Optimization Methods for Target Problems of Control

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Abstract

The present report indicates an array of reachability problems relevant for nonstandard target problems of control. The problems are solved through dynamic optimization techniques for systems with nonintegral costs. This leads to new types of generalized Hamilton-Jacobi-Bellman-type equations in the general case and allows treatment through duality methods of convex analysis and minmax theory in the linear case.

Introduction

Among the principle problems of control theory is the one of *reachability* - the description of the domains in the state space that are reachable with available controls within the preassigned constraints on the controlled process. The last notion was used to solve "classical" problems in optimal control and differential games ([8], [3], [9]). However, recent activities in the field of advanced automation and navigation as well as in scientific computation have promoted new interest in this problem, [10], [13]. A particular question is whether a certain target set or group of sets representing, for example a safety (unsafe) zone or configuration could be reached (or avoided) by a controlled system despite the acting constraints. The posed question is obviously not an optimization problem. However we indicate here some optimization techniques that gives some answers to the question.

This report concentrates on controlled systems subjected to nonstandard functional optimality criteria which produce value functions that allow to define backward reach sets which are actually the solvability sets for various target problems, as well as "forward" reachability sets. Rather than introducing reach sets for *given* instances of time, the interest here is in sets reachable at some instances of time with state constraints true either for *some* instances or for the whole time interval. It formulates an array of various reachability problems and indicates related optimization problems that produce *value functions* whose level sets are the desired reach sets. It then introduces some equations for such value functions that grasp the required properties. These equations are of the generalized Hamilton-Jacobi-Bellman type and allow to treat classes of problems with nonsmooth parameters and solutions. For linear systems explicit formulas for the value functions are given in terms of duality relations of nonlinear analysis. (Such explicit solutions are mostly confined to convex optimization problems, however they are also availabl for some types of nonconvex problems with complementary convex constraints).

A direct calculation of value functions and possibly nonconvex reach sets through either exact HJB equations or through duality relations is complicated. For linear systems a parametrized sequence of HJB equations may be suggested which approximate the exact ones and allows to avoid calculation of viscosity solutions. The level sets for such approximate equations could produce ellipsoids whose intersections allow to externally approximate convex reach sets and whose unions allow to internally approximate the nonconvex reach sets. However, this ia topic for further investigation wiyhin the approaches of [4] - [7]. The reachability schemes indicated here are beyond those indicated in paper [4].

1 The system

Consider a controlled system described by an ordinary differential equation:

$$\dot{x} = f(t, x, u), \tag{1.1}$$

which in particular can be linear,

$$\dot{x} = A(t)x + B(t)u + C(t)v(t), \quad t_0 \leq t \leq \tau, \tag{1.2}$$

Here $x \in \mathbb{R}^n$ is the state , $u \in \mathbb{R}^m$ is the control, with $f(t, x, u)$ continuous in all the variables and satisfying conditions of uniques and extendability of solutions for all starting points and all $t \geq t_0$, whatever be the measurable function $v(t)$ and the control $u(t)$ restricted by hard bounds

$$u(t) \in \mathcal{P}(t), \quad t \geq t_0. \tag{1.3}$$

where $\mathcal{P}(t)$ is a compact set-valued function, continuous in t in the Hausdorff metric. We also require set $f(t, x, \mathcal{P}(t)) = F(t, x)$ to be convex and compact and differential inclusion (DI)

$$\dot{x} \in F(t, x)$$

to have a solution extendable within the intervals under consideration. The tube of solutions to the latter DI that start at set X^* at time τ is denoted as $X(t, \tau, X^*)$. This is the "reach set" of system (1.1).

For linear systems we require the $n \times n$ matrix function $A(t)$ with $n \times p$ and $n \times q$ matrix functions $B(t), C(t)$ to be continuous and $\mathcal{P}(t)$ to be convex. Next are the topics discussed in this paper.

2 The target problems

In this section we present some target problems together with closely related problems of reachability analysis.

