Can positive pseudo-differential operators of diffusive type help stabilize unstable systems?

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Abstract

Diffusive representations of positive pseudo-differential operators (PDOs) can often be used in the analysis of coupled systems, in which their dissipative realization plays a major role. Now, some coupled systems involving a negative PDO can still be stable. Conversely, some unstable systems can be stabilized by positive PDOs, thus requiring some more analytical knowledge: such striking examples will be presented, either in continuous time or in discrete time.

1 Introduction

The impulse response of a pseudo-differential operators $\mathcal{D}$ of diffusive type can be decomposed on a continuous family of purely damped exponentials with weight $\mu_{\mathcal{D}}$. A diffusive *realisation* helps transforming a non-local in time pseudo-differential equation into a first order differential equation on a Hilbert state-space, which allows for stability analysis (see [8, 18, 10, 15]).

This approach reveals useful for both theoretical and numerical treatment of pseudo-differential equations (not only fractional differential ones, as in [10]), even time-varying and non-linear ones (see [16]). An analogous framework with interesting properties alike can be proposed in a discrete-time context, where the key idea is to decompose long-range time-series on a continuous family of purely damped geometric sequences, see e.g. [5, 4, 6].

In the present paper, it will be focused on the spectral analysis of pseudo-differential or pseudo-difference equations, such as those presented in [5, 11], the analysis of which concerns the poles of an infinite-dimensional system of the following form:

$$\partial^2_{tt} X + \varepsilon_1\, \partial_t X + \mathcal{D}_1(\partial_t X) + \varepsilon_2\, \mathcal{A}\partial_t X + \mathcal{D}_2\,(\mathcal{A}\partial_t X) + \mathcal{A}X = 0 \tag{1.1}$$

where $\mathcal{D}_i = \mathcal{D}_{\mu_i} + \frac{d}{dt}\mathcal{D}_{\nu_i}$ are *positive* PDOs (i.e. μ_i and ν_i are positive measures, that characterize the PDO of symbol $\widehat{\mathcal{D}_i}(s) = \int_0^{+\infty} \frac{d\mu_i(\xi) + s\, d\nu_i(\xi)}{s+\xi}$, for $\Re e(s) > 0$) and $\varepsilon_i \geq 0$.

In the case where $\mathcal{A}$ is a Riesz spectral operator, as defined in [3, chap. 2, sec. 3], the solution X can be decomposed onto a Riesz basis $\{\varphi_n\}_{n\geq 0}$; then each time-component $w_n(t)$ satisfies the following pseudo-differential equation:

$$\ddot{w}_n + [\varepsilon_1\, \dot{w}_n + \mathcal{D}_1(\dot{w}_n)] + \lambda_n\, [\varepsilon_2\, \dot{w}_n + \mathcal{D}_2(\dot{w}_n)] + \lambda_n\, w_n = 0 \tag{1.2}$$

where λ_n are the eigenvalues of the Riesz-spectral operator $\mathcal{A}$.

Thus, in order to analyze the dynamics of (1.2), we are naturally led to the following characteristic equation for the poles $s_n \in \mathbb{C}\backslash\mathbb{R}^-$:

$$ {s_n}^2 + \left[\varepsilon_1 + \widehat{\mathcal{D}_1}(s_n)\right]\, s_n + \lambda_n \left[\varepsilon_2 + \widehat{\mathcal{D}_2}(s_n)\right]\, s_n + \lambda_n = 0 \tag{1.3} $$

where λ_n are the complex eigenvalues of the Riesz-spectral operator $\mathcal{A}$. If $\mathcal{A}$ is positive and self-adjoint, all the λ_ns are real and positive numbers, otherwise they can be complex-valued, and even with negative real-part; in which cases, no straightforward energy analysis is available.

For solving (1.3), a distinction will be made between cases where the $\widehat{\mathcal{D}_i}(s)$ are known *analytically* (such as for fractional differential equations, see [17, 14, 10]) or not. Similar examples will be treated in discrete time, with the same distinction.

The main interest of such an analytical knowledge is that it enables to prove the stabilizability of *unstable* systems with*out* using the positivity properties of the PDOs; on the contrary, the characteristic equation (1.3) will be carefully analyzed and even solved *explicitly* in many cases.

The paper is organized as follows:

- the definitions and properties of PDOs of diffusive type are recalled in § 2,
- the use of positive diffusive PDOs in coupled systems is explained in § 3, with many applications where introductory examples of the form (1.2) are used to analyze systems of the form (1.1),
- finally, in § 4, characteristic equations of the form (1.3) will be solved to show that some negative diffusive PDOs can be stability preserving on the one hand, and that some positive diffusive PDOs can stabilize unstable systems on the other hand; in this latter section, striking examples will be treated thoroughly, both in continuous and discrete time.

