Positivity and dissipativity of oscillating diffusive filters, application to the stability of coupled systems.

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Abstract

As for pseudo-differential operators of diffusive type in continuous time, diffusive filters in discrete time have been introduced for the fractional difference filter and for other discretizations of fractional integrals. The impulse response of diffusive filters can be decomposed on a continuous family of geometric sequences with weight μ. Using this diffusive symbol μ in a diffusive *realization* – in the sense of systems theory – helps transforming a non-local in time difference equation into a first order difference equation on an infinite-dimensional state-space, endowed with a Hilbert structure, which allows for positivity, dissipativity, asymptotic and stability analysis.

In the present paper, this framework is extended to filters of the form: $\mathcal{H}(z) = \frac{1}{2}(1 - e^{i\theta}z^{-1})^{-\alpha} + \frac{1}{2}(1 - e^{-i\theta}z^{-1})^{-\alpha}$ for $|z| > 1$. The impulse response of such filters is oscillating with a slowly decreasing amplitude. We show that these filters are a continuous aggregation of *positive* oscillating filters. Hence, these filters are positive and have a dissipative realization. This enables to prove both the external and internal stabilities of some coupled systems involving a rational filter and such an oscillating diffusive filter in the feedback loop, thus extending the results when $\theta = 0$.
Keywords: diffusive representations, discrete time, positivity, dissipativity, Lyapunov functionals, asymptotic analysis, stability analysis.

1 Introduction and definition of oscillating diffusive filters

Discrete-time second order fractional filters such as $\mathcal{H}^{GB}(z) = (1 - 2\cos\theta\, z^{-1} + z^{-2})^{-\alpha}$ are generally used in time-series analysis to model long memory processes with seasonal effects (see [11] who has applied this methodology on sunspot data, exposed in [19]). There are mathematical reasons to use this kind of processes in such cases (see [10]).

The behaviour of similar continuous-time operator has been analysed (see [12], and [13] for a more involved study). An abstract framework can also be found in [15].

Continuous-time diffusive operators are defined as a continuous aggregation of purely damped dynamics, such an idea is not new: it has been used by [17] on fractional operators and on completely monotonic operators in [1], in [8] and in [18] thanks to Bernstein theorem (see [20]). Extensions of this idea to time-varying systems and to non-linear systems can be found in [14]. Various applications exist (see [16]). Some also deal with random processes [3].

In this paper, only causal filters are considered, which enables to use the same notation for transfer functions and for operators. Only real-valued filters are considered, even though the realization might use a complex-valued state.

The following definition of oscillating diffusive filter generalises the systems described in [13], it is very close to equation (45) of [5][1] (or to equation (25) of [5] in a continuous-time framework). The following definition can be reinterpreted as a special case of second order diffusive filters, as exposed in [15].

Definition 1.1. *$\mathcal{H}^\theta$ is an oscillating diffusive filter of angular frequency θ and diffusive symbol $\mu \in L^1(0,1)$ if (1.1) or (1.2) applies to $\mathcal{H}^\theta$.*

$$h_0 \text{ and } \forall n \geq 1 \;\; h_n \;=\; \Re e\left(\int_0^1 e^{in\theta}\mu(\rho)\rho^{n-1}\,d\rho\right); \tag{1.1}$$

$$\mathcal{H}(z) \;=\; h_0 \;+\; \frac{1}{2}e^{i\theta}z^{-1}\int_0^1 \frac{\mu(\rho)d\rho}{1-\rho e^{i\theta}z^{-1}} \;+\; \frac{1}{2}e^{-i\theta}z^{-1}\int_0^1 \frac{\bar{\mu}(\rho)d\rho}{1-\rho e^{-i\theta}z^{-1}}\;, \text{ for } |z|>1 \tag{1.2}$$

