Output-input stability of nonlinear systems and input/output operators

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Abstract

The notion of output-input stability, recently proposed in [2], represents a variant of the minimum-phase property for general smooth nonlinear control systems. In the spirit of the input-to-state stability (ISS) philosophy, the definition of output-input stability requires the state and the input of the system to be bounded by a suitable function of the output and derivatives of the output, modulo a decaying term depending on initial conditions. The present work extends this concept to the setting of input/output operators. We show that output-input stability of a system implies output-input stability of the associated input/output operator, and that under suitable reachability and observability assumptions, a converse result also holds.

1 Motivation

For systems with inputs, two properties of interest are asymptotic stability under zero inputs and bounded state response to bounded inputs. It is well known that for linear systems the first property implies the second one, but for general nonlinear systems this is not the case. The notion of *input-to-state stability* (ISS) introduced in [3] captures both of the above properties; it requires that bounded inputs produce bounded states and inputs converging (or equal) to zero produce states converging to zero.

Dual concepts of detectability result if one considers systems with outputs. For linear systems, one of equivalent ways to define detectability is to demand that the state converge to zero along every trajectory for which the output is identically zero. The notion of *output-to-state stability* (OSS) introduced in [4] is a robust version of this property for nonlinear systems and a dual of ISS; it requires that the state be bounded if the output is bounded and converge to zero if the output converges to zero.

The present line of work is concerned with the *minimum-phase* property of systems with both inputs and outputs. A linear system is minimum-phase if whenever the output is identically zero, both the state and the input must converge to zero; in the frequency domain, this is characterized by stability of system zeros. Byrnes and Isidori [1] provided an important and natural extension of the minimum-phase property to nonlinear systems. According to their definition, the system is minimum-phase if the *zero dynamics*—the internal dynamics of the system under the action of an input that holds the output constantly at zero—are asymptotically stable.

The above remarks suggest that to complete the picture, one should have a robust version of the last property, which should ask the state and the input to be bounded when the output is bounded and to become small when the output is small. Such a concept was proposed in the recent paper [2] under the name of *output-input stability.* This property is in general stronger than the minimum-phase property defined in [1]. Output-input stability can be studied with the help of the tools that have been developed over the years to study ISS, OSS, and related notions (such as Lyapunov-like dissipation inequalities); the minimum-phase property, on the other hand, is investigated using different techniques (such as computation of normal forms and zero dynamics). This makes output-input stability an appropriate alternative notion to use in those situations where the minimum-phase property is insufficient or difficult to check.

In this paper we extend the concept of output-input stability to the setting of input/output operators and study the relationship between output-input stability of a nonlinear system and the corresponding property of its input/output operator. This work parallels the developments of [3], where input-to-state stability of input/output operators is defined and related to the ISS property of state-space systems (see particularly Propositions 3.2 and 7.1 in that paper). The necessary background and definitions are given in Sections 2 and 3, and the main findings are presented in Section 4.

2 Preliminaries

Given a pair of integers $k \geq 0$ and $l > 0$ and a subinterval J of $[0, \infty)$, we denote by $C^k(J, \mathbb{R}^l)$ the space of all k times continuously differentiable functions $w : J \to \mathbb{R}^l$. For example, $C^0([0,\infty), \mathbb{R}^l)$ is the space of all continuous functions $w : [0, \infty) \to \mathbb{R}^l$.

Consider the system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \tag{2.1}$$

where the state x takes values in $\mathbb{R}^n$, the input u takes values in $\mathbb{R}^m$, the output y takes values in $\mathbb{R}^p$ (for some positive integers n, m, and p), and the functions f and h are smooth (C^∞). We restrict admissible input (or "control") signals to be locally bounded piecewise continuous. For every initial condition $x(0)$ and every input $u(\cdot)$, there is a solution $x(\cdot)$ of (2.1) defined on a maximal interval $[0, T_{\max})$, and the corresponding output $y(\cdot)$. An input in $C^k([0,\infty), \mathbb{R}^m)$ produces an output in $C^{k+1}([0, T_{\max}), \mathbb{R}^p)$, with $(k+1)$-st derivative

$$y^{(k+1)}(t) \;=\; H_{k+1}\big(x(t), u(t), \,\ldots, u^{(k)}(t)\big), \qquad t \in [0, T_{\max}) \tag{2.2}$$

where for $i = 0, 1, \ldots$ the functions $H_i : \mathbb{R}^n \times (\mathbb{R}^m)^i \to \mathbb{R}^p$ are defined recursively by the formulas $H_0 := h$ and

$$H_{i+1}(x, u_0, \,\ldots, u_i) \;:=\; \frac{\partial H_i}{\partial x}\, f(x, u_0) + \sum_{j=0}^{i-1} \frac{\partial H_i}{\partial u_j}\, u_{j+1} \tag{2.3}$$

(here the arguments of H_i are $x \in \mathbb{R}^n$ and $u_0, \,\ldots, u_{i-1} \in \mathbb{R}^m$). In fact, if y needs to be differentiated r times before u appears, then an input in $C^k([0,\infty), \mathbb{R}^m)$ produces an output in $C^{k+r}([0, T_{\max}), \mathbb{R}^p)$; see [2] for details.