2 What are PDOs of diffusive type?

Let us introduce the so-called diffusive realizations of PDOs of negative asymptotic order (such as fractional integrals) and PDOs of potive asymptotic order (such as fractional derivatives). These infinite-dimensional formulations will help us prove some *positivity* and also *dissipativity* properties which will be of major help in the study of energy questions.

2.1 Diffusive realizations for PDOs of order less than 0

Let μ a positive measure on $\mathbb{R}^+$, with condition $\int_0^\infty \frac{d\mu(\xi)}{1+\xi} < \infty$.

Consider the infinite-dimensional dynamical system with input v, ouput y and state φ of finite energy (namely $E_\varphi(t) = \frac{1}{2}\int_0^{+\infty} \varphi(\xi,t)^2\, d\mu(\xi) < \infty$):

$$\partial_t \varphi(\xi,t) \;=\; -\,\xi\;\varphi(\xi,t)\,+\,v(t); \quad \varphi(\xi,0)\,=\,0, \quad \forall\,\xi \in \mathbb{R}^+ \tag{2.2}$$

$$y(t) \;=\; \int_0^{+\infty} \varphi(\xi,t)\, d\mu(\xi) \tag{2.3}$$

We have the following properties :

Theorem 2.1. *The input-output relation for system (2.2)–(2.3) is* $y = \mathcal{D}_\mu v = h_\mu \star v$, *with impulse response* h_μ *and transfer function* $\widehat{\mathcal{D}_\mu}$*:*

$$h_\mu(t) \;=\; \int_0^{\infty} e^{-\xi t}\, d\mu(\xi) \tag{2.4}$$

$$\widehat{\mathcal{D}_\mu}(s) \;=\; \int_0^{\infty} \frac{d\mu(\xi)}{s+\xi} \quad \forall s,\, \Re e(s) > 0 \tag{2.5}$$

Moreover, we have the positivity *property of the input-output relation:*

$$\forall\, T > 0, \quad < y, v >_T = \int_0^T y(t)\, v(t)\, dt \ge 0 \tag{2.6}$$

and a corresponding dissipativity *property of the realization (2.2)–(2.3):*

$$\frac{d}{dt} E_\varphi(t) = y(t)\, v(t) - \int_0^{\infty} \xi \varphi(\xi,t)^2\, d\mu(\xi) \le y(t)\, v(t) \tag{2.7}$$

Proof. see [15, 10, 1]. Note that functional spaces must be specified for these infinite-dimensional dynamical systems to make sense; in particular, a classical $V \subset H \subset V'$ framework is needed. □

Example 2.1. *For* $0 < \beta < 1$, *with* $\mu_\beta(\xi) = \frac{\sin(\beta\pi)}{\pi}\,\xi^{-\beta}$, *as* density *of the measure* μ *w.r.t the Lebesgue measure on* $\mathbb{R}^+$, *we get the PDO* $\mathcal{D} = I^\beta$ *(fractional integral of order* β*), with impulse response* $\frac{1}{\Gamma(\beta)} t_+^{\beta-1}$ *and transfer function* $s^{-\beta}$, *a PDO of asymptotic order* $-\beta \in (-1, 0)$.

2.2 Diffusive realizations for PDOs of order less than 1

Let ν a positive measure on $\mathbb{R}^+$, with condition $\int_0^\infty \frac{d\nu(\xi)}{1+\xi} < \infty$.

Now consider the infinite-dimensional dynamical system with input v, ouput z and state ψ of finite energy (namely $E_\psi(t) = \frac{1}{2}\int_0^{+\infty} \xi\,\psi(\xi,t)^2\, d\nu(\xi) < \infty$):

$$\partial_t \psi(\xi,t) \;=\; -\,\xi\;\psi(\xi,t)\,+\,v(t); \quad \psi(\xi,0)\,=\,0, \quad \forall\,\xi \in \mathbb{R}^+ \tag{2.8}$$

$$z(t) \;=\; \int_0^{+\infty} \partial_t\psi(\xi,t)\, d\nu(\xi) = \int_0^{+\infty} \left(-\xi\psi(\xi,t) + v(t)\right)\, d\nu(\xi) \tag{2.9}$$

We have the following properties :

Theorem 2.2. *The input-output relation for system (2.8)–(2.9) is* $z = \frac{d}{dt}\mathcal{D}_\nu v = \frac{d}{dt}(h_\nu \star v)$, *in the sense of distributions, with transfer function* $s\,\widehat{\mathcal{D}_\nu}(s)$.