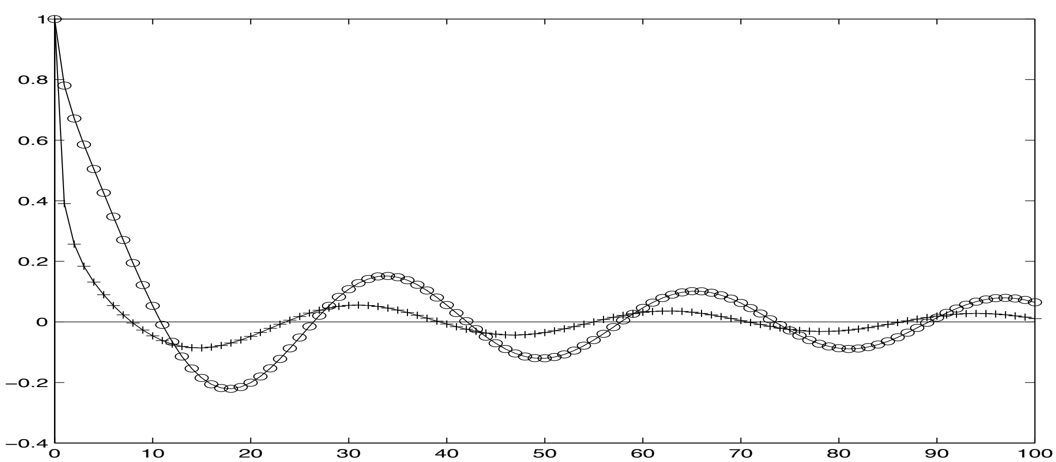

Figure 1: Impulse responses of $\mathcal{H}^{FI,\theta}$ $(+)$ and of $\mathcal{H}^{GB}$ $(\circ)$ for $\alpha = 0.4$ and $\theta = 0.2$. The graph shows that the amplitude of the oscillation is slowly decreasing.

When $\theta = 0$, this definition coincides with the classical definition of diffusive filters [5], an example of which is $\mathcal{H}^{FI} = (1-z^{-1})^{-\alpha}$ (i.e. FI stands for fractional integral). The two examples of oscillating diffusive filters exposed in [5] are

$$\mathcal{H}^{FI,\theta}(z) \;=\; \frac{1}{2}(1-e^{i\theta}z^{-1})^{-\alpha} + \frac{1}{2}(1-e^{-i\theta}z^{-1})^{-\alpha} \tag{1.3}$$

$$\mathcal{H}^{GB}(z) \;=\; (1-2\cos\theta\, z^{-1}+z^2)^{-\alpha} \tag{1.4}$$

[1]The exponent on $e^{i\theta}$ is here n instead of $n-1$ in equation (45) of [5]; in fact, this choice makes the sufficient condition for positivity more easy to write (see §2).

Definition 1.1 applies with $\mu^{FI}(\rho) = \frac{\sin(\alpha\pi)}{\pi}\rho^{\alpha}(1-\rho)^{-\alpha}$ and $\mu^{GB}(\rho) = 2\frac{\sin(\alpha\pi)}{\pi}\rho^{2\alpha}(1-\rho)^{-\alpha}(\rho - e^{-2i\theta})^{-\alpha}$. Both filters have analytical continuations with branching points in $z = 0$, $z = e^{i\theta}$ and $z = e^{-i\theta}$. As pointed out earlier in [12], these branching points entail the non-standard behavior. Indeed, their impulse responses are oscillating at angular frequency θ with a slowly decreasing amplitude. Their impulse responses are shown on figure 1.

Oscillating diffusive filters have infinite-dimensional realizations with a special Markov structure.

Definition 1.2. *The diffusive realization of an oscillating diffusive filter $\mathcal{H}^{\theta}$ with feedthrough h_0 is defined by:*

$$\begin{cases} \varphi_{n+1}(\rho) &= \rho e^{i\theta}\,\varphi_n(\rho) + v_n \quad \textit{with} \quad \rho \in \mathbb{I}\,, \varphi_0 \in \mathbb{H} \;\; \textit{and} \;\; n \geq 0 \\ y_n &= \Re e\left(\int_{\mathbb{I}} \mu(\rho) e^{i\theta}\varphi_n(\rho)\, d\rho\right) + h_0\, v_n \end{cases}$$
$$\textit{and}\ \mathbb{H} = \left\{\varphi \mid \ \textit{supp}(\varphi) \in \mathbb{I}\ \textit{and} \int_{\mathbb{I}} |\mu(\rho)\varphi^2(\rho)|\, d\rho < +\infty\right\}\ \textit{is the Hilbert space}\ \mathbb{H} = L^2(\mathbb{I}, |\mu| d\rho). \tag{1.5}$$

v_n and y_n are the real-valued input and output respectively, φ_n is a function of ρ mapping $\mathbb{I}$ into $\mathbb{C}$, (the state of the system). $\mathbb{I}$ is the smallest closed subset outside which μ is zero.

The following proposition proves that equation (1.5) is a realization of oscillating diffusive filters as defined in 1.1.