Denote $x[t] = x(t, \tau, x^*)$ to be the *system trajectory* which starts from position $\{\tau, x^*\}, \quad x^* \in \mathbb{R}^n$, $\mathcal{M} = \{x \in \mathbb{R}^n : \varphi_1(z) \leq 1\}$ to be the *the target set* and $\mathcal{Y}(t) = \{x \in \mathbb{R}^n : \varphi(t, x) \leq 1\}$ to be the *state constraint*. In the sequel $\varphi(t, x), \varphi_1(x)$ are assumed continuous in the respective variables and closed convex in x.

Problem 2.1. *Given time θ and functions $\varphi(t, x), \varphi_1(x)$, find $W_1[\tau]$ - the set of points x, such that*

$$W_1[\tau] = \{x^* : \exists u(\cdot) \forall t \in [\tau, \theta] \; x[t] \in \mathcal{Y}(t) x[\theta] \in \mathcal{M}.\}$$

Here $W_1[\tau] = \{x : V_1(\tau, x^*) \leq 1\}$ is a level set of the *value function*

$$V_1(\tau, x^*) = \min_u \max\{\max_t \{\varphi(t, x[t]) | t \in [\tau, \theta]\}, \varphi_1(x[\theta])\} \; | u(t) \in \mathcal{P}(t), t \in [\tau, \theta], x[\tau] = x^*\}.$$

$W_1[t]$ is the *backward reach set relative to $\mathcal{M}$ under state constraints* $\mathcal{Y}(t)$, namely, the set of points $\{x^*\}$ for each of which there exists a control $u(t)$ which steers the system to $\mathcal{M}$ under state constraint $\mathcal{Y}(t)$.

If $\varphi(t, x) \equiv \varphi_1(x)$, then $\mathcal{Y}(t) \equiv \mathcal{M}$, and $W_1[\tau]$ is the set of points x^* for each of which there exists a controlled trajectory $x[t] \in \mathcal{M}, \forall t \in [\tau, \theta]$.

Problem 2.2. *Given time θ and functions $\varphi(t, x), \varphi_1(x)$, find $W_2[\tau]$ - the set of points x, such that*

$$W_2[\tau] = \{x \in \mathbb{R}^n : \forall u(\cdot) \forall t \in [\tau, \theta] \; x[t] \in \mathcal{Y}(t) x[\theta] \in \mathcal{M}\}.$$

Here $W_2[\tau] = \{x : V_2(\tau, x^*) \leq 1\}$ is a level set of the *value function*

$$V_2(\tau, x^*) = \max_u \max\{\max_t \{\varphi(t, x[t]) | t \in [\tau, \theta]\}, \varphi_1(x[\theta])\} \; | u(t) \in \mathcal{P}(t), t \in [\tau, \theta], x[\tau] = x^*\}.$$

This is the set of points from which *all* the controlled trajectories reach set $\mathcal{M}$ at time θ and also satisfy the state constraint $\mathcal{Y}(t)$.

If $\varphi(t,x) \equiv \varphi_1(x)$, then $\mathcal{Y}(t) \equiv \mathcal{M}$, and $\mathcal{W}_{1+}[\tau]$ is the set of points for each of which the reach tube *without state constraint* $X(t,\tau,x^*) \in \mathcal{M}, \forall t \in [\tau,\theta]$.

Problems 3.1 and 3.2 respectively reflect the properties of *weak and strong invariance* of the corresponding backward reach sets relative to the equation under consideration and the state constraints.

Problem 2.3. *Given time θ, and functions $\varphi(t,x), \varphi_1(x)$, find $W_3[\tau]$ - the set of points x^*, such that*

$$W_3[\tau] = \{x^* \in \mathbb{R}^n : \exists u(\cdot)\exists t \in [\tau,\theta] \;\; x[t] \in \mathcal{Y}(t)x[\theta] \in \mathcal{M}\}.$$

Here $W_3[\tau] = \{x : V_3(\tau,x) \leq 1\}$ is a level set of the *value function*

$$V_3(\tau,x) = \min_u \max\{\min_t\{\varphi(t,x[t])|t \in [\tau,\theta]\}, \varphi_1(x[\theta])\} \; |u(t) \in \mathcal{P}(t), t \in [\tau,\theta], x[\tau] = x^*\}.$$