Moreover, we have the positivity *property of the input-output relation:*

$$\forall\, T > 0, \quad < z, v >_T = \int_0^T z(t)\, v(t)\, dt \geq 0 \tag{2.10}$$

and a corresponding dissipativity *property of the realization (2.8)–(2.9):*

$$\frac{d}{dt} E_\psi(t) = z(t)\, v(t) - \int_0^\infty (\partial_t \psi(\xi, t))^2 \, d\nu(\xi) \leq z(t)\, v(t) \tag{2.11}$$

Proof. see [15, 10, 1]. Note that functional spaces must be specified for these infinite-dimensional dynamical systems to make sense; in particular, a classical $V \subset H \subset V'$ framework is needed. □

Remark 2.1. *In (2.9), the two parts can* not *be evaluated separately, otherwise the integrals would both diverge. This is very well understood in the appropriate functional framework.*

Example 2.2. *For* $0 < \alpha < 1$, *with* $\nu_\alpha(\xi) = \mu_{1-\alpha}(\xi)$ *as* density *of the measure* ν *w.r.t the Lebesgue measure on* $\mathbb{R}^+$, *we get the PDO* D^α *(fractional derivative of order* α*), with distributional impulse response* $fp(\frac{1}{\Gamma(-\alpha)} t_+^{-\alpha-1})$ *and transfer function* s^α, *a PDO of asymptotic order* $\alpha \in (0,\, 1)$.

Remark 2.2. *Diffusive realisations of fractional integrals* I^β *and derivatives* D^α, *and other pseudo-differential operators, are very important both from theoretical and numerical viewpoints:*

- *from a theoretical aspect, these formulations help understand the very nature of fractional integrals and derivatives (as a particular case of long–memory operators), they also provide natural and straightforward proofs for properties which would otherwise not be so obvious;*

- *from a practical aspect, these formulations help define stable numerical schemes for the approximation of the solution of such systems.*

In § 3, we will use the dissipativity properties (2.7) and (2.11) in order to find sufficient stability conditions for the examples that will be considered. Moreover diffusive realizations of fractional integrals or derivatives through a state of infinite dimension φ or ψ help define a natural state space, together with appropriate energy functionals, which are both most useful for stability considerations.

3 Using positive diffusive PDOs in coupled systems

We now examine a variety of examples where both positivity and dissipativity are useful. The notations are those introduced in the preceding section.

3.1 a fractional diffusion equation

We begin with a simple finite-dimensional example, written with the PDO D^α, but which is treated very much in the same way for other positive PDOs of diffusive type.

Example 3.1. *The dynamical system $D^\alpha w + \omega_0\, w = u$ can be proved to be internally stable, when $\omega_0 > 0$, using $v = w$ in § 2.2, thanks to:*

$$\dot{E}_\psi(t) = w\, D^\alpha w - \int_0^\infty (\partial_t \psi(\xi,t))^2\, d\nu(\xi) = u(t)\, w(t) - \omega_0\, w^2(t) - \int_0^\infty (\partial_t \psi(\xi,t))^2\, d\nu(\xi) \quad (3.3)$$

Then as soon as the input u has stopped, the energy $E_\psi(t)$ starts decreasing; finally, LaSalle invariance principle will help prove that the state ψ of the system goes to 0 as $t \to +\infty$. This proves internal stability; from which external stability might also be proved, thanks to the very simple nature of the infinite-dimensional dynamics.

Consider the following fractional diffusion equation on $\Omega \subset \mathbb{R}^n$:

$$\partial_t^\alpha X - \Delta X = u \quad (3.4)$$

Ω is either the whole space, or a compact subset, in which case some boundary conditions must be prescribed on $\partial\Omega$; they can be of two kinds:

- *conservative* boundary conditions of Dirichlet type ($X(x,t) = u_b(x,t)$) or Neumann type ($\partial_n X(x,t) = u_b(x,t)$);
- *dissipative* boundary conditions of Robin type ($X(x,t) + r(x)\, \partial_n X(x,t) = u_b(x,t)$, with $r(x) > 0$ on $\partial\Omega$) or impedance type ($\partial_t X(x,t) X(x,t) + k(x)\, \partial_n X(x,t) = u_b(x,t)$, with $k(x) > 0$ on $\partial\Omega$); see the second item in problem 3.1.

In each of these cases, the Green formula $\int_\Omega Y\, \Delta X\, dx = -\int_\Omega \nabla Y . \nabla X\, dx + \int_{\partial\Omega} Y\, \partial_n X\, d\sigma$ will be applied carefully and help choose the appropriate functional spaces on Ω.

The stability analysis is then performed on this system using the following energy functional:

$$\mathcal{E}(t) = \frac{1}{2} \int_0^{+\infty} \xi\, \|\psi(.,\xi,t)\|^2_{L^2(\Omega)}\, d\nu(\xi) = \int_\Omega E_\psi(x,t)\, dx \quad (3.5)$$

where $v(x,t) = X(x,t)$ is the input of the diffusive system, and applying the methodology developped in example 3.1.

