver displacement, the governing equation of the loudspeaker is

$$m_D \ddot{x}_D(t) + d\dot{x}_D(t) + k_D x_D(t) = \frac{Bl}{R_{coil}} E_D(t) - A_D P_D(t) \tag{2.5}$$

where $d = \frac{(Bl)^2}{R_{coil}}$, $A_D = \pi r_d^2$, $P_D(t) = p(0,t)$ and $E_D(t)$ is the voltage applied to the loudspeaker. Loudspeaker parameters are in Table 1. The loudspeaker is coupled to the duct by

$$A_D \dot{x}_D(t) = \pi a^2 v(0,t). \tag{2.6}$$

Taking Laplace transforms of the loudspeaker model in (2.5), we obtain

$$\hat{p}(0,s) = \frac{Bl}{R_{coil}} \hat{E}_D(s) - Z_0(s)\hat{v}\,(0,s) \tag{2.7}$$

where

$$Z_0(s) = \frac{\pi a^2}{A_D s}(m_D s^2 + ds + k_D)$$

is the mechanical impedance of the loudspeaker.

Note that when the loudspeaker is undriven ($\widehat{E}_D(s) = 0$), the particle velocity at $x = 0$, $\hat{v}(0, s)$, is not necessarily zero. It is dependent on Z_0, the impedance of the loudspeaker, and $\hat{p}(0, s)$.

Let $V_c(t)$ represent the total volume velocity of the canceller loudspeaker, distributed over a length $2r_c$ of the duct at location $x = x_c$. The volume velocity $V(x, t)$ per unit length is

$$V(x,t) = \begin{cases} 0, & x < x_c - r_c \\ V_c(t)\frac{2}{\pi r_c^2}\sqrt{r_c^2 - (x - x_c)^2}, & x_c - r_c \le x \le x_c + r_c \\ 0, & x_c + r_c \le x \end{cases} . \tag{2.8}$$

The volume velocity is related to the voltage $u(t)$ applied to the loudspeaker by a model identical to that described above for the disturbance loudspeaker:

$$\hat{V}_c(s) = \frac{Bl_c}{R_c Z_c(s)}\hat{u}(s)$$

where

$$Z_c(s) = \frac{m_c s^2 + d_c s + k_c}{s}.$$

The following well-known equations describe the propagation of sound in a one-dimensional duct:

$$\frac{1}{c^2}\frac{\partial p}{\partial t} = -\frac{\partial v}{\partial x}\rho_0 + \frac{1}{\pi a^2}\rho_0 V(x,t) \tag{2.9}$$

$$\rho_0 \frac{\partial}{\partial t} v(x,t) = -\frac{\partial}{\partial x} p(x,t). \tag{2.10}$$

Equations (2.9) and (2.10) with the disturbance and canceller loudspeaker models and the open end at $x = L$ form a boundary value problem that fully describes the sound dynamics in the duct.

Define $k = \frac{s}{c}$,

$$\alpha_0(s) = \frac{Z_0(s) - \rho_0 c A_D}{Z_0(s) + \rho_0 c A_D} \text{ and } \alpha_L(s) = \frac{Z_L(s) - \rho_0 c}{Z_L(s) + \rho_0 c}.$$

The transfer function that relates the pressure measured at x to the voltage applied to the loudspeaker at $x = 0$ is

$$G_d(x,s) = e^{-xk} G_{do}(x,s) \tag{2.11}$$

where

$$G_{do}(x,s) = \frac{Bl\rho_0 c(1 + \alpha_0(s))}{2R_{coil} Z_0(s)(1 - \alpha_0(s)\alpha_L(s)e^{-2Lk})}(1 + \alpha_L(s)e^{2(x-L)k}). \tag{2.12}$$