Proposition 1.1. *These equations can be expressed as an $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$-system where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are* continuous *linear operators, respectively from $\mathbb{H}$ to $\mathbb{H}$, from $\mathbb{R}$ to $\mathbb{H}$, from $\mathbb{H}$ to $\mathbb{R}$ and from $\mathbb{R}$ to $\mathbb{R}$. This system has internal asymptotic stability for the topology associated to $\mathbb{H}$ in that the free-evolution of the state φ_n vanishes for initial condition φ_0 in $\mathbb{H}$.*

Proof. The output y_n is the convolution of the impulse response h_n by the input v_n: $y_n = \sum_{k=0}^{n-1} h_{n-k}v_k + h_0 v_n$. With (1.1) and after exchanging the sum and the integral, this expression becomes: $y_n = \int_{\mathbb{I}} \mu(\rho) \sum_{k=0}^{n-1} \rho^{n-k} e^{i(n-k)\theta} u_k\, d\rho + h_0 u_n$. Let $\varphi_n(\rho) = \sum_{k=0}^{n-1} \rho^{n-k} e^{i(n-k)\theta} u_k$ and $\varphi_0(\rho) = 0$, thus (1.5) is proved. □

When $\theta = 0$, this realization coincides with the classical diffusive realization (cf [5]).

2 Positivity issue and energy stability of coupled system

Positivity means that the input-output relation $(v_n \mapsto y_n)$ of a causal filter satisfies $\sum_n v_n y_n \geq 0$. As for finite-dimensional filters, it has been shown in [4, appendix B] that for diffusive filters also, positivity reads $\forall |z| \geq 1\ \Re e\,(\mathcal{H}(z)) \geq 0$, under a technical assumption $\mu(\rho)\ln(\frac{1}{1-\rho}) \in L^1(0,1)$; this proof can be extended to oscillating diffusive filters.

The following proposition gives a sufficient condition on oscillating diffusive filters for positivity. The key idea of the proof is that this condition enables to express such filters as a continuous aggregation of *positive* filters.

Proposition 2.1. *Let $\mathcal{H}^\theta$ be an oscillating diffusive filter with diffusive symbol μ. If μ is a* real-valued *and* positive *function such that $h_0 \geq \int_0^1 \frac{\mu(\rho)}{1+\rho} d\rho$, then $\mathcal{H}^\theta$ is positive: $\forall |z| \geq 1$, $\Re e\left(\mathcal{H}^\theta(z)\right) \geq 0$.*

Proof. Simple algebraic computations lead to

$$\mathcal{H}^\theta(z) = h_0 - \int_0^1 \frac{\mu(\rho)}{1+\rho} d\rho + \frac{1}{2}\int_0^1 \frac{\mu(\rho)}{1+\rho} \frac{1+e^{i\theta}z^{-1}}{1-\rho e^{i\theta}z^{-1}} d\rho + \frac{1}{2}\int_0^1 \frac{\mu(\rho)}{1+\rho} \frac{1+e^{-i\theta}z^{-1}}{1-\rho e^{-i\theta}z^{-1}} d\rho \quad (2.6)$$

$\frac{1+z^{-1}}{1-\rho z^{-1}}$ are positive filters, (indeed when z lies outside the unit disk, $\arg(1+z^{-1})$ and $\arg(1-\rho z^{-1})$ lie both in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and have the same sign). Substituting $e^{i\theta}z$ to z and then substituting $e^{-i\theta}z$ to z proves that $\forall |z| > 1$, $\Re e\left(\frac{1+e^{i\theta}z^{-1}}{1-\rho e^{i\theta}z^{-1}}\right) \geq 0$ and $\Re e\left(\frac{1+e^{-i\theta}z^{-1}}{1-\rho e^{-i\theta}z^{-1}}\right) \geq 0$.
Hence (2.6) shows that $\mathcal{H}^\theta$ is a continuous aggregation of positive filters. □

$\mathcal{H}^{FI,\theta}$ fulfills the sufficient condition of proposition 2.1, whereas $\mathcal{H}^{GB}$ does not. Now both filters are positive (see their Nyquist diagrams on figure 2). This suggests that proposition 2.1 is quite restrictive, remark 2.1 enlights why it is difficult to extend the result.