3.2 a fractionally damped diffusion equation

We begin with a simple finite-dimensional example.

Example 3.2. *The dynamical system $\dot{w} + p\, D^\alpha w + \omega_0\, w = u$ can be proved to be internally stable, when $p, \omega_0 > 0$, using $v = w$ in § 2.2, thanks to the balance on $E(t) = \frac{1}{2} w^2(t) + p\, E_\psi(t)$:*

$$\dot{E}(t) = u(t)\, w(t) - \omega_0\, w^2(t) - \int_0^\infty (\partial_t \psi(\xi,t))^2\, d\nu(\xi) \quad (3.6)$$

Then as soon as the input u has stopped, the global energy $E(t)$ starts decreasing; finally, LaSalle invariance principle will help prove that the augmented state (w, ψ) of the system goes to $(0,0)$ as $t \to +\infty$. This proves internal stability.

Consider the following fractionally damped diffusion equation on $\Omega \subset \mathbb{R}^n$:

$$\partial_t X + p\,\partial_t^\alpha X - \Delta X = u \tag{3.7}$$

The problem can be tackled with the following global energy functional:

$$\mathcal{E}(t) = \frac{1}{2}\|X(.,t)\|^2_{L^2(\Omega)} + \frac{p}{2}\int_0^{+\infty} \xi\,\|\psi(.,\xi,t)\|^2_{L^2(\Omega)}\,d\nu(\xi) \tag{3.8}$$

where $v(x,t) = X(x,t)$ is the input of the diffusive system; then, the methodology developped in example 3.2 is applied.

3.3 a fractional wave equation

We begin with a simple finite-dimensional example, written with the PDOs D^α and I^β, but which is treated very much in the same way for other positive PDOs of diffusive type.

Example 3.3. *The dynamical system $D^\alpha \dot{w} + \varepsilon\,\dot{w} + q\,I^\beta \dot{w} + \omega_0^{1+\alpha}\,w = u$ can be proved to be internally stable, when $\varepsilon, q > 0$, using $v = \dot{w}$ in both § 2.1 and § 2.2, thanks to the balance on $E(t) = \frac{\omega_0^{1+\alpha}}{2} w^2(t) + E_\psi(t) + q\,E_\varphi(t)$:*

$$\dot{E}(t) = u(t)\,\dot{w}(t) - \varepsilon\,\dot{w}^2(t) - \int_0^\infty (\partial_t \psi(\xi,t))^2\,d\nu(\xi) - q\int_0^\infty \xi(\varphi(\xi,t))^2\,d\mu(\xi) \tag{3.9}$$

Then as soon as the input u has stopped, the global energy $E(t)$ starts decreasing; finally, LaSalle invariance principle will help prove that the augmented state (w, φ, ψ) of the system goes to $(0,0,0)$ as $t \to +\infty$. This proves internal stability.

Consider the following fractional wave equation on $\Omega \subset \mathbb{R}^n$:

$$\partial_t^{1+\alpha} X + \varepsilon\,\partial_t X + q\,\partial_t^{1-\beta} X - \Delta X = u \tag{3.10}$$

It can be analyzed with the following global energy functional:

$$\mathcal{E}(t) = \frac{1}{2}\|\nabla X(.,t)\|^2_{L^2(\Omega)} + \int_\Omega E_\psi(x,t)\,dx + q\int_\Omega E_\varphi(x,t)\,dx \tag{3.11}$$

where $v(x,t) = \partial_t X(x,t)$ is the input of the diffusive system; the principle developped in example 3.3 are then applied. Note that the Robin and impedance boundary conditions need a special care in the analysis (see the second item in problem 3.1).

3.4 a fractionally damped wave equation

The simple viscoelastically damped oscillator, already presented in [10], is first recalled.

Example 3.4. *Consider the following second order stable ($\varepsilon > 0$) system perturbed ($pq \neq 0$) by some fractional dampings of order $1+\alpha \in]1,2[$ and $1-\beta \in]0,1[$:*

$$\ddot{w} + p\,D^{\alpha}\dot{w} + \varepsilon\,\dot{w} + q\,I^{\beta}\dot{w} + \omega^2\,w = u \tag{3.12}$$

The system is asymptotically internally stable $\forall p > 0$ and $\forall q > 0$, which is proved thanks to the global energy functional of the augmented system:

$$E(t) = \frac{1}{2}\dot{w}^2(t) + \frac{\omega^2}{2}w^2(t) + p\,E_{\psi}(t) + q\,E_{\varphi}(t) \tag{3.13}$$

with $v = \dot{w}$ as input for both § 2.1 and § 2.2. Some computations then lead to:

$$\dot{E}(t) = \dot{w}u - \varepsilon\,\dot{w}^2 - p\int_0^{+\infty}(\partial_t\psi)^2(\xi,t)\,d\nu(\xi) - q\int_0^{+\infty}\xi\,\varphi^2(\xi,t)\,d\mu(\xi) \tag{3.14}$$

Then, as soon as the input u has stopped, the global energy starts decreasing; finally, LaSalle invariance principle will help prove that the augmented state $(w,\dot{w},\psi,\varphi)$ of the global system goes to $(0,0,0,0)$ as $t \to +\infty$.

