An Example of Output Regulation for a Distributed Parameter System with Infinite Dimensional Exosystem

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Abstract

In this short paper we present an example of the geometric theory of output regulation applied to solve a tracking problem for a plant consisting of a boundary controlled distributed parameter system (heat equation on a rectangle) with unbounded input and output maps and signal to be tracked generated by an infinite dimensional exosystem. The exosystem is neutrally stable but with an infinite (unbounded) set of eigenmodes distributed along the imaginary axis. For this reason the standard methods of analysis do not apply.

1 Plant and Exosystem

We consider the temperature in a two-dimensional unit square, $\Omega = [0,1] \times [0,1]$, with coordinates $x = (x_1, x_2)$ and boundary of Ω denoted by $\partial\Omega$. The temperature distribution across the region is governed by the Heat Equation. In order to avoid technical difficulties which do not add any useful information concerning the main point of the paper, we will arrange for the heat plant to be stable by assuming that some intervals of $\partial\Omega$ will have homogeneous Dirichlet boundary conditions, i.e., the temperature will be held at 0 on those intervals. This part of the boundary will be denoted by $\mathcal{S}_D$, and it will be important that, by our assumption, S_D will consist of a finite union of intervals of positive length. Next, we will designate another part of the boundary, on which we have Neumann boundary conditions, by $\mathcal{S}_N = \partial\Omega \backslash \mathcal{S}_D$. We designate p non-overlapping input intervals $\mathcal{S}_j$, for $j = 1, \ldots, p$, and p

non-overlapping output intervals, $\widehat{\mathcal{S}}_j$, for $j = 1, \ldots, p$, with each $\mathcal{S}_j$ and being $\widehat{\mathcal{S}}_j$ a subset of $\mathcal{S}_N$. We point out that the intersections $\mathcal{S}_i \cap \widetilde{\mathcal{S}}_j$ are not necessarily empty. Indeed, in the case of co-located actuators and sensors the $\mathcal{S}_i$ and $\widetilde{\mathcal{S}}_j$ will coincide. Finally, we define the set, $\mathcal{S}_0 = \mathcal{S}_N \backslash \cup \mathcal{S}_j$. A general depiction of the layout of these sections (in the case $\mathcal{S}_i \cap \widetilde{\mathcal{S}}_j = \emptyset$, $i, j = 1, \ldots, p$) is portrayed in Figure 1, below.

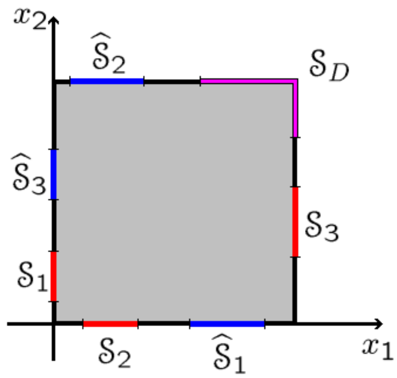


Figure 1: Layout of the Various Intervals of the Boundary on Ω

The controlled heat plant is given by the following initial-boundary value problem:

$$\frac{\partial}{\partial t} z(x,t) = \Delta z(x,t), \; x \in \Omega, \; t \geq 0, \; \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \tag{1.1}$$

$$z(x,0) = z_0(x),$$

$$z(x,t)\Big|_{x \in \mathcal{S}_D} = 0, \tag{1.2}$$

$$\frac{\partial z}{\partial \nu}(x,t)\Big|_{x \in \mathcal{S}_j} = u_j(t), \; j = 1, ..., p, \quad \frac{\partial z}{\partial \nu}(x,t)\Big|_{x \in \mathcal{S}_0} = 0. \tag{1.3}$$

Here $u_j(t)$ are the inputs. We define the state space of the plant as $\mathcal{Z} = L^2(\Omega)$.

In this work it will be assumed that the p outputs are given by the average temperature over the small regions $\widehat{\mathcal{S}}_j$ of the boundary, i.e.,

$$y_j(t) = \frac{1}{|\widehat{\mathcal{S}}_j|} \int_{\widehat{\mathcal{S}}_j} z(x,t) d\sigma_x, \quad y = [y_1(t), \cdots, y_p(t)]^T = Cz \tag{1.4}$$

where $|\widehat{\mathcal{S}}_j|$ denotes the length of the interval $\widehat{\mathcal{S}}_j$ and where $d\sigma_x$ is “surface measure” on the boundary of Ω. From this, we may note that the output C operator is defined in the following way

$$Cz(t) = \begin{bmatrix} \frac{1}{|\widehat{\mathcal{S}}_1|}\int_{\widehat{\mathcal{S}}_1} z(x,t)d\sigma_x \\ \vdots \\ \frac{1}{|\widehat{\mathcal{S}}_p|}\int_{\widehat{\mathcal{S}}_p} z(x,t)d\sigma_x \end{bmatrix}. \tag{1.5}$$