This is the set of points such that each controlled trajectory reaches set $\mathcal{M}$ and also satisfies the state constraint $\mathcal{Y}(t)$ *for some* instant $t \in [\tau,\theta]$. If $\varphi(t,x) \equiv \varphi_1(x)$, then $\mathcal{Y}(t) \equiv \mathcal{M}$, and $W_3[\tau]$ is the set of points x^* for each of which there exists a controlled trajectory $x[t] \in \mathcal{M}$, *for some* $t \in [\tau,\theta]$. This is the union $\cup\{X(t,\tau,x^*)|t \in [\tau,\theta]\}$ of reach sets without state constraints.

Problem 2.4. *Given time θ and functions $\varphi(t,x), \varphi_1(x)$, find $W_4[\tau]$ - the set of points x, such that*

$$W_4[\tau] = \{x \in \mathbb{R}^n : \forall u(\cdot), \exists t \in [\tau,\theta] \; x[t] \in \mathcal{Y}(t)x[\theta] \in \mathcal{M}\}.$$

Here $W_4[\tau] = \{x : V_4(\tau,x) \leq 1\}$ is a level set of the *value function*

$$V_4(\tau,x^*) = \max_u \max\{\min_t\{\varphi(t,x[t])|t \in [\tau,\theta]\}, \varphi_1(x[\theta])\} \; |u(t) \in \mathcal{P}(t), t \in [\tau,\theta], x[\tau] = x^*\}.$$

If $\varphi(t,x) \equiv \varphi_1(x)$, then $\mathcal{Y}(t) \equiv \mathcal{M}$, and $W_4[\tau]$ is the set of points for each of which *each* of the controlled trajectories $x[t] \in \mathcal{M}$, *for some* $t \in [\tau,\theta]$.

Problems 3.3 and 3.4 respectively reflect the weak and strong possibilities of reaching the target set at some instant of time within the interval $[\tau,\theta]$.

The sets W_1, W_2, W_3, W_4 are the possible types of *backward reach sets* or *solvability sets* for target problems. Other possible options for such problems are beyond the scope of the present paper. Note that in general, for a linear system (1.1), the sets W_1, W_2 are closed convex, while W_3, W_4 are closed, but need not be convex.

A similar array of target problems is connected with forward reachability. In the forthcoming problems $3.5 - 3.8$ we denote $x[t] = x(t,t_0,x^*)$. The properties of function $\varphi_0(x)$ are similar to $\varphi_1(x)$.

Problem 2.5. *Given time* θ *and functions* $\varphi(t,x), \varphi_0(x)$*, find the* value function

$$V_5(\theta,x) = \min_u \max\{\max_t\{\varphi(t,x[\theta]), \varphi_0(x^*)|t \in [t_0,\theta]\ u(t) \in \mathcal{P}(t), x[\theta] = x\}\}.$$

Here $\mathcal{X}_1[\theta] = \{x : V_5(\theta,x) \leq 1\}$ is the set of points for each of which there exists a controlled trajectory which starts at time t_0 from a certain $x^* \in \mathcal{X}_0 = \{x^* : \varphi_0(x^*) \leq 1\}$ and ensures $x[t] \in \mathcal{Y}(t), \forall t \in [t_0,\theta]$. It is the conventional reach set under state constraints.

Problem 2.6. *Given time* θ *and functions* $\varphi(t,x), \varphi_0(x)$*, find the* value function*,*

$$V_6(\theta,x) = \max_u \max\{\max_t\{\varphi(t,x[t], \phi_0(x^*)|t \in [\tau,\theta]\ u(t) \in \mathcal{P}(t), x[\theta] = x\}.$$

Here $\mathcal{X}_2[\theta] = \{x : V_6(\theta,x) \leq 1\}$ is the set of points x such that $\forall u\ \exists x^*$ the respective trajectory $x[t] \in \mathcal{Y}(t), \forall t \in [t_0,\theta]$ and $x[\theta] = x$. If $\mathcal{X}_0$ is a singleton, then $\mathcal{X}_{1+}[t]$ is the reach tube without state constraints and $\mathcal{X}_{1+}[t] \subseteq \mathcal{Y}(t)$ for all $t \in [t_0,\theta]$.