Consider the following fractional wave equation on $\Omega \subset \mathbb{R}^n$, (a particular 1-dimensional example of which has been studied in [12]):

$$\partial^2_{tt}X + p\,\partial_t^{1+\alpha}X + \varepsilon\,\partial_t X + q\,\partial_t^{1-\beta}X - \Delta X = u \tag{3.15}$$

Such systems can be recast in the framework presented in [18], where diffusive representations were first introduced (see also [8] on completely monotonic kernels and the use of Bernstein theorem). The stability can be analyzed with the following global energy functional:

$$\mathcal{E}(t) = \frac{1}{2}\|\partial_t X\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla X(.,t)\|^2_{L^2(\Omega)} + p\int_{\Omega}E_{\psi}(x,t)\,dx + q\int_{\Omega}E_{\varphi}(x,t)\,dx \tag{3.16}$$

with $v(x,t) = \partial_t X(x,t)$ as input of the diffusive systems; the elementary principles of example 3.4 are then translated in an appropriate manner.

Remark 3.1. *The four applications developped in this section have been presented with* uniform *coefficients versus the space variable x and with* constant *coefficients with respect to the time variable t; moreover, all of them were* linear *systems.*

A treatment of non-homogeneous *systems is straightforward, provided the space-varying coefficients $c(x)$ satisfy $0 < m_c \leq c(x) \leq M_c$; thus, the Laplacian and its boundary conditions can be taken as non-homogeneous, but also the fractional orders $\alpha(x)$, $\beta(x)$ or more generally the measures $\nu(x,\xi)$, $\mu(x,\xi)$ can be taken as non-homogeneous.*

A treatment of time-varying *or* non-linear *systems is certainly more involved, and we refer to [16] for these extensions.*

Problem 3.1. *In this section, principles have been derived so as to show the variety of non-standard dynamics that can be analyzed, but more involved technical aspects that have not been treated must not be overlooked. Let us cite the most important ones:*

1. *The Sobolev spaces on* Ω *must be chosen accordingly to the boundary conditions.*

2. *When the boundary conditions are of Robin or impedance type, some extra boundary term do appear in the energy balance, and must be taken into account properly: either as extra positive term in the definition of the global energy* $\mathcal{E}$*, or as extra negative term in the energy balance* $\frac{d}{dt}\mathcal{E}$*; for Robin type it reads* $\frac{1}{2}\int_{\partial\Omega} r(x)(\partial_n X(x,t))^2\, d\sigma$*, whereas for impedance type it reads* $\frac{1}{2}\int_{\partial\Omega} k(x)(\partial_n X(x,t))^2\, d\sigma$*.*

3. *When using LaSalle invariance principle on infinite-dimensional state spaces, a technical condition of* precompactness *of the trajectories in the energy space must be fulfilled, which is generally not easy to check properly. Especially the non-linear case would be even more difficult to tackle than the linear case.*

4. *Then, asymptotic internal stability need not imply external stability for infinite-dimensional systems in general, and a careful study must be carried out; however, the very simple nature of the diffusive realizations (well-posedness, approximate controllability and approximate observability) should help to prove the external stability, at least on the four introductory examples.*

4 Using positive diffusive PDOs to stabilize unstable systems

From this variety of applications and introductory examples, it could be thought that *positive* coupling is a necessary condition for stability; we will first show that it is not true, on a very simple example. Then, we will give some examples in which a positive PDO can stabilize an unstable system, provided some conditions are fulfilled.

4.1 Positivity of the PDO is not necessary for stability

Let us take the simplest case study in example 3.2, that is $\dot{w} + p\, D^{\frac{1}{2}} w + w = u$, the transfer function of which is given by $\mathcal{H}_p(s) = (s + p\sqrt{s} + 1)^{-1}$. It is already clear that, $\forall p \geq 0$, there are no poles in the closed right-half plane. Moreover, as the fractional differential system is of commensurate orders (even fractional), the stability analysis can be carried out using the roots of the polynomial $P(\sigma) = \sigma^2 + p\,\sigma + 1$. The necessary and sufficient stability condition reads (see [10]) $|\arg(\sigma)| > \frac{\pi}{4}$. Now the poles can be computed exactly, namely those $s = \sigma^2$ for the roots σ of the polynomial P which lie in the sector $|\arg(\sigma)| < \frac{\pi}{2}$ (otherwise, there are still roots in the σ-plane, but no poles in the s-plane).

