AN ADDENDUM TO THE PROBLEM OF STOCHASTIC OBSERVABILITY

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Abstract

In this paper we introduce a concept of stochastic observability for a class of linear stochastic systems subjected both to multiplicative white noise and Markovian jumping. The definition of stochastic observability adopted here extends to this framework the definition of uniform observability of a time varying linear deterministic system. By several examples we show that the concept of stochastic observability introduced in this paper is less restrictive than those introduced in some previous papers.

Also we show that the concept of stochastic observability introduced in this paper, does not imply always the stochastic detectability as it would be expected.

Keywords. Linear stochastic systems, Markovian jumping, multiplicative white noise, stochastic observability.

1 Introduction

Both the observability property and controlability property of a linear system plaied a crucial role in solving a wide class of control problems. Here we recall that starting with the pioneer paper of Kalman [7], the observability and controlability properties provide sufficient conditions guaranteeing the existence of the stabilizing solution of a matrix Riccati differential equation which is connected with the linear quadratic problem and filtering problem [12, 13].

In stochastic framework the concept of stochastic observability was introduced in order to provide conditions which guarantee the existence of the stabilizing solutions of the matrix Riccati differential equations of stochastic control [9]. For the linear systems subjected to Markovian jumping some definitions of stochastic observability were introduced in [8, 6].

In this paper we introduce the concept of stochastic observability for a class of linear stochastic systems subjected both to multiplicative white noise and Markovian jumping. The definition of stochastic observability adopted here extends to this framework the definition of uniform observability of a time varying linear deterministic system [7, 1]. By several examples we show that the concept of stochastic observability introduced in this paper is less restrictive than those introduced by [8] and [6].

Also we show that the concept of stochastic observability introduced in this paper, does not imply always the stochastic detectability as it would be expected. Finally we show that the stochastic observability introduced here guarantees the positivity of the observability Gramian (if it exists) and additionaly any positive solution of corresponding Riccati equation is stabilizing as it happens in deterministic case.

2 Stochastic observability

Consider the system:

$$dx(t) = A_0(\eta(t))x(t)dt + \sum_{k=1}^{r} A_k(\eta(t))x(t)dw_k(t) \quad (2.1)$$

$$y(t) = C_0(\eta(t))x(t) \quad (2.2)$$

$x(t) \in \mathbf{R}^n, y \in \mathbf{R}^p, w(t) = (w_1(t) \ ... \ w_r(t))^*, t \geq 0$ is a standard Wiener process on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$; $\eta(t), t \geq 0$ is a right continuous homogenous Markov chain, with the state space set $\mathcal{D} = \{1, 2, ..., d\}$ and the probability transition matrix $P(t) = e^{Qt}, t > 0, Q = [q_{ij}]$ with $\sum_{j=1}^{d} q_{ij} = 0, i \in \mathcal{D}, q_{ij} \geq 0, i \neq j$ (see[2]).

Throughout this paper assume that $\{w(t)\}_{t\geq 0}$ and $\{\eta(t)\}_{t\geq 0}$ are independent stochastic processes, and $\mathcal{P}\{\eta(0) = i\} > 0$ for all $i \in \mathcal{D}$. For each $t_0 \geq 0$ and $x_0 \in \mathbf{R}^n, x(t, t_0, x_0)$ stands for the solution of (2.1) with initial condition $x(t_0, t_0, x_0) = x_0$. (For precise definition of the solution see [5, 11]).

Let $\Phi(t, t_0)$ be the fundamental matrix solution of (2.1). That is the j-th column of $\Phi(t, t_0)$ is $x(t, t_0, e_j)$, $j = 1, 2, ..., n$, where $e_j = (0 \ \ 0 \ ...0 \ \ 1 \ \ 0 \ ... \ 0)^*$ being vector of canonic basis of $\mathbf{R}^n$.

Now we are able to introduce the definition of stochastic observability.