Problem 2.7. *Given time* θ*, and functions* $\varphi(t,x), \varphi_0(x)$*, find the* value function*,*

$$V_7(\theta,x) = \min_u \max\{\{\min_t\{\varphi(t,x[t]|\ t \in [\tau,\theta]\}, \varphi_0(x^*)\}\ u(t) \in \mathcal{P}(t), x[\theta] = x\}.$$

Here $\mathcal{X}_3[t] = \{x : V_7(t,x) \leq 1\}$ is the set of points for which there exists a control $u(\cdot)$ and a starting point $x^* \in \mathcal{X}_0$ such that the respective trajectory $x[t] \in \mathcal{Y}(t)$ *for some* $t \in [t_0,\theta]$ and $x[\theta] = x$.

Problem 2.8. *Given time* $t_0 \leq \theta$ *and functions* $\varphi(t,x), \varphi_0(x)$*, find the* value function*,*

$$V_8(\theta,x) = \max_u \max\{\min_t\{\varphi(t,x[t])|t \in [t_0,\theta], \varphi_0(x^*)\}\ u(t) \in \mathcal{P}(t), x[\theta] = x\}.$$

Here $\mathcal{X}_4[\theta] = \{x : V_8(\theta,x) \leq 1\}$ is the set of points x such that for each control $u(\cdot)$ there exists a vector $x^* \in \mathcal{X}_0$ which ensure that the respective trajectory $x[t] \in \mathcal{Y}(t)$, *for some* $t \in [t_0,\theta]$ and $x[\theta] = x$.

Note that in general, for a linear system (1.1), the sets X_1, X_2 are closed convex, while X_3, X_4 are closed, but need not be convex.

The given array of problems may also include backward reachability for linear systems under *complementary convex constraints*. Here is an example of a *reach-evasion set*.

Problem 2.9. . *Given time* θ *and functions* $\varphi(t,x), \varphi_0(x)$*, find* $\mathcal{W}[\tau]$ *- the set of points* x^**, such that*

$$\mathcal{W}[\tau] = \{x^* \in \mathbb{R}^n : \exists u(\cdot), \exists t \in [t_0,\theta]\ x[t] \in \overline{\mathcal{Y}(t)}, x[\theta] \in \mathcal{M}\}.$$

where $\overline{\mathcal{Y}}$ *denotes the closure of the open complement of closed convex set* $\mathcal{Y}$ *and* $\mathcal{W}[t] = \{x^* : \mathcal{V}(t,x^*) \geq 1\}$ *is the complement of the open level set* $\{x : \mathcal{V}(t,x^*) < 1\}$ *of the* value function

$$\mathcal{V}(\tau,x^*) = \max_u \min\{\min_t\{\varphi(t,x[t])|t \in [\tau,\theta]\}, -\varphi_1(x[\theta])+2\}\ |u(t) \in \mathcal{P}(t), t \in [\tau,\theta], x[\tau] = x^*\}.$$

Here $\mathcal{W}[\tau]$ is the set of points for each of which there exists a controlled trajectory $x[t] \in \overline{\mathcal{Y}(t)}, \ \ \forall t \in [\tau, \theta]$ and $x[\theta] \in \mathcal{M}$. Therefore, it is the set of points from which it is possible to *avoid* the domain $int\mathcal{Y}(t)$ for all t while *reaching* the target set $\mathcal{M}$ (which is assumed to lie beyond $\mathcal{Y}(\theta) : \mathcal{M} \cap \mathcal{Y}(\theta) = \emptyset$). In general $\mathcal{W}[\tau]$ is a nonconvex set.

3 Solution methods. The HJB Equations

In the general case the respective value functions may be calculated through the generalized HJB equation. We shall indicate such equations for problems 3.1, 3.5.