A very simple computation shows that the *necessary and sufficient* condition for stability of $\mathcal{H}_p(s)$ reads: $p > -\sqrt{2}$. In the case when $-\sqrt{2} < p < 0$, the system is oscillating: its impulse response can be decomposed into a damped sinusoid and a purely diffusive part (in the sense of (2.4)) with long-memory asymptotics (see also [10, 1]).

This *negative* coupling does introduce oscillations, though stable ones.

A similar counter-example can be built to example 3.4: the sufficient condition $p > 0$, $q > 0$ is *not* necessary at all: still when $\alpha = 1/2$, $\sigma^4 + p\,\sigma^3 + \varepsilon\,\sigma^2 + q\,\sigma + \omega^2$ does not have positive coefficients when taking 2 roots in $\frac{\pi}{4} < |\arg(\sigma)| < \frac{\pi}{2}$ and -1,-1.

4.2 A family of first order examples in continuous time

Let us study the family of transfer functions $\mathcal{H}_p(s) = (s + p\,s^\alpha - s_0)^{-1}$, with $0 < \alpha < 1$ and $s_0 \in \mathbb{C}$. We would like to analyze the influence of the coupling parameter p, especially when the original system (for $p = 0$) is unstable, that is $\Re e(s_0) > 0$.

The following careful analysis can be carried out:

Theorem 4.1. *When $p \to +\infty$, the roots of $s + p\,s^\alpha - s_0$ in $\mathbb{C}\backslash\mathbb{R}^-$ are given by:*

- *if $|\arg(s_0)| \leq \alpha\frac{\pi}{2}$, $s_p \sim p^{-\frac{1}{\alpha}} {s_0}^{\frac{1}{\alpha}} \to 0$ with $\arg(s_p) \sim \frac{\arg(s_0)}{\alpha}$, hence $\Re e(s_p) \geq 0$ and this pole is* un*stable.*

- *if $\alpha\frac{\pi}{2} < |\arg(s_0)| \leq \alpha\pi$, $s_p \sim p^{-\frac{1}{\alpha}} {s_0}^{\frac{1}{\alpha}} \to 0$ with $\arg(s_p) \sim \frac{\arg(s_0)}{\alpha}$, hence $\Re e(s_p) < 0$ and this pole is stable.*

- *if $\alpha\pi < |\arg(s_0)| \leq \pi$, there exists p^* such that $\forall p \geq p^*$, the function $s \mapsto s + p\,s^\alpha - s_0$ has no root in $\mathbb{C}\backslash\mathbb{R}^-$.*

  Moreover, with $s_0 = r_0\,e^{i\theta_0}$, we can compute $p^* = {r_0}^\alpha \frac{\sin\theta_0}{(\sin\alpha\pi)^{1-\alpha}\,(\sin(\theta_0-\alpha\pi))^\alpha}$, and the limit point $s^* = -\xi^* = -r_0\,\frac{\sin(\theta_0-\alpha\pi)}{\sin\alpha\pi}$ on the cut is attained from above when $\Im m(s_0) > 0$ and from below when $\Im m(s_0) < 0$.

Proof. Some straightforward asymptotic analysis is performed, care is taken with the definition of s^α (see [10]), some tedious computations are needed for p^* and ξ^*. □

Note that there is no root going to infinity, that is $|s_p| \to +\infty$.

Example 4.1. *In order to illustrate theorem 4.1, we compute the root locus of the transfer function $\mathcal{H}_p(s)$ when $\alpha = \frac{1}{2}$, in which case, it is only needed to find the roots of a second order polynomial for each p, which can be done easily with Matlab. The results are shown on figure 1 with different original poles: for $p = 0$, s_0 is either stable or unstable. The quite complex situation described by theorem 4.1 is fully confirmed on this simulation, and especially when $\Re e(s_0) < 0$, the existence of a limit point $-\xi^*$ on the cut $\mathbb{R}^-$ is clearly seen.*

Remark 4.1. *We know from [2, proposition 3.1] that such fractional differential systems only have a* finite *number of poles, but we do not know how many there are in this example,*