Define $J(z) = 2J(1,z)/z$ where $J(1,z)$ indicates the Bessel function of the first kind of order 1. The transfer function that relates pressure measured at x to the volume velocity generated by the canceller loudspeaker at $x = x_c$ is

$$\frac{Bl\rho_0 c}{2R_{coil}Z_c(s)\pi a^2(1-\alpha_0(s)\alpha_L(s)e^{-2Lk})}\tilde{G}(s)$$

where

$$\tilde{G}(s) = \begin{cases} \left(\alpha_L(s)e^{(-2L+x)k} + e^{-kx}\right)\left(e^{x_c k}J(-ir_c k) + \alpha_0(s)e^{-x_c k}J(ir_c k)\right) & 0 < x_c \leq x \\ \left(\alpha_L(s)e^{(x_c-2L)k}J(-ir_c k) + e^{-kx_c}J(ir_c k)\right)\left(e^{xk} + \alpha_0(s)e^{-xk}\right) & x \leq x_c < L \end{cases}. \tag{2.13}$$

This model has been validated by comparing its predicted with the actual frequency response [6, 8].

The speaker cone radius r_c is very small, and so the terms $J(z)$ in the above function are close to the constant value 1 over the frequency range of interest. Approximating $J(z)$ by the constant value 1, we obtain

$$G_c(x,s) = e^{-|x-x_c|k}G_{co}(s) \tag{2.14}$$

where defining

$$R(s) = \frac{Bl\rho_0 c}{2R_{coil}Z_c(s)\pi a^2(1-\alpha_0(s)\alpha_L(s)e^{-2Lk})}, \tag{2.15}$$

$$G_{co}(s) = R(s)\begin{cases} \left(1+\alpha_L(s)e^{2(x-L)k}\right)\left(1+\alpha_0(s)e^{-2x_c k}\right) & 0 < x_c \leq x \\ \left(1+\alpha_L(s)e^{2(x_c-L)k}\right)\left(1+\alpha_0(s)e^{-2xk}\right) & x \leq x_c < L \end{cases}. \tag{2.16}$$

This is the same transfer function obtained if the canceller loudspeaker is regarded as a point source of total volume velocity located at $x = x_c$. The spatial effects of this loudspeaker has a negligable effect on the system frequency response over the frequency range of interest.

3 Control Problem Formulation

In order to determine the best controlled pressure a measurement of the pressure at the sensing point x_m is made and used as input to a controller C. The controller calculates the voltage u which is applied to the loudspeaker at the control point x_c. The Laplace transform of pressure at x, $P(x)$, due to a disturbance voltage d at $x = 0$ and the control voltage u at $x = x_c$ will be

$$P(x) = G_d(x_m)d + G_c(x)u.$$

Since $u = CP(x_m)$,

$$P(x_m) = (I - G_c(x_m)C)^{-1}G_d(x_m)d.$$

Thus, at an arbitrary point x,

$$P(x) = G_d(x)d + G_c(x)C(I - G(x_m)C)^{-1}G_d(x_m)d. \tag{3.17}$$

We require a controller C so that the closed loop is stable. Since $G_c(x_m) \in \mathcal{H}_\infty$ any stabilizing controller can be written $Q(I+G_c(x_m)Q)^{-1}$ for some $Q \in \mathcal{H}_\infty$ *e.g.* [7]. Conversely, any controller of this form stabilizes $G_c(x_m)$. The function Q is known as the Youla parameter. In this problem Q is the transfer function from the noise at x_m due to the disturbance d to the control signal u. Substituting into (3.17),

$$P(x) = [G_d(x) + G_c(x)QG_d(x_m)]\, d. \tag{3.18}$$

Since $\|\alpha_0\|_\infty \leq 1$ and $\|\alpha_L\|_\infty \leq 1$, $G_{do}(x,s)$ and $G_{co}(x,s)$ in (2.12) and (2.16) respectively are outer functions. Thus, the representations (2.11) and (2.14) give inner/outer factorizations of $G_d(x)$ and $G_c(x)$ respectively. Both families of outer functions are strictly proper. Note that $\alpha_0(0) = 1$ and $|\alpha_0(j\omega)| < 1$ at all other frequencies. Also, $\alpha_L(0) = -1$ and $|\alpha_L(j\omega)| < 1$ at all other frequencies and also $\lim_{|s|\to\infty} \alpha_L(s) = 0$. Thus, the functions $G_d(x)$ and $G_c(x)$ have a double zero at 0 and no other zeros on the imaginary axis.