Introducing a standard formulation, we write the plant (1.1)-(1.3) in abstract form as

$$\frac{d}{dt}z = Az + Bu, \quad y(t) = Cz, \tag{1.6}$$

where $A : \mathcal{D}(A) \subset \mathcal{Z} \to \mathcal{Z}$, and $C : \mathcal{D}(C) \subset \mathcal{Z} \to \mathcal{Y} = \mathbb{R}^p$ are unbounded densely defined linear operators and $B : \mathcal{U} = \mathbb{R}^p \to \widetilde{H}^{-1}(\Omega)$ where $\widetilde{H}^{-\alpha}(\Omega)$ denotes the dual of $H^{\alpha}(\Omega)$, $\alpha > 0$ (see, e.g., [11]). $\widetilde{H}^{-\alpha}(\Omega)$ can be identified with a subspace of the space of distributions $H^{-\alpha}(\mathbb{R}^n) = [H^{\alpha}(\mathbb{R}^n)]^* \subset \mathcal{D}(\mathbb{R}^n)^*$:

$$\widetilde{H}^{-\alpha}(\Omega) = \left\{ f \in H^{-\alpha}(\mathbb{R}^n) \, : \, \mathrm{supp}(f) \subseteq \overline{\Omega} \right\}.$$

(See definition of B in (1.8)-(1.11) below.)

The operator $A = \Delta$ with domain

$$\mathcal{D}(A) = \left\{ \varphi \in \mathcal{Z} \, : \, \frac{\partial \varphi}{\partial \nu}\bigg|_{x\in \mathcal{S}_N} = 0, \quad \varphi\big|_{x\in sS_D} = 0 \right\}$$

is an unbounded self-adjoint operator in the Hilbert space $\mathcal{Z} = L^2(\Omega)$ whose spectrum consists of real eigenvalues $\{\zeta_k\}_{k=1}^{\infty}$ satisfying

$$\zeta_{k+1} \le \zeta_k, \quad \zeta_k \xrightarrow{k\to\infty} -\infty, \tag{1.7}$$

and with associated orthonormal eigenfunctions $\varphi_k(x)$ satisfying

$$A\varphi_k = \zeta_k \varphi_k, \qquad \langle \varphi_n, \varphi_m \rangle = \delta_{nm}.$$

(Here and below we denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\Omega)$).

The input operator is defined by

$$Bu(\eta) = \sum_{i=1}^{p} u_i(t) \frac{1}{|\mathcal{S}_i|} \int_{\mathcal{S}_i} \eta(x) d\sigma_x, \tag{1.8}$$

where $Bu \in \widetilde{H}^{-1}(\Omega)$ is a distribution and $\eta \in \mathcal{D}(\mathbb{R}^n)$ is a test function. Therefore,

$$Bu = \sum_{i=1}^{p} u_i b_i, \tag{1.9}$$

where b_i is the distribution which acts on a test function $\eta \in \mathcal{D}(\mathbb{R}^n)$ by the rule

$$b_i(\eta) = \frac{1}{|\mathcal{S}_i|} \int_{\mathcal{S}_i} \eta(x) d\sigma_x, \tag{1.10}$$

and

$$u = \begin{bmatrix} u_1 & \cdots & u_p \end{bmatrix}^T \in \mathcal{U} = \mathbb{R}^p. \tag{1.11}$$

We note that $b_i \in \widetilde{H}^{-1/2-\epsilon}(\Omega)$ for $\epsilon > 0$.

The exosystem is given by the one-dimensional wave equation on the interval $[0,1]$ (with spatial coordinate ξ) and with homogeneous Dirichlet boundary conditions.

$$\frac{\partial^2}{\partial t^2} w(\xi,t) = \frac{\partial^2}{\partial \xi^2} w(\xi,t),\ \xi \in (0,1),\ t \in \mathbb{R} \tag{1.12}$$

$$w(0,t) = w(1,t) = 0$$

$$w(\xi,0) = \psi_0(\xi), \quad \frac{\partial}{\partial t} w(\xi,0) = \psi_1(\xi). \tag{1.13}$$

For this exosystem we are interested in reference outputs $y_j^{\text{ref}}(t)$ given as the displacements at a set of p points ξ_p in the interval $(0,1)$

$$y_j^{\text{ref}}(t) = w(\xi_j,t),\ 0 < \xi_j < 1, \quad y_{\text{ref}} = \widetilde{Q} w = \left[y_1^{\text{ref}}(t), \cdots, y_p^{\text{ref}}(t) \right]^T, \tag{1.14}$$

where $\widetilde{Q}w$ would be defined as

$$\widetilde{Q} w = \begin{bmatrix} w(\xi_1,t) \\ \vdots \\ w(\xi_p,t) \end{bmatrix}.$$