Definition 2.1 We say that the system (2.1)-(2.2) is stochasticaly observable, or the triple $(C_0, A_0, A_1, ..., A_r; Q)$ is observable if there exist $\beta > 0, \tau > 0$ such that

$$E[\int_t^{t+\tau} \Phi^*(s,t)C_0^*(\eta(s))C_)(\eta(s))\Phi(s,t)ds|\eta(t) = i] \geq \beta I_n, \quad (2.3)$$

$i \in \mathcal{D}, (\forall)t \geq 0, E[\cdot|\eta(t) = i]$ being the conditional expectation with respect to the event $\eta(t) = i$.

Sometimes we shall write $(C_0, \mathbf{A}; Q)$ is observable instead of $(C_0, A_0, A_1, ..., A_r; Q)$ is observable.

Remark 2.1

a) In the particular case $\mathcal{D} = \{1\}$ and $A_k = 0 \ 1 \leq k \leq r$ the inequality (2.3) reduces to the well known definition of uniform observability for a time varying linear deterministic system (see e.g. [7, 1]).

b) If $\mathcal{D} = \{1\}$ the equations (2.1)-(2.2) become

$$dx(t) = A_0 x(t)dt + \sum_{k=1}^{r} A_k x(t) dw_k(t) \tag{2.4}$$

$$y(t) = C_0 x(t). \tag{2.5}$$

In this case (2.3) is

$$E[\int_t^{t+\tau} \Phi^*(s,t) C_0^* C_0 \Phi(s,t) ds] \geq \beta I_n \tag{2.6}$$

for all $t \geq 0, \Phi(s,t)$ being now the fundamental matrix solution of (2.4).

c) In the case $d \geq 2, A_k(i) = 0, 1 \leq k \leq r, i \in \mathcal{D}$, (2.1)-(2.2) reduce to

$$\dot{x}(t) = A_0(\eta(t)) \tag{2.7}$$

$$y(t) = C_0(\eta(t))x(t) \tag{2.8}$$

and in this case if (2.3) holds with $\Phi(s,t)$ standing for the fundamental matrix solution of (2.7) we say that the triple $(C_0, A_0; Q)$ is observable.

3 Some auxiliary results

3.1 Liapunov type operators and exponential stability

Let $\mathcal{S}_n \subset \mathbf{R}^{n \times n}$ be the space of symmetric matrices and $\mathcal{S}_n^d = \mathcal{S}_n \otimes \mathcal{S}_n \otimes ... \otimes \mathcal{S}_n$ (d factors).

Let $\mathcal{L} : \mathcal{S}_n^d \to \mathcal{S}_n^d$ be defined by

$$(\mathcal{L}X)(i) = A_0(i)X(i) + X(i)A_0^*(i) + \sum_{k=1}^{r} A_k(i)X(i)A_k^*(i) + \sum_{j=1}^{d} q_{ij} X(j) \tag{3.9}$$

$(\forall)\ i \in \mathcal{D}, X = (X(1)\ ...\ X(d)) \in \mathcal{S}_n^d$. The operator $\mathcal{L}$ will be termed as the Liapunov type operator defined by the system $(A_0, A_1, ..., A_r; Q)$.

It is easy to see that the adjoint operator $\mathcal{L}^*$ is given by

$$(\mathcal{L}^*X)(i) = A_0^*(i)X(i) + X(i)A_0(i) + \sum_{k=1}^{r} A_k^*(i)X(i)A_k(i) + \sum_{j=1}^{d} q_{ij} X(j) \tag{3.10}$$

$\mathcal{L}_0 : \mathcal{S}_n^d \to \mathcal{S}_n^d$ is the linear operator obtained from (3.1) taking $A_k(i) = 0,\ 1 \leq k \leq r, i \in \mathcal{D}$.

Consider the linear differential equation on $\mathcal{S}_n^d$

$$\frac{d}{dt} S(t) = \mathcal{L} S(t) \tag{3.11}$$

and set $e^{\mathcal{L}t}$ the linear evolution operator defined on $\mathcal{S}_n^d$ by (3.3).

The following result was proved in [3] in a more general setting.