Typically, disturbance signals contain a number of frequencies, with an upper bound on the frequency content. Also, passive noise reduction methods do not work at low frequencies. Thus, the control objective is to reduce the system response to low frequency disturbances. In order to minimize

$$\frac{\|P(x,t)\|_{L_2(0,\infty)}}{\|d(t)\|_{L_2(0,\infty)}}$$

over all disturbances in the given frequency range we must solve

$$\mu = \inf_{Q \in \mathcal{H}_\infty} \|W_1 \frac{P(x)}{d}\|_\infty \tag{3.19}$$

where $W_1 \in \mathcal{H}_\infty$ is strictly proper. Using (3.18) and the factorization of $G_d(x), G_c(x)$, we obtain the optimization problem

$$\mu = \inf_{Q \in \mathcal{H}_\infty} \|W_1(G_{do}(x)e^{(|x-x_c|+x_m-x)k} + G_{co}(x)QG_{do}(x_m))\|_\infty \tag{3.20}$$

where the norm is

$$\|f\|_\infty = \sup_w |f(jw)|.$$

Define

$$\tilde{Q}(s) = G_{co}(x)QG_{do}(x_m).$$

This problem has the form

$$\inf_{\tilde{Q} \in \mathcal{H}_\infty} \|A + B\tilde{Q}\|_\infty$$

and is an example of a *model-matching problem*. The problem is find a $\tilde{Q} \in \mathcal{H}_\infty$ so that the model $B\tilde{Q}$ most closely matches A.

4 Performance

If in the cost to be minimized (3.20), we have $|x - x_c| + x_m - x > 0$, then the model

$$A = W_1 G_{do}(x) e^{(|x-x_c|+x_m-x)k} \notin \mathcal{H}_\infty.$$

If we choose $\tilde{Q}$ so that the cost $\mu = 0$, then $\tilde{Q} \notin \mathcal{H}_\infty$ and $Q \notin \mathcal{H}_\infty$. Since the family of all stabilizing controllers is parametrized by $Q \in \mathcal{H}_\infty$, this implies an unstable closed loop. The amount of time delay determines the performance achievable with a stable closed loop. The optimal performance can be calculated using the technique in [4, App.B].

In order to have optimal performance $\mu = 0$, we must have the measurement point $x_m < x_c$. For any choice of the performance point $x > \frac{x_c+x_m}{2}$, we can calculate a controller so that the performance $\mu = 0$. For points $x < \frac{x_c+x_m}{2}$, achievable noise reduction is limited by the time delay.

In the experimental apparatus, the canceller speaker was placed about 2/3 along the duct, at $x_c = 2.32$. The measurement microphone was placed at $x_m = L/2 = 1.77$. This is generally referred to as a "feedforward" configuration, since $x_m < x_c$. However, because the noise waves at x_c travel in both directions, any signal generated by the loudspeaker at x_c affects the pressure at x_m. This is the transfer function $G_c(x_m)$. Thus, even in a "feedforward" configuration there is feedback.

For any $x > x_c$, we have the model-matching problem

$$\begin{aligned} \mu &= \inf_{Q \in \mathcal{H}_\infty} \| W_1 (G_{do}(x) e^{(x_m - x_c))k} + G_{co}(x) Q G_{do}(x_m)) \|_\infty \\ &= \inf_{Q \in \mathcal{H}_\infty} \| W_1 (G_{do}(x) e^{(x_m - x_c)k} + \tilde{Q}) \|_\infty. \end{aligned}$$

Choosing

$$\tilde{Q} = -G_{do}(x) e^{(x_m - x_c)k}$$

yields optimal performance $\mu = 0$. This is at the performance point $x > x_c$.