Once again we recast, in this case the exosystem, as an abstract dynamical system in an infinite dimensional state space in the usual way by first introducing new dependent variables

$$W = \begin{bmatrix} w \\ \frac{\partial}{\partial t} w \end{bmatrix} \equiv \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \quad W(0) = \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} \equiv W_0, \quad S = \begin{bmatrix} 0 & 1 \\ \dfrac{\partial^2}{\partial \xi^2} & 0 \end{bmatrix} \tag{1.15}$$

and then writing the exosystem as:

$$\frac{d}{dt} W = SW, \quad W(0) = W_0 \tag{1.16}$$

with reference outputs

$$y_{\text{ref}} = QW, \tag{1.17}$$

where

$$Q = \begin{bmatrix} \widetilde{Q} & 0 \end{bmatrix}.$$

The state space for (1.16) is

$$\mathcal{W} = H_0^1(0,1) \times L^2(0,1),$$

which is a Hilbert space with inner product defined by

$$\left\langle \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \right\rangle_{\mathcal{W}} = \langle \varphi_1', \psi_1' \rangle_{L^2(0,1)} + \langle \varphi_2, \psi_2 \rangle_{L^2(0,1)}, \tag{1.18}$$

The spectrum of the operator S consists of eigenvalues

$$\lambda_n = n\pi i,\ \ n = \pm 1, \pm 2, ... \tag{1.19}$$

and associated normalized eigenfunctions (i.e., $\|\Phi_\ell\|_{\mathcal{W}} = 1$)

$$\Phi_\ell = \begin{bmatrix} \dfrac{1}{\lambda_\ell} \\ 1 \end{bmatrix} \sin(\ell\pi\xi),\ \ \ell = \pm 1, \pm 2, ...$$

Thus the exosystem is infinite dimensional with simple eigenvalues along the imaginary axis, and from (1.7) and (1.19) we note that the respective spectra of A and S are disjoint.

2 Output Regulation Problem

Our objective is to regulate the plant so that its outputs track the reference outputs generated by the exosystem. To achieve this goal, we follow the program given in [4] and seek u as a feedback of the state of the exosystem

$$u = \Gamma W,$$

where Γ is a linear map from $\mathcal{W}$ to $\mathcal{U}$ $(= \mathbb{R}^p)$. With this, the plant becomes

$$\frac{d}{dt} z = Az + B\Gamma W \tag{2.1}$$

$$z(0) = z_0$$

and the exosystem given in (1.16) is

$$\begin{aligned} \frac{d}{dt} W &= SW \\ W(0) &= W_0. \end{aligned} \tag{2.2}$$

Now that we have defined the plant and exosystem coupled through the feedback $u = \Gamma W$, we now define the associated composite system consisting of the inter-connection of these systems, which will allow us to define the Main Problem and to solve that problem, once defined. The composite system, is given by

$$\frac{d}{dt} \begin{bmatrix} z \\ W \end{bmatrix} = \begin{bmatrix} A & B\Gamma \\ 0 & S \end{bmatrix} \begin{bmatrix} z \\ W \end{bmatrix} \tag{2.3}$$

$$\begin{bmatrix} z \\ W \end{bmatrix} (0) = \begin{bmatrix} z_0 \\ W_0 \end{bmatrix}.$$

Our tracking problem is formulated in terms of the error, defined as an output of the composite system given by the difference between the measured output and the reference output, i.e.,

$$e(t) = y(t) - y_{\rm ref}(t),$$

where y and $y_{\rm ref}$ are defined in (1.4) and (1.17). Written another way,

$$y = Cz$$

and

$$y_{\rm ref} = QW,$$

so that the error can be written in terms of (1.4) and (1.17) as

$$e(t) = y(t) - y_{ref}(t) = [C, -Q] \begin{bmatrix} z \\ W \end{bmatrix} (t). \tag{2.4}$$

We are now ready to present our main problem.

Problem 2.1 (The Main Problem). *Find a feedback control $u = \Gamma W$ so that for every initial condition of the plant and exosystem, the error satisfies*

$$e(t) \longrightarrow 0 \ \textit{as} \ t \longrightarrow \infty. \tag{2.5}$$

3 Formal Solution

From the work [3] we know that the open loop system defines a regular linear system [13] and for any finite dimensional exosystem the following result from [4] provides necessary and sufficient conditions for the solvability of the associated regulator problem.