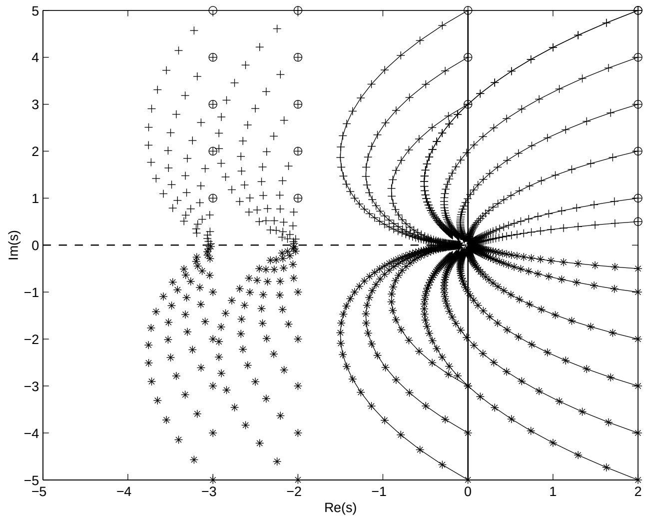

Figure 1: Influence of the coupling parameter p on a family of stable or unstable systems with pole s_0. Poles of the system $\mathcal{H}_p(s) = [s + p\, s^\alpha - s_0]^{-1}$ for $p \geq 0$, and various stable or unstable s_0.

even though it can be conjectured that there is at most *one pole – at least, this happens to be the case for* $\alpha = \frac{1}{2}$.

Remark 4.2. *The impulse reponse of such a system is composed of a finite-dimensional part, the stability of which is given by the location of the corresponding poles, and a purely diffusive part (in the sense of equation (2.4), which is always stable, though not exponentially). In our case,* μ *is a measure with density, namely (with a slight abuse of notations):*

$$\mu(\xi) = p\,\frac{\sin\alpha\pi}{\pi}\,\frac{\xi^\alpha}{(\xi + s_0)^2 + p^2\xi^{2\alpha} - 2p\cos\alpha\pi\,\xi^\alpha(\xi + s_0)} \tag{4.4}$$

The asymptotics of the diffusive part is proportional to $t^{-1-\alpha}$ *thanks to the Watson lemma, thus always BIBO-stable: that is why we are not very much concerned with it.*

4.3 A family of first order examples in discrete time

Let us study the family of transfer functions $\mathcal{H}_p(z) = [(1 - z_0\, z^{-1}) + p\,(1 - z^{-1})^\alpha]^{-1}$, with $0 < \alpha < 1$ and $z_0 \in \mathbb{C}$. We would like to analyze the influence of the coupling parameter p, especially when the original system (for $p = 0$) is unstable, that is $|z_0| > 1$.

The following careful analysis can be carried out:

Theorem 4.2. *When* $p \to +\infty$*, the roots of* $(1 - z_0\, z^{-1}) + p\,(1 - z^{-1})^\alpha$ *in* $\mathbb{C}\backslash[0,\, 1]$ *are given by:*

- *if* $|\arg(z_0) - 1| \leq \alpha\frac{\pi}{2}$, $z_p \to 1$ *with* $\arg(z_p - 1) \sim \frac{\arg(z_0 - 1)}{\alpha}$, *hence* $\Re e(z_p - 1) \geq 0$ *implies* $|z_p| > 1$ *and this pole is* unstable.

- *if* $\alpha\frac{\pi}{2} < |\arg(z_0-1)| \leq \alpha\pi$*,* $z_p \to 1$ *with* $\arg(z_p-1) \sim \frac{\arg(z_0-1)}{\alpha}$*, hence* $|\arg(z_p-1)| > \frac{\pi}{2}$ *implies* $|z_p| < 1$ *and this pole is stable.*

- *if* $\alpha\pi < |\arg(z_0-1)| \leq \pi$ *and* $(1-\alpha)\pi < |\arg(-z_0)| \leq \pi$*, then there exists* p^* *such that* $\forall p \geq p^*$*, the function* $z \mapsto (1-z_0\,z^{-1}) + p\,(1-z^{-1})^\alpha$ *has no root in* $\mathbb{C}\backslash[0,\,1]$*. Moreover* p^* *can be computed exactly, and the limit point* $z^* = \rho^*$ *located on the cut is attained from above when* $\Im m(z_0) > 0$ *and from below when* $\Im m(z_0) < 0$*.*

- *if* $|\arg(-z_0)| \leq (1-\alpha)\pi$*,* $z_p \to 0$ *and this pole is stable.*

Proof. It is fully based on theorem 4.1, with the change of variable $s = 1 - z^{-1}$: we first use this theorem for the analysis of the roots of $s \mapsto z_0\,s + p\,s^\alpha - (z_0 - 1)$; we then use the same theorem for the analysis of the roots of $s \mapsto (1-z_0)\,\frac{1}{s} + p\,\frac{1}{s}^{1-\alpha} + z_0$.