Suppose $\varphi_0(x) = d^2(x, \mathcal{X}_0), \varphi_1(x) = d^2(x, \mathcal{M}), \varphi(t, x) = d^2(x, \mathcal{Y}(t))$. Starting with problem 3.5, denote $V_5(t, x) = V_5(t, x|V_5(t_0, x^0))$, emphasizing the dependence of $V_5(t, x)$ on the boundary condition $V_5(t_0, x^0) = \varphi_0(x)$.

Theorem 3.1. *Value function $V_5(t, x)$ satisfies the* **principle of optimality***, which has the semigroup form:*

$$V_5(\tau, x|V_5(t_0, x^0)) = V_5(\tau, x|V_5(t, \cdot|V_5(t_0, \cdot)), \tag{3.1}$$

with $t_0 \leq t \leq \tau$.

This property is established through a conventional argument [1] and its consequence is a similar property for respective reach sets. Relation (3.1) yields the following "forward" HJB equation.

Denote

$$\mathcal{H}(t, x, V, u) = V_t(t, x) + (V_x(t, x), f(t, x, u)).$$

Then the HJB equation is

$$V_{5t}(t, x) + \max_u(V_{5x}, f(t, x, u)) = 0, \tag{3.2}$$

when $V_5(t, x) \neq \varphi(t, x)$ and

$$\max_u\{\min\{\mathcal{H}(t, x, V_5, u), \mathcal{H}(t, x, \varphi, u)\}|u \in \mathcal{P}(t)\}\} = 0, , \tag{3.3}$$

when $V_5(t, x) = \varphi(t, x)$.The boundary condition is

$$V_5(t_0, x) = \max\{\varphi(t_0, x), \varphi_0(t_0, x)\}$$

.

Here V_t, V_x stand for the partial derivatives of $V(t,x)$, if these exist. Otherwise (3.2), (3.3) is a symbolic relation for the generalized HJB equation which has to be described in terms of subdifferentials, Dini derivatives or their equivalents. However the typical situation is that *V is not differentiable.* The treatment of equations (3.2), (3.3)then involves the notion of generalized "viscosity" solutions for this equation or their equivalents, [1],[12].

Similarly, if we deal with Problem 3.1, we will have a "backward" HJB equation for $V_1(t,x)$, namely,

$$V_{1t}(t,x) + \min_u (V_{1x}, f(t,x,u)) = 0, \tag{3.4}$$

when $V_1(t,x) \neq \varphi(t,x)$ and

$$\min_u \{\max\{\mathcal{H}(t,x,V_1,u), \mathcal{H}(t,x,\varphi,u)\} | u \in \mathcal{P}(t)\}\} = 0,, \tag{3.5}$$

when $V_1(t,x) = \varphi(t,x)$.The boundary condition is

$$V_1(t_0,x) = \max\{\varphi(\theta,x), \varphi_1(\theta,x)\}$$

.

Taking Problem 3.2 we will have

$$V_{2t}(t,x) + \max_u (V_{2x}, f(t,x,u)) = 0, \tag{3.6}$$

when $V_{2x}(t,x) \neq \varphi(t,x)$ and

$$\max_u \{\max\{\mathcal{H}(t,x,\mathcal{V}_{2x},u), \mathcal{H}(t,x,\varphi,u)\} | u \in \mathcal{P}(t)\}\} = 0,, \tag{3.7}$$

when $V_2(t,x) = \varphi(t,x)$.The boundary condition is

$$V_2(t_0,x) = \max\{\varphi(\theta,x), \varphi_1(\theta,x)\}$$

.