Theorem 3.1. *The state feedback regulator problem (with finite dimensional exosystem) is solvable if and only if the regulator equations*

$$\Pi S - A\Pi = B\Gamma \tag{3.1}$$

$$C\Pi - Q = 0 \tag{3.2}$$

are solvable for bounded linear operators $\Pi : \mathcal{W} \to \mathcal{D}(A) \hookrightarrow \mathcal{Z}$ and $\Gamma : \mathcal{W} \to \mathcal{U}$.

This result cannot be applied directly to a problem of output regulation with infinite dimensional exosystem. However, mimicking the proof given in [4] in the present setting we can obtain formulas which allow us to solve the problem once we introduce an appropriate modification of the reference signals to be tracked.

We turn to the formal derivation of explicit formulas for Π and Γ solving the regulator equations (3.1), (3.2). First we note that the first regulator equation (3.1) applied to a general Φ_j gives,

$$\Pi S\Phi_j - A\Pi\Phi_j = B\Gamma\Phi_j$$

and since (λ_j, Φ_j) is an eigenpair of S this equation simplifies to

$$(\lambda_j I - A)\Pi\Phi_j = B\Gamma\Phi_j. \tag{3.3}$$

From (1.19) we know that the eigenvalues of S are contained in the resolvent set of A (i.e., $\sigma(A) \cap \sigma(S) = \emptyset$), which implies that the term $(\lambda_j I - A)$ in (3.3) is invertible. This allows us to solve (3.3) for $\Pi\Phi_j$

$$\Pi\Phi_j = (\lambda_j I - A)^{-1} B\Gamma\Phi_j. \tag{3.4}$$

Applying C to both sides and using (3.2), the second regulator equation, we have

$$Q\Phi_j = C(\lambda_j I - A)^{-1} B\Gamma\Phi_j \tag{3.5}$$

where in (3.5) $C(\lambda I - A)^{-1}B$ is the *transfer function* of the plant which we denote by

$$G(\lambda) = C(\lambda I - A)^{-1}B.$$

Thus we can rewrite (3.5) as

$$Q\Phi_j = G(\lambda_j)\Gamma\Phi_j.$$

In order to solve this equation explicitly for $\Gamma\Phi_j$, we need to invert $G(\lambda_j)$. In this case we obtain

$$\Gamma\Phi_j = G(\lambda_j)^{-1} Q\Phi_j. \tag{3.6}$$

Assuming that this can be done for every λ_j and using the expansion

$$W = \sum_{j=-\infty}^{\infty} \langle W, \Phi_j \rangle \Phi_j, \quad \text{for every} \quad W \in \mathcal{W},$$

we can solve, at least formally, (3.6) for $u = \Gamma W$ in terms of elements we know,

$$u = \Gamma W = \sum_{j=-\infty}^{\infty} \langle W, \Phi_j \rangle G(\lambda_j)^{-1} Q\Phi_j. \tag{3.7}$$

4 Main Technical Difficulties

There are two main difficulties associated with obtaining the formal solution for $u = \Gamma W$ given in (3.7):

1. Invertibility of the transfer function G at the eigenvalues of S,

2. Convergence of the resulting infinite sum.

At this point these difficulties have not been fully resolved for the general problems defined in Sections 2 and 3. To obtain better insight into these problems let us write a formal explicit expression for the transfer function,

$$G(\lambda) = C(\lambda I - A)^{-1}B,$$

given as

$$G(\lambda) = C(\lambda I - A)^{-1}B = \sum_{k=1}^{\infty} \frac{\langle B, \varphi_k \rangle C\varphi_k}{\lambda - \zeta_k}, \tag{4.1}$$

where we recall that B and C are defined in (1.8)-(1.11) and (1.4), (1.5).

Recall that the transmission zeros for a transfer function $G(\lambda)$ are defined as the set of complex numbers λ satisfying $\det G(\lambda) = 0$. As we can see from (3.7) (and is also well known in the finite dimensional linear case [10]) solvability of the regulator problem requires that the eigenvalues of S are not transmission zeros of the plant. Unfortunately, it not usually easy to find the transmission zeros explicitly although there has been some work in this direction. For example, in the SISO case it has been shown in [12] in special cases the transmission zeros are real and interlace with the negative eigenvalues. In the SISO case with co-located actuators and sensors (i.e., when the input regions and output regions are the same, $\mathcal{S}_j = \widehat{\mathcal{S}}_j$, for all j), it can be shown that our systems satisfy the necessary conditions to conclude that no eigenvalue of S is a transmission zero of the transfer function and our first technical problem is resolved.