We will now calculate the Youla parameter Q. Define for small $\epsilon > 0$,

$$G_{co}^{\epsilon}(x) = \frac{s+\epsilon}{s} R(s) \left(\alpha_L(s) e^{2(x-L)k} + 1 + \epsilon \right) \left(\alpha_0(s) e^{-2x_c k} + 1 \right) \tag{4.21}$$

where $R(s)$ is defined in (2.15). Note that $G_{co}^{\epsilon} \approx G_{co}$ except that G_{co} is modified so that $G_{co}^{\epsilon}(0) \neq 0$. Also, let $W_R \in \mathcal{H}_\infty$ be a strictly proper function that is close to 1 over the frequency range $[0, R]$ where the weight W_1 is significant, and then rolls off so that $G_{co}^{\epsilon}(x)^{-1} G_{do}(x_m)^{-1} W_R$ is proper. We then define $Q_\epsilon \in \mathcal{H}_\infty$

$$\begin{aligned} Q_\epsilon(x, x_m) &= \tilde{Q} G_{co}^{\epsilon}(x)^{-1} G_{do}(x_m)^{-1} W_R \\ &= -e^{(x_m - x_c)k} G_{do}(x) G_{co}^{\epsilon}(x)^{-1} G_{do}(x_m)^{-1} W_R. \end{aligned}$$

Since the weight W_1 is strictly proper, by choosing ϵ small enough and R large enough we can find Q_ϵ so that arbitrarily small performance μ is obtained. Since $G_{co}(0) = G_{do}(0) = 0$ and

0 is the only imaginary axis zero, the optimal performance is not constrained by imaginary axis zeros. This is in contradiction to the simpler model with constant impedences at $x = 0$ and $x = L$ analyzed in [4]. In that model the constant impedences led to imaginary axis zeros that limited performance.

In [4] the controller in this feedforward configuration $x_m < x_c$ was independent of x for all $x > x_c$. Indicate the performance point used in controller design by x_p. Uniform levels of noise reduction were obtained for all $x > x_p$ in [4], while the noise level increased for $x < x_p$. This is also the case here, although now the controller depends on the performance point x_p. In Figures (1-2) the controlled and uncontrolled pressure at various points along the duct are shown for different choices of performance point. The weight W_1 is any strictly proper function with $|W_1(j2\pi f)| = 1$ for $f < 1000$ and $\epsilon = 10^{-5}$. Clearly, although arbitrarily small noise levels can be obtained at a point, this is at the expense of the noise at points closer to the disturbance.

This controller was designed to optimize performance, with no consideration of robustness. Suppose a given controller C stabilizes a plant $P \in \mathcal{H}_\infty$. It is a well-known result *e.g.* [3] that this controller will stabilize all plants in the family

$$\tilde{P} = (1 + \Delta)P$$

where $\tilde{P} \in \mathcal{H}_\infty$ and

$$\|\Delta\|_\infty \leq \|W_2\|_\infty$$

if and only if

$$\sup_\omega \left| \left[W_2 PC(I - PC)^{-1} \right] (j\omega) \right| < 1.$$

Figure 3 shows the magnitude of $G_c(x_m)C(I - G_c(x_m)C)^{-1}$ with the controller derived above. Since this magnitude is large, even at low frequencies, very little uncertainty in the model can be allowed and still maintain closed loop stability.

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References

[1] L. L. Beranek, *Acoustics,* American Institute of Physics, Inc., New York, 1986.

[2] H. Levine and J. Schwinger, "On the Radiation of Sound from an Unflanged Circular Pipe," *Phys. Rev.*, **73**, pp. 383-406, 1948.

[3] K.A. Morris, *Introduction to Feedback Control*, Academic Press, 2001.

[4] K.A. Morris, "Noise Reduction Achievable by Point Control", *ASME Journal on Dynamic Systems, Measurement and Control*, Vol. 120, No. 2, 1998, pg. 216-223.

[5] P.M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, Inc., New York, 1953.

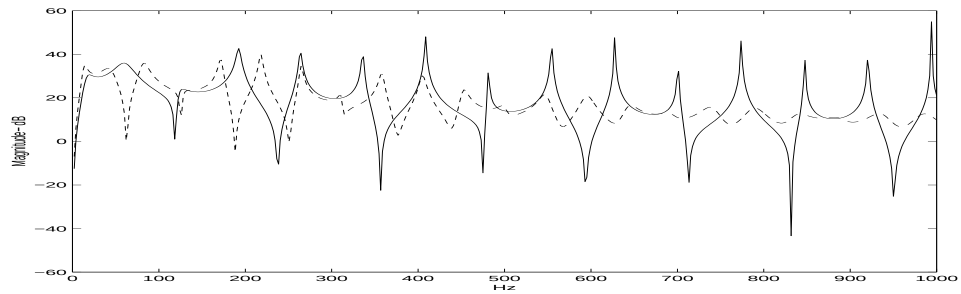
[6] E. Obasi, *Active Noise Control in a Duct using a Finite-Dimensional H_∞-Design*, M.Math Thesis, University of Waterloo, 2002.