We let $\|\cdot\|_J$ denote the supremum norm of a signal restricted to an interval $J \subset [0, \infty)$, i.e., $\|z\|_J := \sup\{|z(s)| : s \in J\}$, where $|\cdot|$ is the standard Euclidean norm. Given an $\mathbb{R}^l$-valued signal z and a nonnegative integer k, we denote by $\mathbf{z}^k$ the $\mathbb{R}^{l(k+1)}$-valued signal

$$\mathbf{z}^k := (z_1, \dot{z}_1, \ldots, z_1^{(k)}, \ldots, z_l, \dot{z}_l, \ldots, z_l^{(k)})^T$$

provided that the indicated derivatives exist.

According to Definition 1 of [2], the system (2.1) is called *output-input stable* if there exist a positive integer N, a class $\mathcal{KL}$ function[1] β, and a class $\mathcal{K}_\infty$ function γ such that for every initial state $x(0)$ and every input $u \in C^{N-1}([0,\infty),\mathbb{R}^m)$ the inequality

$$\left|\begin{pmatrix} u(t) \\ x(t) \end{pmatrix}\right| \le \beta(|x(0)|,t) + \gamma(\|\mathbf{y}^N\|_{[0,t]}) \tag{2.4}$$

holds for all t in the domain of the corresponding solution. (The continuous differentiability assumption on the input can be weakened if the function H_N is independent of u_{N-1}.) The class of output-input stable systems includes all affine systems in global normal form with ISS internal dynamics and also all left-invertible linear systems whose transmission zeros have negative real parts. We refer the reader to [2] for a detailed discussion and applications of output-input stability, which can be viewed as a generalization of the notion of a minimum-phase linear system.

3 Input/output operators

We now extend the above concept of output-input stability to input/output operators. By an *input/output* (I/O) *operator* we mean a causal mapping

$$F : C^k([0,\infty),\mathbb{R}^m) \to \bigcup_{T_{\max}>0} C^{k+r}([0,T_{\max}),\mathbb{R}^p) \tag{3.1}$$

where k is a nonnegative integer and m, r, and p are positive integers. "Causal" means that if $y = F(u)$, then $y(t)$ does not depend on the values $u(s)$, $s > t$. Let us call an I/O operator (3.1) *output-input stable* if there exist a positive integer $N \le k + r$, a class $\mathcal{KL}$ function β_u, and a class $\mathcal{K}_\infty$ function γ_u such that for every input $u \in C^k([0,\infty),\mathbb{R}^m)$ and every pair of times $t \ge T$ in the domain of the corresponding output $y = F(u)$ we have

$$|u(t)| \le \beta_u(\|\mathbf{y}^N\|_{[0,T)}, t - T) + \gamma_u(\|\mathbf{y}^N\|_{[T,t]}). \tag{3.2}$$

When studying the relationship between state-space systems and the associated I/O operators from the point of view of output-input stability, we will need to consider input signals obtained by concatenating two continuous inputs. To this end, we denote by $\widehat{C}^k([0,\infty),\mathbb{R}^m)$ the space of functions $u : [0,\infty) \to \mathbb{R}^m$ which either belong to $C^k([0,\infty),\mathbb{R}^m)$ or have a single discontinuity at some time T, satisfy $u(T) = \lim_{s\to T^+} u(s)$, and are k times continuously differentiable everywhere else. Then we can consider a more general I/O operator

$$\widehat{F} : \widehat{C}^k([0,\infty),\mathbb{R}^m) \to \bigcup_{T_{\max}>0} \widehat{C}^{k+r}([0,T_{\max}),\mathbb{R}^p) \tag{3.3}$$

where the space $\widehat{C}^{k+r}([0,T_{\max}),\mathbb{R}^p)$ is defined similarly and the location T of the discontinuity is assumed to be preserved under the action of $\widehat{F}$ (and if the input is continuous on $[0,T_{\max})$, then so is the output). Let us say that this operator is *output-input stable* if:

[1]Recall that a function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of *class* $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. If $\alpha \in \mathcal{K}$ is unbounded, then it is said to be of *class* $\mathcal{K}_\infty$. A function $\beta : [0,\infty)\times[0,\infty) \to [0,\infty)$ is said to be of *class* $\mathcal{KL}$ if $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each fixed $t \ge 0$ and $\beta(s,t)$ decreases to 0 as $t \to \infty$ for each fixed $s \ge 0$.