Assuming that $\det(G(\lambda_\ell)) \neq 0$ for all ℓ, for a given problem under consideration, then (at least formally) as we have seen in (3.7), for any $\Phi \in \mathcal{W}$, we have

$$\Gamma\Phi \overset{?}{=} \sum_{\ell=-\infty}^{\infty} \langle \Phi, \Phi_\ell \rangle_{\mathcal{W}} \, G(\lambda_\ell)^{-1} \, Q\Phi_\ell. \tag{4.2}$$

Unfortunately, it is still not clear that this infinite sum exists.

This brings us to our next fundamental difficulty. Using methods from classical elliptic boundary value problems, it can be shown that

$$|G(\lambda_\ell)| \sim C_1 \exp(-C_2\sqrt{|\lambda_\ell|}) \xrightarrow{\ell\to\infty} 0, \tag{4.3}$$

i.e., the transfer function, evaluated at the eigenvalues λ_ℓ, decays exponentially as $\ell \to \infty$. Thus we see that it is extremely difficult for the sum in (4.2) to exist. Indeed, for the sum to exist $\langle \Phi, \Phi_\ell \rangle_{\mathcal{W}} Q\Phi_\ell$ must be rapidly decreasing in order to compensate for the rapid increase of the terms from $G(\lambda_\ell)^{-1}$. From a physical point of view, this corresponds to the well known fact that a parabolic system does not want to oscillate rapidly so it is difficult for its output to track a rapidly oscillating output from a hyperbolic system. In order to make the tracking problem solvable, we must somehow damp high-order oscillations. There are several ways to deal with this problem. One could truncate high order oscillations by choosing Q so that

$$Q\Phi_\ell = 0 \quad \text{for} \quad |\ell| > N, \tag{4.4}$$

or, equivalently, we could restrict initial data for the exosystem to the span of the eigenfunctions $\{\Phi_\ell\}_{\ell=-N}^{N}$. But this approach really amounts to starting with a finite dimensional exosystem. Rather than take this approach we change our definiton of the reference signal in (1.14). Namely, we follow an appraoch suggested in [1] and consider reference signals obtained by convolution with special kernels that rapidly damp high order oscillations.

$$y_r(t) = Q_\rho^{(m)} W(t) = \int_0^1 k_\rho^{(m)}(\xi_0 - \xi) W_1(\xi, t)\, d\xi, \quad \xi_0 \in (0,1) \tag{4.5}$$

where for any $0 < \rho < 1$, and any $m = 1, 2, \cdots$ we define

$$k_\rho^{(m)}(\xi) = \frac{C_m}{\rho} \begin{cases} \exp\left((-1)^{m+1}\left(\dfrac{\rho^2}{\xi^2 - \rho^2}\right)^m\right) & , \quad 0 \le |\xi| < \rho \\ 0 & , \quad |\xi| \ge 1 \end{cases} \tag{4.6}$$

with C_m chosen so that $\int_0^1 k_\rho^{(m)}(\xi)\, d\xi = 1$ (see Figure 2).

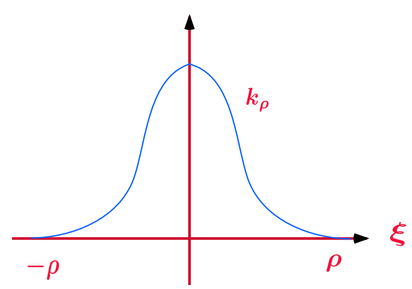


Figure 2: Graph of $k_\rho^{(m)}$

Notice that for every $m \in \mathbb{Z}^+$ the kernels converge to the Dirac δ-function:

$$k_\rho^{(m)}(\xi) \xrightarrow{\rho \to 0} \delta(\xi).$$

The following nontrivial result has been established in our forthcoming paper [1], using the theory developed in [7], Chapter IV and direct estimates of high order derivatives, $\frac{d^q}{d\xi^q} k_\rho^{(m)}$ for large q.

Theorem 4.1. *For every m there exists a $C_m > 0$ so that*

$$\left| Q_\rho^{(m)} \Phi_\ell \right| \le \frac{C_m}{\rho} e^{-a_\rho |\ell|^{1/\alpha}} \tag{4.7}$$

where $\alpha = \dfrac{(m+1)}{m}$ *and* $a_\rho = \dfrac{1}{2}\left(\dfrac{\pi\rho}{2m}\right)^{1/\alpha}$.