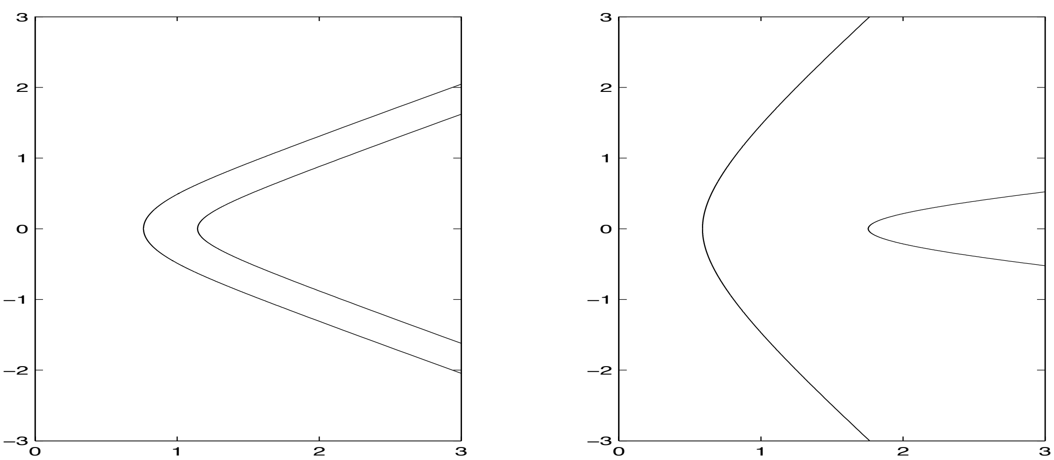

Figure 2: Nyquist diagram of $\mathcal{H}^{FI,\theta}$ on the left-hand side and of $\mathcal{H}^{GB}$ on the right-hand side.

Remark 2.1. *Extension of proposition 2.1 to an oscillating diffusive filter with a positive measure as diffusive symbol, $\mu(\rho) = 2\delta(\rho - 1)$, and feedthrough $h_0 = 1$, shows that $\mathcal{R}^\theta(z) = \frac{1}{2}\frac{1+e^{i\theta}z^{-1}}{1-e^{i\theta}z^{-1}} + \frac{1}{2}\frac{1+e^{i\theta}z^{-1}}{1-e^{i\theta}z^{-1}} = \frac{1-z^{-2}}{1-2\cos(\theta)z^{-1}+z^{-2}}$ is a positive filter and has two poles on the unit circle. In fact in [2], a necessary condition for such second order rational filters to be positive is derived from an asymptotic analysis: $(1-e^{i\theta}z^{-1})\mathcal{R}^\theta(z)$ must have a positive real limit when $z \to e^{i\theta}$ with $|z| \geq 1$. And indeed it has, since: $(1-e^{i\theta}z^{-1})\mathcal{R}^\theta(z) \to 1$.*

That pure oscillating filters can be positive is not specific to discrete time: $\frac{s}{s^2+\theta^2}$ *is also a positive causal oscillating operator. Indeed its inverse is* $s + \frac{\theta^2}{s}$*, which is the sum of two positive operators.*

Positivity is a property that can be used to prove energy stability of coupled systems. The classical positivity theorem (cf : [21]) assumes the input strict positivity of a system and the positivity of another, or it assumes the positivity of a system and the output strict positivity of the other. The following proposition is based on the same idea and requires weaker assumptions. Note that for a filter $\mathcal{H}$, input strict positivity means that there exists κ such that $\Re e\left(\mathcal{H}\right) \geq \kappa$ and output strict positivity means that there exists $\kappa > 0$ such that $\Re e\left(\mathcal{H}\right) \geq \kappa|\mathcal{H}(z)|^2$. Moreover the input strict positivity of $\mathcal{H}$ is equivalent to the output strict positivity of the inverse of $\mathcal{H}$, namely, $\Re e\left(\frac{1}{\mathcal{H}(z)}\right) \geq \kappa$.

Proposition 2.2. *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be two positive filters.*
If there exists $\mathbb{K}_1$ *and* $\mathbb{K}_2$ *a partition*[2] *of the exterior of the unit disc* $\mathbb{E}$ *such that* $z \in \mathbb{K}_1 \Rightarrow \Re e\left(\mathcal{H}_1(z)\right) \geq \kappa_1$ *(i.e. input strict positivity on* $\mathbb{K}_1$*) and* $z \in \mathbb{K}_2 \Rightarrow \Re e\left(\mathcal{H}_2(z)\right) \geq \kappa_2|\mathcal{H}_2(z)|^2$ *(i.e. output strict positivity on* $\mathbb{K}_2$*) with* κ_1 *and* κ_2 *any two positive constants.*
Then $\mathcal{H}^S = \frac{\mathcal{H}_2}{1+\mathcal{H}_1\mathcal{H}_2}$ *is energy-stable:* $|\mathcal{H}^S(z)| \leq \frac{1}{\min(\kappa_1,\kappa_2)}$.