1. The restriction of $\widehat{F}$ to $C^k([0,\infty),\mathbb{R}^m)$ is output-input stable in the sense of the previous definition.
2. For every input $u \in \widehat{C}^k([0,\infty),\mathbb{R}^m)$ with a discontinuity at T and every $t \geq T$ in the domain of the corresponding output $y = \widehat{F}(u)$, the inequality (3.2) holds[2].

4 Systems and their operators

We now take $\widehat{F}$ to be the I/O operator that describes the input/output mapping of the system (2.1), with $x(0) = 0$. We want to understand the relationship between output-input stability of the system and of the operator.

System → operator

First, suppose that the system (2.1) is output-input stable. As the domain of $\widehat{F}$, we can take $\widehat{C}^{N-1}([0,\infty),\mathbb{R}^m)$, so that we have

$$\widehat{F} : \widehat{C}^{N-1}([0,\infty),\mathbb{R}^m) \to \bigcup_{T_{\max}>0} \widehat{C}^N([0,T_{\max}),\mathbb{R}^p).$$

Here N is the positive integer that appears in (2.4).

Proposition 1 *If the system* (2.1) *is output-input stable, then the I/O operator $\widehat{F}$ is also output-input stable.*

Proof. Take an input $u \in \widehat{C}^{N-1}([0,\infty),\mathbb{R}^m)$, the output $y = \widehat{F}(u) \in \widehat{C}^N([0,T_{\max}),\mathbb{R}^p)$, and a pair of times (T,t) satisfying $0 \leq T \leq t < T_{\max}$, where T coincides with the discontinuity of u if one exists and is arbitrary otherwise. By time invariance, (2.4) implies

$$|u(t)| \leq \beta(|x(T)|, t-T) + \gamma(\|\mathbf{y}^N\|_{[T,t]}).$$

Applying (2.4) again, this time to $x(T^-) = x(T)$, and recalling that $x(0) = 0$, we obtain

$$|u(t)| \leq \beta(\gamma(\|\mathbf{y}^N\|_{[0,T)}), t-T) + \gamma(\|\mathbf{y}^N\|_{[T,t]}).$$

Therefore, (3.2) holds with $\beta_u(s,t) := \beta(\gamma(s),t)$ and $\gamma_u := \gamma$, hence $\widehat{F}$ is output-input stable. □

Remark 1 A natural question to ask is whether output-input stability of an I/O operator F of the form (3.1) automatically implies output-input stability of the extended operator $\widehat{F}$. We do not know the answer to this question in general, although some positive results can be obtained for I/O operators associated with certain classes of state-space systems. □

[2] The vector $\mathbf{y}^N(T)$ is to be interpreted as $\lim_{s\to T^+} \mathbf{y}^N(s)$.

Operator → system

Obtaining a converse result is more interesting. Suppose that $\widehat{F}$ is output-input stable. We take the domain of $\widehat{F}$ to be $\widehat{C}^{N-1}([0,\infty),\mathbb{R}^m)$, where N is the positive integer that appears in (3.2). We impose the following two assumptions on the system (2.1).

Assumption 1 (*strong finite-time observability with output derivatives*). There exist a number $\varepsilon > 0$ and two class $\mathcal{K}_\infty$ functions α_1 and α_2 such that for every $x(0)$, every input $u \in C^{N-1}([0,\infty),\mathbb{R}^m)$, and every $t \in [0, T_{\max} - \varepsilon)$, where $[0, T_{\max})$ is the maximal interval of existence of the corresponding solution of (2.1), we have

$$|x(t)| \le \alpha_1(\|u\|_{[t,t+\varepsilon]}) + \alpha_2(\|\mathbf{y}^N\|_{[t,t+\varepsilon]}). \tag{4.1}$$

Remark 2 In contrast with the strong observability property considered in [3], finite time intervals are used here. On the other hand, (4.1) is weaker in the sense that the right-hand side contains derivatives of the output. Alternative definitions of observability can also be explored in this context; cf. Remark 4 below. □

Assumption 2 (*reachability with bounded overshoot*). There exists a class $\mathcal{K}_\infty$ function α_3 such that for each $\xi \in \mathbb{R}^n \backslash \{0\}$ it is possible to find a time $T > 0$ and a control input $u \in C^{N-1}([0,T),\mathbb{R}^m)$ which steers the system (2.1) from state 0 at time $t = 0$ to state ξ at time $t = T$ in such a way that the corresponding output y satisfies

$$\|\mathbf{y}^N\|_{[0,T)} \le \alpha_3(|\xi|). \tag{4.2}$$

Remark 3 Under appropriate conditions, this property can be derived from a strong reachability property of the kind considered in [3]. Namely, assume that there exists a class $\mathcal{K}_\infty$ function α_4 such that for each $\xi \in \mathbb{R}^n \backslash \{0\}$ it is possible to find a time $T > 0$ and a control input $u \in C^{N-1}([0,T),\mathbb{R}^m)$ which steers the system (2.1) from state 0 at time $t = 0$ to state ξ at time $t = T$ and satisfies