Theorem 4.1 leads to an important corollary concerning the series (4.2). If $m \ge 2$ then $Q_\rho^{(m)} \Phi_\ell \to 0$ as $\ell \to \infty$ so fast that the series (4.2) converges in spite of the fast growth of $G(\lambda_\ell)^{-1}$. Indeed, according to (4.3), $G(\lambda_\ell)^{-1} \sim \exp(c_2 |\ell|^{1/2})$ as $\ell \to \infty$, while, according to (4.7), $G(\lambda_\ell)^{-1} \sim \exp(-a_\rho |\ell|^{m/(m+1)})$ as $\ell \to \infty$. If $m \ge 2$ then

$$\frac{m}{(m+1)} > \frac{1}{2}$$

and, therefore, the decay of $Q_\rho^{(m)} \Phi_\ell$ is faster than the growth of $G(\lambda_\ell)^{-1}$. We refer to [1] for details.

5 Numerical Example

In this section we provide a numerical example which illustrates the above theory. For the numerical example we consider a problem with 2 inputs, 2 outputs and 2 reference signals to be tracked. More specifically, in (1.3), (1.4) we have $p = 2$.

Thus, as is portrayed in Figure 3, we see that $\mathcal{S}_j$ and $\widehat{\mathcal{S}}_j$ for $j = 1, 2$ are given as intervals along the boundaries $x_1 = 0$ and $x_2 = 0$, respectively. Namely, we let $x_1^1 = 1/4$, $x_1^2 = 3/5$, $x_2^1 = 1/3$, $x_2^2 = 2/3$, $\nu_1^1 = 1/8$, $\nu_1^2 = 1/8$, $\nu_2^1 = 1/8$, and $\nu_2^2 = 1/8$. Define

$$\begin{aligned}
\mathcal{S}_1 &= \{(0, x_2) \,:\, x_2 \in [x_2^1 - \nu_2^1, x_2^1 + \nu_2^1]\} \\
\mathcal{S}_2 &= \{(0, x_2) \,:\, x_2 \in [x_2^2 - \nu_2^2, x_2^2 + \nu_2^2]\} \\
\widehat{\mathcal{S}}_1 &= \{(x_1, 0) \,:\, x_1 \in [x_1^1 - \nu_1^1, x_1^1 + \nu_1^1]\} \\
\widehat{\mathcal{S}}_2 &= \{(x_1, 0) \,:\, x_1 \in [x_1^2 - \nu_1^2, x_1^2 + \nu_1^2]\}
\end{aligned}$$

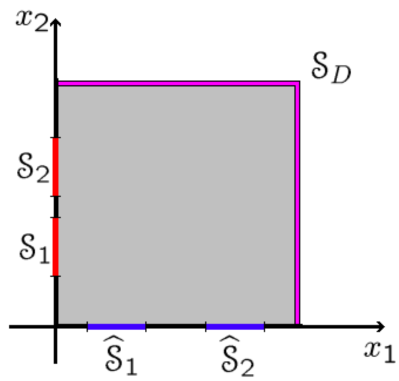


Figure 3: MIMO Plant with $\mathcal{S}$ Regions

The eigenvalues for the heat problem are given by

$$\zeta_{nm} = \mu_n^2 + \mu_m^2,$$

with associated eigenfunctions given by

$$\varphi_{nm} = \varphi_n(x_1)\varphi_m(x_2),$$

where

$$\varphi_j(\xi) = \sqrt{2}\cos(\mu_j \xi), \text{ and } \mu_j = \left(\frac{2j-1}{2}\right)\pi.$$

$$G_{i,j}(\lambda) = \langle (\lambda I - A)^{-1} b_i, c_j \rangle$$

$$= 4 \sum_{n,m=1}^{\infty} \frac{\big(\sin(\mu_m \widetilde{d}_i) - \sin(\mu_m \widetilde{c}_i)\big)\big(\sin(\mu_n \widetilde{b}_j) - \sin(\mu_n \widetilde{a}_j)\big)}{\mu_n \mu_m (\widetilde{b}_j - \widetilde{a}_j)(\widetilde{d}_i - \widetilde{c}_i)(\lambda - \zeta_{nm})} \tag{5.1}$$

$$= \sum_{n,m} \frac{\langle b_j, \varphi_{nm} \rangle \langle \varphi_{nm}, c_i \rangle}{\lambda_\ell - \zeta_{nm}}. \tag{5.2}$$

For this special case, in the Masters Thesis [8], Jeff Hood has given complete details verifying that the infinite sum defining $G(\lambda)$ converges uniformly and absolutely for all $\lambda \in \rho(A)$.