Finally we indicate the HJB equation for Problem 3.9. Then

$$\mathcal{V}(t,x) + \min_u (\mathcal{V}_x, f(t,x,u)) = 0, \tag{3.8}$$

when $\mathcal{V}(t,x) \neq \varphi(t,x)$ and

$$\min_u \{\min\{\mathcal{H}(t,x,\mathcal{V}_x,u), \mathcal{H}(t,x,\varphi,u)\} | u \in \mathcal{P}(t)\}\} = 0,, \tag{3.9}$$

when $\mathcal{V}(t,x) = \varphi(t,x)$.The boundary condition is

$$\mathcal{V}(t_0, x) = \min\{\varphi(\theta, x), -\varphi_1(\theta, x) + 2\}.$$

The HJB equations for the other problems of Section 3 are produced in a similar way. They follow from respective versions of the Principle of Optimality. The calculation of solutions to these equations in the general case is not simple and requires additional investigation. However, in the case of linear systems the value functions $V_1 - V_8, \mathcal{V}$ may be described through duality relations of convex analysis and related branches of optimization theory.

4 Solution Methods. Duality Techniques of Optimization Theory

In this section we indicate solution methods to the problems of this paper for the case of *linear systems*where the value functions could be found through the techniques of *convex analysis, semidefinite programming and minmax theory*, [2], [11]. We describe an example - the formula for calculating $\mathcal{V}(\tau, x)$ which allows to find the *reach-evasion set*. This shows the type of relations encountered here.

Suppose $y = Nx,\ z = Mx,\ y \in \mathbb{R}^m, z \in \mathbb{R}^q,\ \varphi(x) = (y, \mathbf{N}y),\ \varphi_1(x) = (z - m, \mathbf{M}(z - m)),\ \mathbf{N} = \mathbf{N}' > 0,\ \mathbf{M} = \mathbf{M}' > 0$, $\Lambda(t)$ is a nondecreasing scalar function of unit variation on $[\tau, \theta],\ l(t)$ is a continuous m-vector function from a compact class of such functions defined on the same interval, $s[t] = s(t, \theta, p|\Lambda(\cdot), l(\cdot))$ is the solution to the generalized equation

$$ds' = -s'A(t)dt - l'(t)N\beta d\Lambda(t),\ \ s'(\theta) = -\alpha p'M,$$

set $\mathcal{P}(t) = E(0, P(t))$ is an ellipsoid with center 0 and shape matrix $P = P' > 0$, symbol $\rho(q|\mathcal{P}) = \max\{(q,u)|u \in \mathcal{P}\}$ stands for the support function of convex compact set $\mathcal{P}$, so that $\rho^2(q|E(0,P)) = (q, P^{-1}q),\ \alpha, \beta > 0, \alpha + \beta = 1$. Then

$$\mathcal{V}(\tau, x) =$$

$$= \max_{l(\cdot)} \min_{\Lambda(\cdot)} \min_{p} \min_{\alpha,\beta} \{s'[\tau]x + \int_{\tau}^{\theta} (s'[t]B(t)P^{-1}(t)B'(t)s[t])^{1/2} + s'[t]v(t))dt +$$

$$-\alpha \int_{\tau}^{\theta} l'(t)\mathbf{N}^{-1}l(t)d\Lambda(t) + \beta(p'\mathbf{M}^{-1}p) + 2\beta\}.$$

The level set $\mathcal{W}[\tau]$ of this function is nonconvex. In contrast with this property the value function

$$V_1(t, x) =$$

$$= \max_{l(\cdot)} \max_{\Lambda(\cdot)} \max_{p} \min_{\alpha,\beta} \{ s'[\tau]x - \int_{\tau}^{\theta} (s'[t]B(t)P^{-1}(t)B'(t)s[t])^{1/2} + s'[t]v(t))dt -$$

$$-\alpha \int_{\tau}^{\theta} l'(t)\mathbf{N}^{-1}l(t)d\Lambda(t) - \beta(p'\mathbf{M}^{-1}p)\}.$$

for Problem 1 has a convex level set.

The problems of this section allow ellipsoidal approximations to the reach sets under dicussion having in view approximations of convex reach sets by intersections of ellipsoids and nonconvex reach sets by unions of ellipsoids along the framework of papers [4] - [7].

5 Conclusion

This paper presents the basics of optimization techniques for nonstandard target and reachability problems motivated by new trends in automation and navigation. The solutions are given in the form of generalized HJB equations or, in the linear case, in the form of duality relations of convex analysis and minmax theory.

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