Geometrically, the region $\alpha\pi < |\arg(z_0-1)| \leq \pi$ and $(1-\alpha)\pi < |\arg(-z_0)| \leq \pi$ is easy to draw, and in the case $\alpha = \frac{1}{2}$ it is the vertical strip delimited by the two branching points $z = 0$ and $z = 1$. □

Remark 4.3. *The remarks concerning the finite number of poles on the one hand and the stability, though not of geometric type, of the diffusive part do also apply in this discrete-time context.*

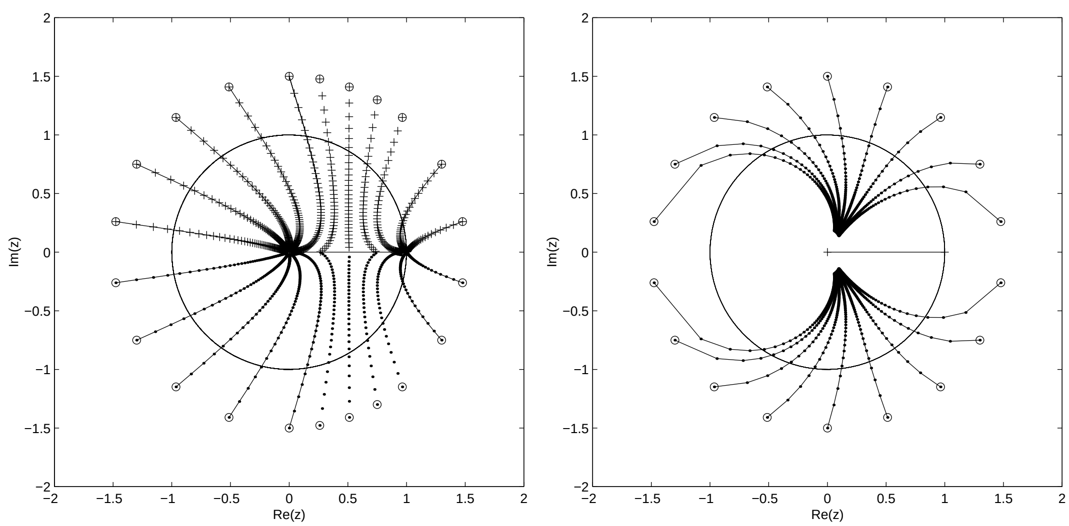

Figure 2: Influence of the coupling parameter p on two families of unstable systems with pole z_0: (a) Poles of the system $\mathcal{H}_p(z) = \left[(1-z_0\,z^{-1}) + p\,(1-z^{-1})^{\frac{1}{2}}\right]^{-1}$ for $p \geq 0$, and various unstable z_0. (b) Poles of the system $\mathcal{H}_p^2(z) = \left[(1-z_0\,z^{-1})(1-\bar{z}_0\,z^{-1}) + p\,(1-z^{-1})^{\frac{1}{2}}\right]^{-1}$ for $p \geq 0$, and various unstable z_0.

Example 4.2. *In order to illustrate theorem 4.2, we compute the root locus of the transfer function* $\mathcal{H}_p(z)$ *when* $\alpha = \frac{1}{2}$*, in which case, it is only needed to find the roots of a second* *order polynomial for each p, which can be done easily with Matlab. The results are shown on figure 2(a) with different original poles: for* $p = 0$, z_0 *is unstable. The very complex situation described by theorem 4.2 is fully confirmed on this simulation, and especially when* z_0 *belongs to the vertical strip* $0 \leq \Re e(z_0) \leq 1$, *the existence of a limit point* ρ^* *on the cut* $[0, 1]$ *is clearly seen.*

4.4 Families of second order examples

So far in this section, we have analyzed examples of first order systems with complex coefficients, it is then easy to build some second order systems with real coefficients by considering the products $\mathcal{H}_{p,s_0,\alpha}(s)\,\mathcal{H}_{p,\bar{s}_0,\alpha}(s)$ in continuous time, or $\mathcal{H}_{p,z_0,\alpha}(z)\,\mathcal{H}_{p,\bar{z}_0,\alpha}(z)$ in discrete time.

Example 4.3. *We compute the root locus of the transfer function:*

$$\mathcal{H}_p^2(z) = \frac{1}{(1 - z_0\,z^{-1})(1 - \bar{z}_0\,z^{-1}) + p\,(1 - z^{-1})^{\frac{1}{2}}} \tag{4.5}$$

in which case, it is only needed to find the roots of a fourth order polynomial for each p, which can be done easily with Matlab. The results are shown on figure 2(b) with different original poles: for $p = 0$, z_0 *is unstable. From this simulation only, it seems that the stabilization of the unstable system will always be achieved; now the analysis of this system appears to be more complicated.*

5 Conclusion

Positive PDOs of diffusive type are useful to study quite a great variety of systems with non-standard dynamics, through an energy analysis on coupled systems. We have also shown that the stability condition given by this methodology is only sufficient, but not necessary. In fact, negative PDOs can make a stable system oscillate, even though it is kept stable; conversely, positive PDOs can help stabilize unstable systems, under some conditions on the parameters, which can be expressed quite nicely when the diffusive transfers at stake are known analytically.

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