We have chosen an initial condition for the heat equation given by $\varphi(x) = x^2(1-2x)$. For the exosystem (one dimensional wave equation) we have taken initial conditions

$$W_0 = \begin{bmatrix} x(1-x) \\ \sin(2x) \end{bmatrix}.$$

The representation for the solution to the wave equation

$$\frac{d}{dt} W = SW, \quad W(0) = W_0 = \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix}$$

is given as

$$W = e^{St} W_0 = \sum_{n=-\infty}^{\infty} e^{\lambda_n t} \langle W_0, \Phi_n \rangle_{\mathcal{W}} \Phi_n,$$

or

$$W = \sum_{n=-\infty}^{\infty} e^{\lambda_n t} \left[\langle \psi_0', \Phi_n^{1'} \rangle_{L_2} + \langle \psi_1, \Phi_n^2 \rangle_{L_2} \right] \Phi_n.$$

From this we obtain the explicit formula for the solution $w = W_1$

$$w(\xi, t) = \sqrt{2} \sum_{n=1}^{\infty} \left[\langle \psi_0, \Xi_n \rangle \cos(n\pi t) + \langle \psi_1, \Xi_n \rangle \frac{\sin(n\pi t)}{n\pi} \right] \sin(n\pi\xi),$$

where $\Xi_n(\xi) = \sqrt{2}\sin(n\pi\xi)$ for $n = 1, 2, 3, \ldots$.

The reference outputs y_{ref} are thus approximations to

$$y_r = \widetilde{Q} W = \begin{bmatrix} w(\xi_1, t) \\ w(\xi_2, t) \end{bmatrix}$$

by

$$y_{\text{ref}}(t) = Q_\rho^{(2)} W(t) = \sum_{\ell=-\infty}^{\infty} e^{\lambda_\ell t} \langle W_0, \Phi_\ell \rangle Q_\rho^{(2)} \Phi_\ell, \tag{5.3}$$

where

$$Q_\rho^{(2)}\Phi_j = \begin{bmatrix} \int_0^1 k_\rho^{(2)}(\xi_1-\xi)Q_\rho^{(2)}\Phi_\ell(\xi)\,d\xi \\ \int_0^1 k_\rho^{(2)}(\xi_2-\xi)Q_\rho^{(2)}\Phi_\ell(\xi)\,d\xi \end{bmatrix}.$$

In our numerical example we have set $\xi_1 = 1/4$ and $\xi_2 = 3/4$.

In the numerical simulation we have also truncated the infinite sum and computed

$$u(t) = \Gamma W(t) = \sum_{\ell=-N}^{N} e^{\lambda_\ell t}\langle W_0, \Phi_\ell\rangle G(\lambda_\ell)^{-1} Q_\rho^{(2)}\Phi_\ell, \tag{5.4}$$

with $N = 25$.

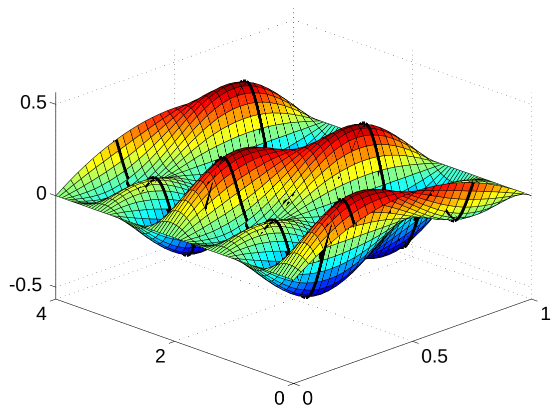


Figure 4: Solution Surface with Output Curves

The outputs are depicted as solid curves on the solution surface for the exosystem in Figure 4. The outputs and errors are plotted in Figures 5. The error $e_2(t) = y_2(t) - y_{r,2}(t)$ peaks much larger near $t = 0$ but both are essentially zero by $t = 1$. The sup norm error of the each of the channels of the error $e_1(t) = y_1(t) - y_{r,1}(t)$ and $e_2(t) = y_2(t) - y_{r,2}(t)$ are given by $0.4206e - 03$ and $0.4389e - 03$ on the interval $2 \le t \le 4$.