Proposition 3.1
a) $e^{\mathcal{L}t}X \geq e^{\mathcal{L}_0 t}X \geq 0$, $e^{\mathcal{L}^* t}X \geq e^{\mathcal{L}_0^* t}X \geq 0$, $(\forall)$ $X = (X(1)\ X(2)\ ...X(d)) \in \mathcal{S}_n^d, X(i) \geq 0, i \in \mathcal{D}, t \geq 0$.
b) $[e^{\mathcal{L}^*(t-s)}X](i) = E[\Phi^*(t,s)X(\eta(t))\Phi(t,s)|\eta(s) = i]$ $(\forall)$, $t \geq s > 0, i \in calD, X = (X(1)\ X(2)\ ...\ X(d)) \in \mathcal{S}_n^d$.

Definition 3.1 We say that the zero solution of the equation (2.1) is mean square exponentially stable (MSES for shortness), or that the system $(A_0, A_1, ...A_r; Q)$ is stable if there exist $\alpha > 0, \beta > 0$ such that

$$E[|x(t)|^2] \leq \beta e^{-\alpha t}|x(0)|^2, \quad (\forall)t \geq 0, x(0) \in \mathbf{R}^n.$$

The following result provides necessary and sufficient conditions for the MSES of the zero solution of (2.1).

Theorem 3.1 *The following are equivalent:*
(i) The solution $x(t) = 0$ of the equation (2.1) is MSES.
(ii) $\lim_{t\to\infty} E[|x(t)|^2] = 0$ *for any solution $x(t)$ of (2.1).*
(iii) The eigenvalues of the operator $\mathcal{L}$ are located in the half plane $Re\ \lambda < 0$.
(iv) There exists $H = (H(1)\ H(2)\ ...\ H(d)) \in \mathcal{S}_n^d$ with $H(i) > 0, i \in \mathcal{D}$, such that the equation

$$\mathcal{L}X + H = 0$$

has a solution $X = (X(1)\ ...\ X(d)), X(i) > 0, i \in \mathcal{D}$.
(v) There exists H as before such that the equation

$$\mathcal{L}^*X + H = 0$$

has a solution $X = (X(1)\ ...\ X(d)), X(i) > 0, i \in \mathcal{D}$.
(vi) There exists $X = (X(1)\ ...\ X(d)), X(i) > 0$ such that $\mathcal{L}^*X < 0$.
For detailed proof see [3].

3.2 Stochastic detectability

Definition 3.2 We say that the system (2.1)-(2.2) is stochasticaly detectable, or that the triple $(C, \mathbf{A}; Q)$ is detectable if there exist $L = (L(1)\ L(2)\ ...\ L(d))$, $L(i) \in \mathbf{R}^{n\times p}$ such that the zero solution of the equation

$$dx(t) = (A_0(\eta(t)) + L(\eta(t))C_0(\eta(t)))x(t)dt + \sum_{k=1}^{r} A_k(\eta(t))x(t)dw_k(t)$$

is MSES. L will be termed stabilizing injection.

Proposition 3.2 *The following are equivalent:*
(i) The triple $(C_0, \mathbf{A}; Q)$ is detectable.
(ii) The system of linear equations

$$A_0^*(i)X(i) + X(i)A_0(i) + C_0^*(i)\Gamma^*(i) + \Gamma(i)C_0(i) + \sum_{k=1}^{r} A_k^*(i)X(i)A_k(i) + \sum_{j=1}^{d} q_{ij}X(j) + I_n = 0 \tag{3.12}$$

has a solution $X = (X(1) \ldots X(d)) \in \mathcal{S}_n^d$, $\Gamma = (\Gamma(1) \ldots \Gamma(d))$, $X(i) > 0, \Gamma(i) \in \mathbf{R}^{n\times p}, i \in \mathcal{D}$.
(iii) The linear matrix inequality

$$A_0^*(i)X(i) + X(i)A_0(i) + C_0^*(i)\Gamma^*(i) + \Gamma(i)C_0(i) + \sum_{k=1}^{r} A_k^*(i)X(i)A_k(i) + \sum_{j=1} dq_{ij}X(j) < 0 \tag{3.13}$$

has a solution $X = (X(1)\ X(2) \ldots X(d)) \in \mathcal{S}_n^d$, $\Gamma = (\Gamma(1)\ \Gamma(2) \ldots \Gamma(d))$, $X(i) > 0$, $\Gamma(i) \in \mathbf{R}^{n\times p}$, $i \in \mathcal{D}$. Moreover, if $(X, \Gamma), X > 0$ is a solution of (3.4), then $L = (L(1)\ L(2) \ldots L(d))$, $L(i) = X^{-1}(i)\Gamma(i)$ provides a stabilizing injection.