$$\|\mathbf{u}^{N-1}\|_{[0,T)} \le \alpha_4(|\xi|). \tag{4.3}$$

Assume also that $h(0) = 0$ and that the system (2.1) is $\mathcal{K}$-*stable* in the sense that for some $\gamma \in \mathcal{K}$ (can take $\gamma \in \mathcal{K}_\infty$ with no loss of generality) we have

$$\|x\|_{[0,t]} \le \gamma(\|u\|_{[0,t]}) \tag{4.4}$$

along all solutions of (2.1). Then, combining (4.3) and (4.4) and using the formulas (2.2) and (2.3), we can arrive at (4.2). This simplifies if N is such that $\mathbf{y}^N$ is independent of the derivatives of u, because then we can replace (4.3) by

$$\|u\|_{[0,T)} \le \alpha_4(|\xi|)$$

thus recovering the strong reachability condition imposed in [3]. □

Take an arbitrary $\xi \in \mathbb{R}^n$. If $\xi = 0$, let $T = 0$. Otherwise, by Assumption 2 we can find a time $T > 0$ and an input in $C^{N-1}([0,T),\mathbb{R}^m)$ which steers the system (2.1) from state 0 at time $t = 0$ to state ξ at time $t = T$ so that the inequality (4.2) holds. Apply an arbitrary input in $C^{N-1}([T,\infty),\mathbb{R}^m)$ for $t \ge T$, and denote the concatenated input signal by u. Note that

$u \in \widehat{C}^{N-1}([0,\infty),\mathbb{R}^m)$. Let y be the resulting output, and take an arbitrary $t \in [T, T_{\max} - \varepsilon)$ where ε is provided by Assumption 1. The inequalities (3.2) and (4.2) give

$$|u(t)| \le \beta_u(\alpha_3(|\xi|), t - T) + \gamma_u(\|\mathbf{y}^N\|_{[T,t]}). \tag{4.5}$$

Moreover, we conclude from (4.5) that

$$\|u\|_{[t,t+\varepsilon]} \le \beta_u(\alpha_3(|\xi|), t - T) + \gamma_u(\|\mathbf{y}^N\|_{[T,t+\varepsilon]}).$$

Combined with (4.1), this yields

$$|x(t)| \le \beta_x(|\xi|, t - T) + \gamma_x(\|\mathbf{y}^N\|_{[T,t+\varepsilon]}) \tag{4.6}$$

where

$$\beta_x(s,t) := \alpha_1(2\beta_u(\alpha_3(s),t)), \qquad \gamma_x(s) := \alpha_1(2\gamma_u(s)) + \alpha_2(s).$$

By time invariance, the inequalities (4.5) and (4.6) imply that for every $x(0)$ and every input $u \in C^{N-1}([0,\infty),\mathbb{R}^m)$ the solution of (2.1) satisfies the inequalities

$$|u(t)| \le \beta(|x(0)|, t) + \gamma(\|\mathbf{y}^N\|_{[0,t]}) \tag{4.7}$$

$$|x(t)| \le \beta(|x(0)|, t) + \gamma(\|\mathbf{y}^N\|_{[0,t+\varepsilon]}) \tag{4.8}$$

for all $t \in [0, T_{\max} - \varepsilon)$, where $\beta(s,t) := \max\{\beta_u(\alpha_3(s),t), \beta_x(s,t)\}$ and $\gamma(s) := \max\{\gamma_u(s), \gamma_x(s)\}$. We summarize as follows.

Proposition 2 *If the I/O operator $\widehat{F}$ is output-input stable and Assumptions 1 and 2 hold, then the system* (2.1) *has the properties expressed by the inequalities* (4.7)–(4.8).

Remark 4 Note that, due to the presence of "non-causal" ε in (4.8), we do not exactly recover output-input stability of (2.1). This difference would disappear if we strengthened Assumption 1 by requiring that x can be bounded in terms of the *instantaneous* values of the output and its first N derivatives. This new assumption would be rather restrictive, though. □

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References

[1] C. I. Byrnes and A. Isidori. Local stabilization of minimum phase nonlinear systems. *Systems Control Lett.*, 11:9–17, 1988.

[2] D. Liberzon, A. S. Morse, and E. D. Sontag. Output-input stability and minimum-phase nonlinear systems. *IEEE Trans. Automat. Control*, 47:422–436, 2002.

[3] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control*, 34:435–443, 1989.

[4] E. D. Sontag and Y. Wang. Output-to-state stability and detectability of nonlinear systems. *Systems Control Lett.*, 29:279–290, 1997.