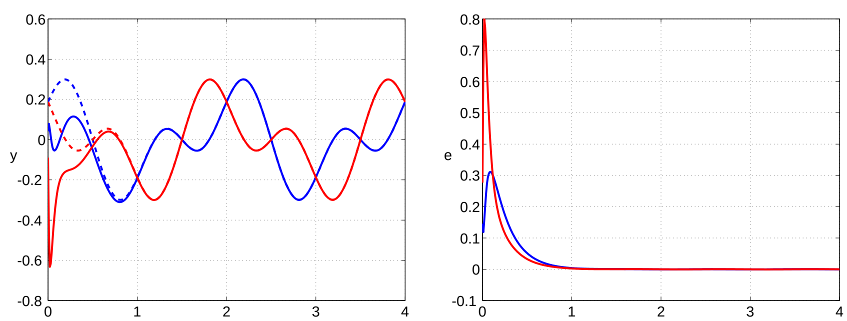
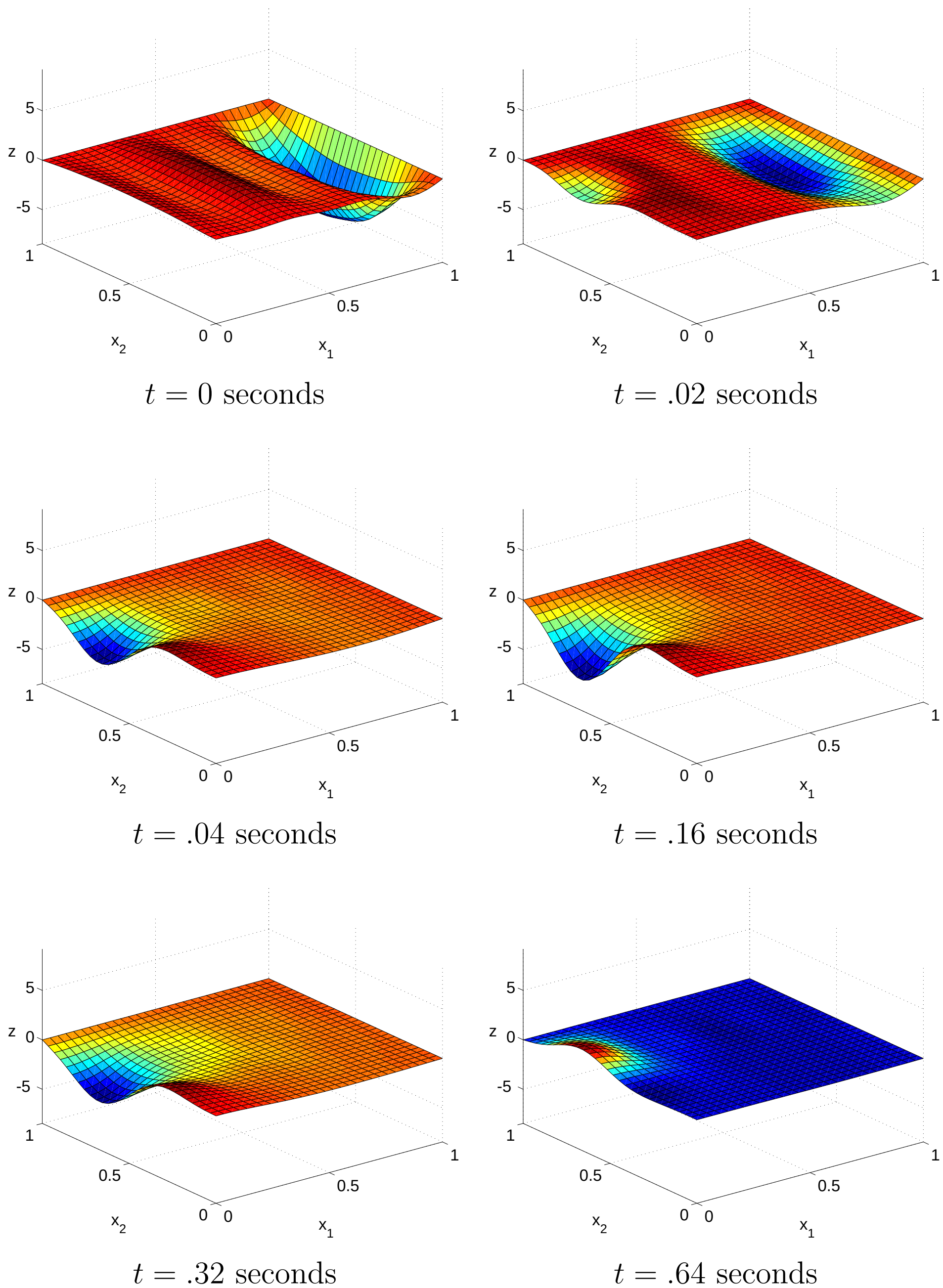


Figure 5: Output Tracking and Tracking Error

The outputs and errors are plotted in Figures 5. The error $e_2(t) = y_2(t) - y_{r,2}(t)$ peaks much larger near $t = 0$ but both are essentially zero by $t = 1$. The sup norm error of the each of the channels of the error $e_1(t) = y_1(t) - y_{r,1}(t)$ and $e_2(t) = y_2(t) - y_{r,2}(t)$ are given by $0.4206e-03$ and $0.4389e-03$ on the interval $2 \leq t \leq 4$.

Next, in Figure 6, we present several snapshots of the solution to the controlled heat problem at various times.



$t = 0$ seconds | $t = .02$ seconds

$t = .04$ seconds | $t = .16$ seconds

$t = .32$ seconds | $t = .64$ seconds

Figure 6: Solution Surface at Various Times

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