Proof. It stems from simple computations on $\mathcal{H}^S = \frac{1}{\frac{1}{\mathcal{H}_2}+\mathcal{H}_1}$. □

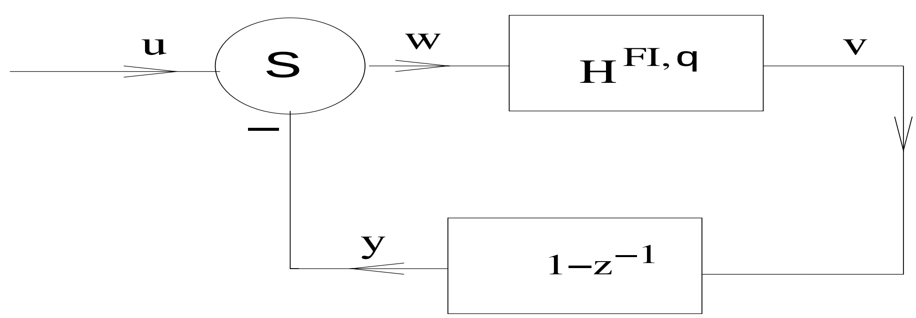

Figure 3: $\mathcal{H}^{S1}$ is the interconnection of two positive subsystems.

The coupled system $S1$ with $\mathcal{H}^{S1}(z) = \frac{\mathcal{H}^{FI,\theta}(z)}{1+(1-z^{-1})\mathcal{H}^{FI,\theta}(z)}$ is shown on figure 3. Proposition 2.2 proves the energy stability of $\mathcal{H}^{S1}$ because $(1 - z^{-1})$ is input strictly positive outside a neighborhood of $z = 1$ and $\mathcal{H}^{FI,\theta}(z)$ is input strictly positive on a small neighborhood of $z = 1$. Its impulse response is shown on figure 5, it is oscillating and the amplitude of the oscillations are slowly decreasing.

The coupled system $\mathcal{H}^{S2}(z) = \frac{\mathcal{R}^{\theta}(z)}{1+\mathcal{H}^{FI}(z)\mathcal{R}^{\theta}(z)}$ is shown on figure 4. Proposition 2.2 proves the energy stability of $\mathcal{H}^{S2}(z) = \frac{\mathcal{R}^{\theta}(z)}{1+\mathcal{H}^{FI}(z)\mathcal{R}^{\theta}(z)}$ because of the input strict positivity of $\mathcal{H}^{FI}$. Its impulse response is also shown on figure 5, it vanishes quickly. The reason is that $\mathcal{R}^{\theta}$ has

[2] $\mathbb{E} = \mathbb{K}_1 \bigcup \mathbb{K}_2$ and $\mathbb{K}_1 \bigcap \mathbb{K}_2 = \emptyset$

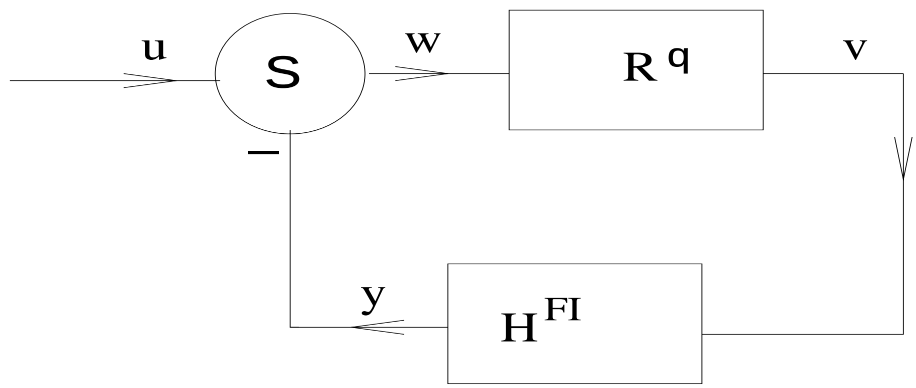

Figure 4: $\mathcal{H}^{S2}$ is the interconnection of two positive subsystems.

a zero of order two at $z = 1$ that kills the singularity of $\mathcal{H}^{FI}$. This issue is studied more in depth in [4, chapter 6] on coupling systems involving non-oscillating diffusive filters.