[7] Vidyasagar, M., *Control System Synthesis: A Factorization Approach*, MIT Press, Cambridge, 1985.

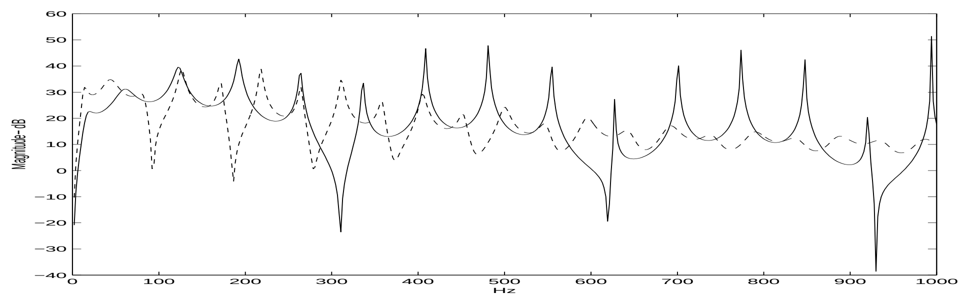
[8] B.J. Zimmer, S.P. Lipshitz, K.A. Morris and J. Vanderkooy, "An Improved Acoustic Model for Active Noise Control in a Duct", *ASME Journal of Dynamic Systems, Measurement and Control*, submitted.

L	duct length	3.54 m
a	duct radius	.101m
ρ	density of air	1.20 km/m^3
c	speed of sound in air	341 m/s
R_2	end impedence parameter	$\rho_0 c/\pi a^2$ mks ac. Ω
R_1	end impedence parameter	$0.504R_2$ mks ac.Ω
C	end impedence parameter	$5.44a^3/\rho_0 c^2$ m^5/N
M	end impedence parameter	$0.1952\rho_0/a$ kg/m^4
m_D	disturbance speaker's cone mass	.015 kg
k_D	disturbance speaker's cone suspension stiffness	810.87 N/m
R_{coil}	electrical resistance of voice coil (disturbance)	6.0 Ω
Bl	$B \cdot l$ disturbance product of magnetic voice coil motor (disturbance)	5.6 N/A
r_d	disturbance speaker's effective radius	.087 m
m_c	canceller speaker mass parameter	.006394 kg
k_c	canceller speaker stiffness parameter	673.7 N/m
R_c	electrical resistance of voice coil (canceller)	6.05 Ω
Bl_c	$B \cdot l$ disturbance product of magnetic voice coil motor (canceller)	5.68 N/A
d_c	canceller speaker's damping parameter	1.247 kg /s
r_c	canceller speaker's effective radius	.06 m

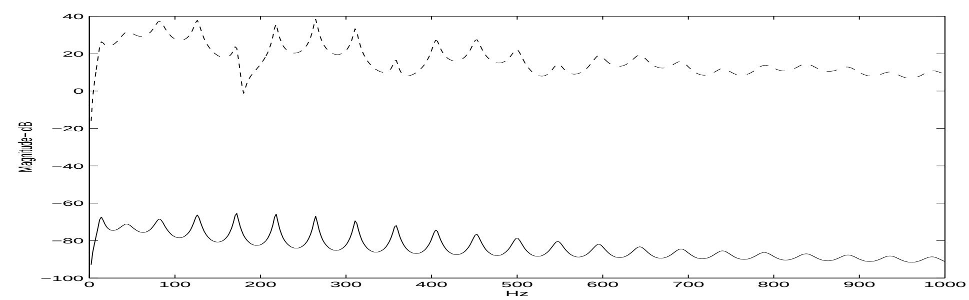
Table 1: Parameters



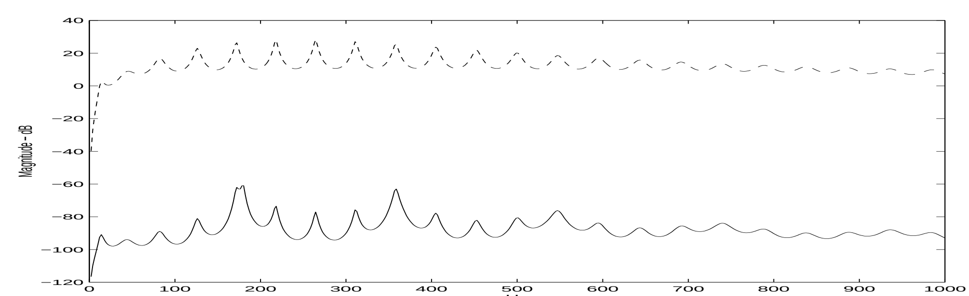
(a) $x = .25L$



(b) $x = .5L$

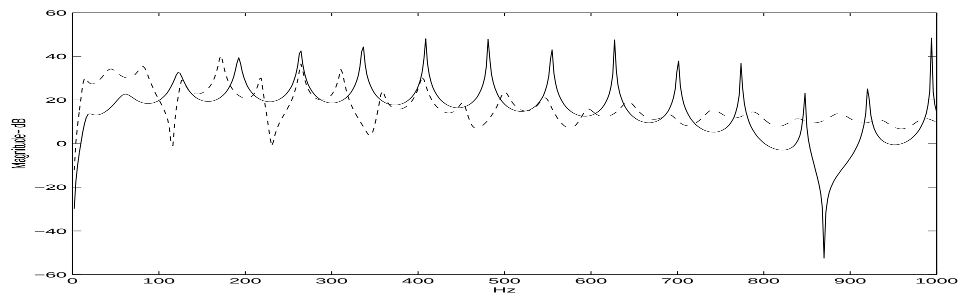


(c) $x = .75L$

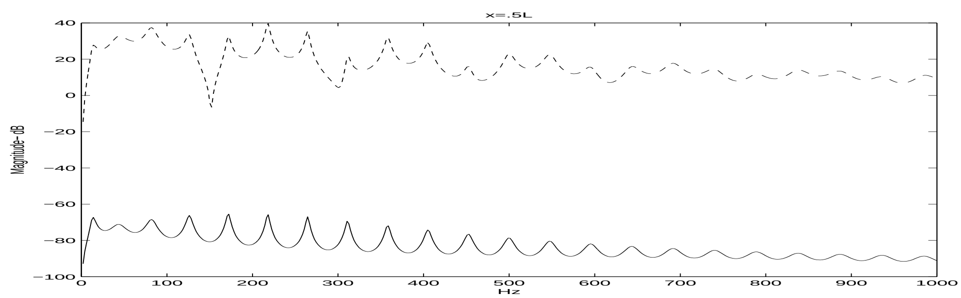


(d) $x = L$

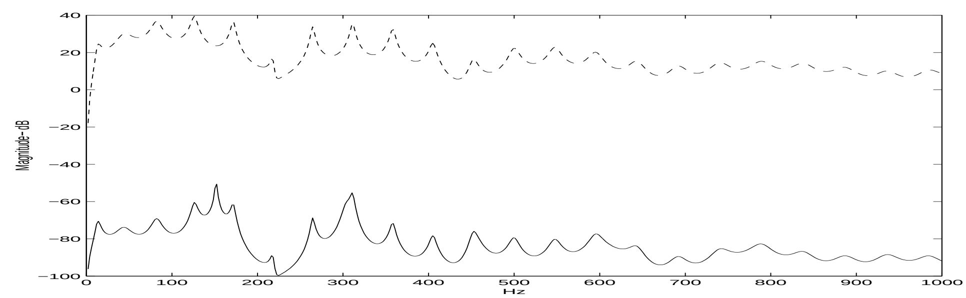
Figure 1: Uncontrolled (- -) and Controlled (—) Noise: $x_m = .5L$, $x_c = .65L$, $x_p = .75L$.