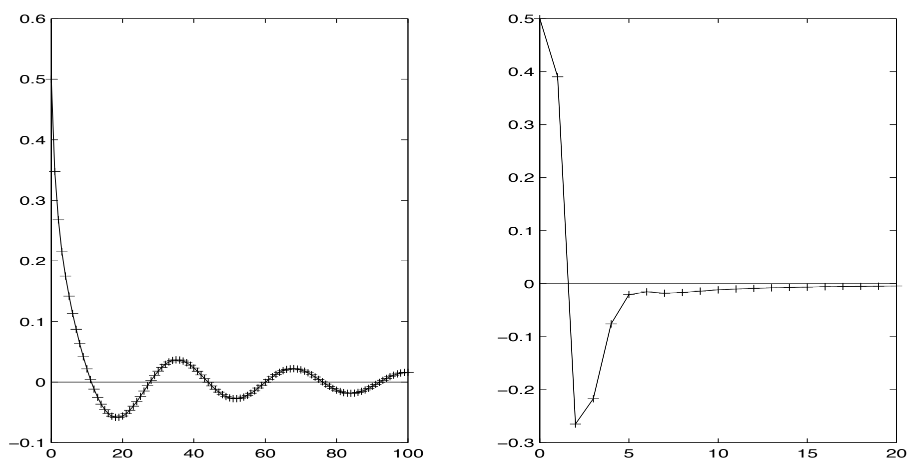

Figure 5: Impulse responses of two coupled systems $\mathcal{H}^{S1}(z) = \frac{\mathcal{H}^{FI,\theta}(z)}{1+(1-z^{-1})\mathcal{H}^{FI,\theta}(z)}$ on the left-hand side and $\mathcal{H}^{S2}(z) = \frac{\mathcal{R}^{\theta}(z)}{1+\mathcal{H}^{FI}(z)\mathcal{R}^{\theta}(z)}$ on the right-hand side. The impulse response of $\mathcal{H}^{S1}$ has slowly decreasing oscillations whereas the impulse response of $\mathcal{H}^{S2}$ vanishes quickly.

3 Dissipativity and internal stability of coupled systems

Dissipativity is related to a realization of a filter. It does imply positivity of the filter. The reverse is sometimes also true. For rational filters, the Kalman-Yacubovich-Popov lemma states the dissipativity of any minimal realization of any positive rational stable filter. The proof can be found in [2] or in [9]. For diffusive filters, [6] shows that positivity implies the dissipativity of the diffusive realization when the diffusive symbol is of constant sign. Now for oscillating diffusive filters, the following proposition shows the dissipativity of the diffusive realization when the sufficient condition of proposition 2.1 is fulfilled.

Proposition 3.1. *Let $\mathcal{H}$ be an oscillating diffusive filter with diffusive symbol μ and with feedthrough h_0.*
If μ is real-valued and if $e_0 = h_0 - \int_0^1 \frac{\mu(\rho)d\rho}{1+\rho} \geq 0$,
then the diffusive realization of $\mathcal{H}$ is dissipative: there exists a Lyapunov *functional which satisfies:*

a. V is positive and coercive: $V(\varphi) > 0$ when $\varphi \neq 0$, and $V(0) = 0$ and $V(\varphi) \to +\infty$ as $\|\varphi\|_{\mathbb{H}} \to +\infty$.

b. $V(\varphi_{n+1}) - V(\varphi_n) \leq v_n y_n$

In fact $V(\varphi) = \frac{1}{2}\int_{\mathbb{I}} \mu(\rho)|\varphi(\rho)|^2\, d\rho = \frac{1}{2}\|\varphi\|^2_{\mathbb{H}}$.

Sketch of the proof. Equation (2.6) shows that $\mathcal{H}^\theta - e_0$ is a continuous aggregation with weight $\frac{\mu(\rho)}{1+\rho}$ of second order dissipative filters, namely $\mathcal{H}^\theta_\rho(z) = \frac{1}{2}\frac{1+e^{i\theta}z^{-1}}{1-\rho e^{i\theta}z^{-1}} + \frac{1}{2}\frac{1+e^{-i\theta}z^{-1}}{1-\rho e^{-i\theta}z^{-1}}$. $V_\rho(\varphi_n) = \frac{1+\rho}{2}|\varphi_n|^2$ reveal their dissipativity. The expected result then follows. $\square$

3.1 Analysis of system $S1$

The following system is a minimal representation of the input-output relation $v_n \mapsto y_n = v_n - v_{n-1}$ with state $\mathbf{X}_n = v_{n-1}$

$$\begin{cases} \mathbf{X}_{n+1} = 0 \times \mathbf{X}_n + v_n \text{ with } \mathbf{X}_0 \in \mathbb{R} \\ y_n = -\mathbf{X}_n + v_n \end{cases}$$

This system is dissipative for $E(\mathbf{X}_n) = \frac{1}{2}\mathbf{X}_n^2$. Indeed $E(\mathbf{X}_{n+1}) - E(\mathbf{X}_n) = \frac{1}{2}v_n^2 - \frac{1}{2}v_{n-1}^2 \leq v_n^2 - v_n v_{n-1} = v_n y_n$.