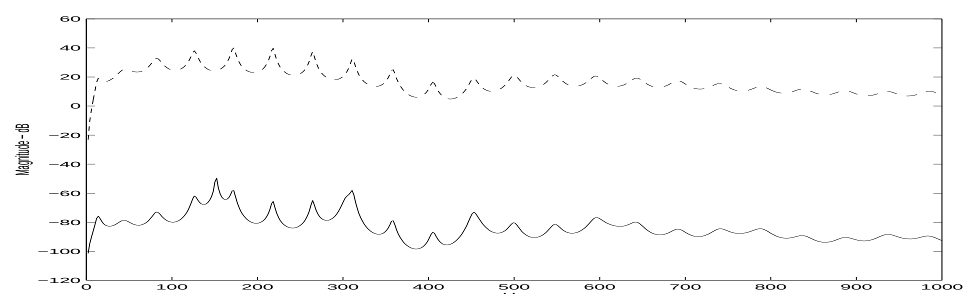
(a) $x = .6L$



(b) $x = .7L$



(c) $x = .8L$



(d) $x = .9L$

Figure 2: Uncontrolled (- -) and Controlled (—) Noise: $x_m = .5L$, $x_c = .65L$, $x_p = .7L$.

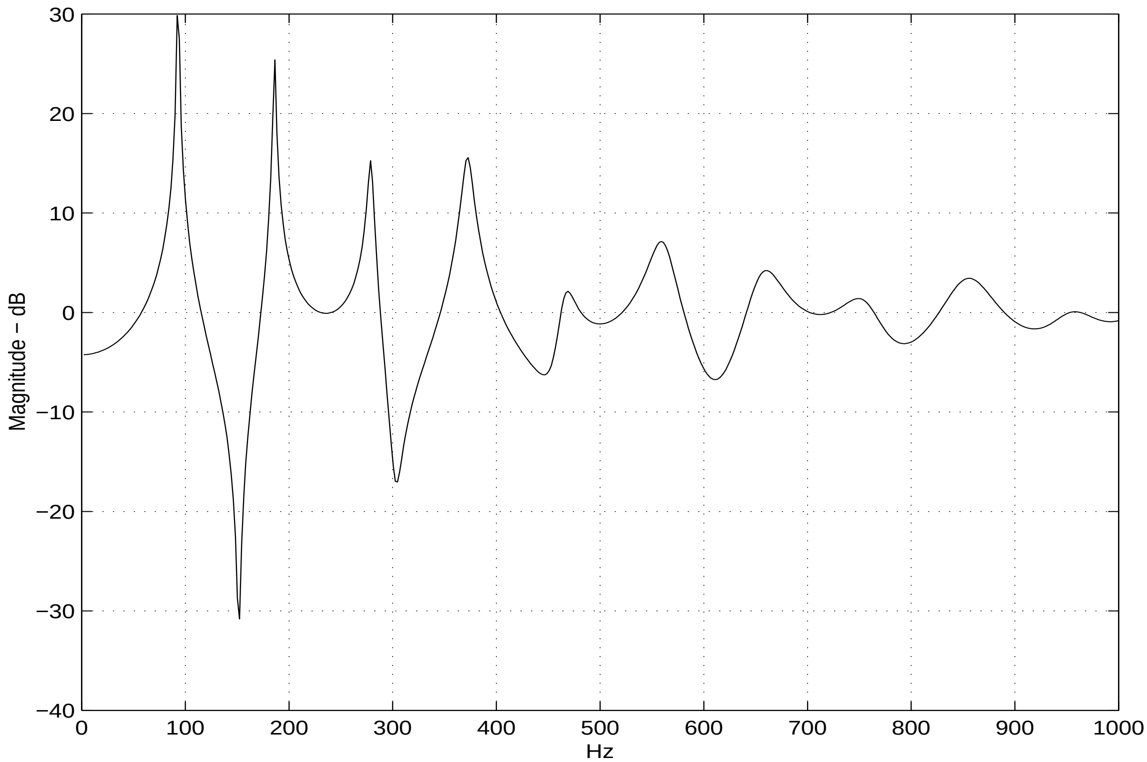


Figure 3: Robustness: $|G_c(x_m)(I + G_c(x_m)C)^{-1}|$, $x_m = .5L$, $x_c = .65L$, $x = .7L$, $\epsilon = 10^{-5}$