A realization of $\mathcal{H}^{S1}$ is

$$\begin{cases} \varphi_{n+1} = \rho e^{i\theta}\varphi_n(\rho) + w_n \text{ with } \varphi_0 \in \mathbb{H} \\ v_n = \Re e\left(\int_0^1 \mu(\rho) e^{i\theta}\varphi_n(\rho)\, d\rho\right) + h_0 w_n \\ y_n = v_n - v_{n-1} \\ w_n + y_n = 0 \end{cases} \tag{3.7}$$

Figure 6 is a simulation of (3.7), it shows the evolution of the state for an initial condition φ_0. This figure illustrates that the sequence of functions φ_n vanishes on any compact subset contained in $[0, 1)$. However, $\varphi_n(1)$ does not tend towards zero.

3.2 Analysis of System $S2$

From remark 2.1 and definition 1.2, a minimal realization of $\mathcal{R}^\theta$ with state $\mathbf{X}_n$ is

$$\begin{cases} \mathbf{X}_{n+1} = e^{i\theta}\mathbf{X}_n + w_n \text{ with } \mathbf{X}_0 \in \mathbb{C} \\ v_n = \Re e\left(e^{i\theta}\mathbf{X}_n\right) + w_n \end{cases}$$

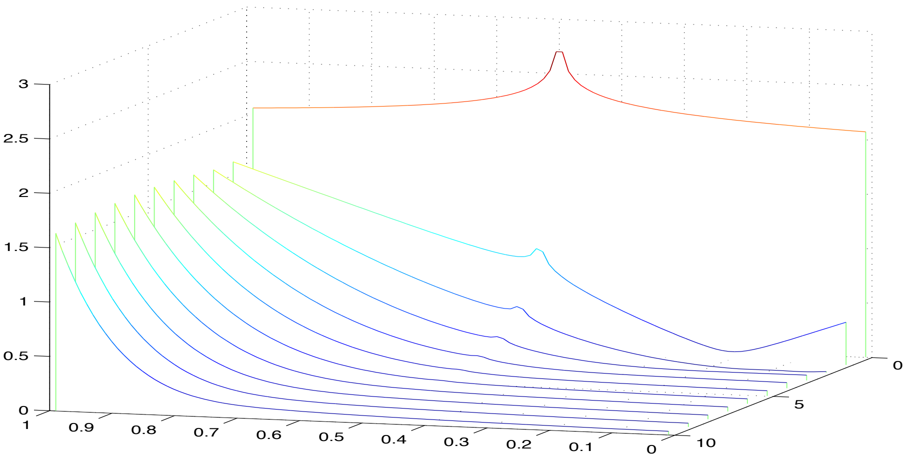

Figure 6: Simulation of a realization of $\mathcal{H}^{S1}$ for $\alpha = 0.4$, $\varphi_0(\rho) = 1 + |\rho - 0.5|^{-0.1}$ and $\theta = 0.2$. The sequence of functions $\varphi_n(\rho)$ is shown on the vertical axis with time n on the right and ρ on the left.

Application of proposition 3.1 to $\mu(\rho) = 2\delta(1-\rho)$ this system is dissipative for $\mathbf{E}(\mathbf{X}_n) = \frac{1}{2}|\mathbf{X}_n|^2$.

A realization of $\mathcal{H}^{S2}$ is

$$\begin{cases} \mathbf{X}_{n+1} = e^{i\theta}\mathbf{X}_n + w_n \text{ with } \mathbf{X}_0 \in \mathbb{C} \\ v_n = \Re e\left(e^{i\theta}\mathbf{X}_n\right) + w_n \\ \varphi_{n+1} = \rho\varphi_n + v_n \text{ with } \varphi_0 \in \mathbb{H} \\ y_n = \int \mu\varphi_n d\rho + h_0 v_n \\ w_n + y_n = 0 \end{cases} \tag{3.8}$$

Figure 7 is a simulation of (3.8), it shows the evolution of the state for an initial condition φ_0. The sequence of functions φ_n vanishes on any compact subset contained in $[0,1)$ at a geometric speed. Unlike on figure 6, $\varphi_n(1)$ also tends to zero but with a bigger speed.

The following theorem proves the dissipativity of both realizations (3.7) and (3.8), thanks to dissipativity of $\mathcal{H}^{FI,\theta}$, $1 - z^{-1}$ and $\mathcal{H}^{FI}$, $\mathcal{R}^\theta$.

Theorem 3.1. *Let $\mathcal{H}_\Phi$ be an interconnection between a dissipative oscillating diffusive filter with state φ_n (i.e. a function) and a dissipative finite dimensional filter with state $\mathbf{X}_n$ (a vector). The state of $\mathcal{H}_\Phi$ is $\Phi_n = (\mathbf{X}_n, \varphi_n)$. The internal stability of $\mathcal{H}_\Phi$ is revealed by the Lyapunov function $\mathcal{E}(\Phi_n) = V(\varphi_n) + \mathbf{E}(\mathbf{X}_n)$ in the sense that:*

$$\forall\epsilon > 0,\ \exists\gamma,\ \textit{such that } \|\Phi_0\|_{\mathbb{H}_\Phi} \leq \gamma \Rightarrow \forall n \geq 0,\ \|\Phi_n\|_{\mathbb{H}_\Phi} \leq \epsilon$$

where $\mathbb{H}_\Phi$ is the extended Hilbert space inferred from $\|\Phi\|^2_{\mathbb{H}_\Phi} = \|\varphi\|^2 + \mathbf{X}^T\mathbf{X}$.

Sketch of the proof. $\mathcal{E}(\Phi_n)$ decreases along the free trajectories of $\mathcal{H}_\Phi$: $\mathcal{E}(\Phi_{n+1}) - \mathcal{E}(\Phi_n) = V(\varphi_{n+1}) - V(\varphi_n) + \mathbf{E}(\mathbf{X}_{n+1}) - \mathbf{E}(\mathbf{X}_n) \leq 0$. $\square$

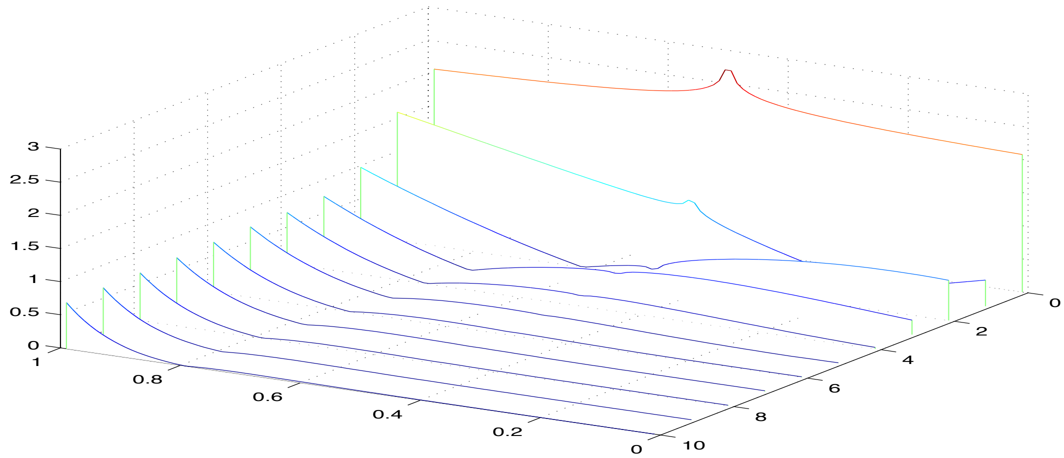

Figure 7: Simulation of a realization of $\mathcal{H}^{S2}$ for $\alpha = 0.4$, $\varphi_0(\rho) = 1 + |\rho - 0.5|^{-0.1}$ and $\theta = 0.2$. The sequence of functions φ_n is shown on the vertical axis with n on the right and ρ on the left.

4 Conclusion and prospects

A sufficient condition on diffusive symbols of oscillating diffusive filters has been stated. It ensures positivity and dissipativity of these systems. It also reveals the input-output energy stability and the internal stability of such systems, when coupled with dissipative finite dimensional systems.

The sufficient condition seems a little restrictive, since it concerns only real-valued diffusive symbols. Now the cut joining the branching points $z = 0$ and $z = e^{i\theta}$ needs not be a straight line. Concerning $\mathcal{H}^{GB}$, it is conjectured that the cut can be chosen so that the corresponding diffusive symbol is positive and fulfills the sufficient condition.

Other prospects are to extend the results of [6], [7] to oscillating diffusive filters and to see whether these claims are true:

- The diffusive realization of a strictly positive diffusive filter is asymptotically stable under a technical condition.
- A positive oscillating diffusive filter coupled with a positive rational filter is the sum of a stable rational filter and an other oscillating diffusive filter that is BIBO stable. This coupled system is therefore BIBO stable.
- In the realization of these coupled systems, the asymptotic behaviour of the input and output of the diffusive filters can be determined by the asymptotics of μ and φ_0.

In [4], there is another result that may also be extended.

- Gaussian white noise filtered by oscillating diffusive filters produce long-memory processes with seasonal effect: the autocorrelation is oscillating with slowly decreasing amplitude.

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