4 Main results

Based on Proposition 3.1 (b) we obtain:

Proposition 4.1 *The following are equivalent:*
(i) The system (2.1)-(2.2) is stochastically observable.
(ii) There exists $\tau > 0$ such that

$$\int_0^\tau e^{\mathcal{L}^* s}\tilde{C}ds > 0 \tag{4.14}$$

$\tilde{C} = (\tilde{C}(1) \ldots \tilde{C}(d))$, $\tilde{C}(i) = C_0^*(i)C_0(i), i \in \mathcal{D}$.
(iii) There exists $\tau > 0$ such that $X_0(\tau) > 0$, where $X_0(t)$ is the solution of the problem with initial value

$$\frac{d}{dt}X_0(t) = L^*X_0(t) + \tilde{C}, \qquad X(0) = 0. \tag{4.15}$$

Proof. $(i) \leftrightarrow (ii)$ follows from Proposition 3.1 (b). Since

$$X_0(t) = \int_0^t e^{\mathcal{L}^*(t-s)}\tilde{C}ds = \int_0^t e^{\mathcal{L}^* s}\tilde{C}ds, \ \ t \geq 0$$

it follows that $(iii) \leftrightarrow (ii)$. The proof is complete.

Further from Porposition 4.1 and Proposition 3.1 (a) we obtain:

Proposition 4.2 *The following hold:*
(i) If for each $i \in \mathcal{D}$, the pair $(C_0(i), A_0(i))$ is observable (in deterministic sense) then the triple $(C_0, A_0; Q)$ is observable.
(ii) If $(C_0, A_0; Q)$ is observable, then $(C_0, \mathbf{A}; Q)$ is observable.

Proposition 4.3 *Let $X_0(t)$ be the solution of the problem with initial value (4.2). If there exists $\tau > 0$ such that $X_0(\tau) > 0$ then $X_0(t) > 0$ for all $t > 0$.*
Proof. For each $t > 0$, we write the representation

$$X_0(t) = (X_0(t,1),\ X_0(t,2), ..., X_0(t,d)) = \int_0^t e^{\mathcal{L}^*(t-s)} \tilde{C} ds.$$

Since $e^{\mathcal{L}^*(t-s)} : \mathcal{S}_n^d \to \mathcal{S}_n^d$ is a positive operator, we deduce that $X_0(t) \geq 0$, for all $t \geq 0$. Moreover if $t \geq \tau$ we have $X_0(t) \geq X_0(\tau)$, therefore if $X_0(\tau) > 0$, we have $X_0(t) > 0$ for all $t \geq \tau$. It remains to show that $X_0(t) > 0, 0 < t < \tau$. To this end we show that $detX_0(t,i) > 0,\ 0 < t < \tau,\ i \in \mathcal{D}$. Indeed, since $detX_0(t,i) = det\left\{\int_0^t e^{\mathcal{L}^*(t-s)} \tilde{C} ds)(i)\right\}$, we deduce that $t \to detX_0(t,i)$ is an analytic function.

The set of its zeros on $[0, \tau]$ has no accumulation point. In this way it will follow that there exist $\tau_1 > 0$ such that $detX_0(t,i) > 0$ for all $t \in (0, \tau_1]$. Invoking again the monotonicity of the function $t \to X_0(t)$ we conclude that $X_0(t) > 0$ for all $t \geq \tau_1$, and the proof ends.

Remark 4.1 From Proposition 4.1 and Proposition 4.3 it follows that the stochastic observability for the system (2.1)-(2.2) may be checked by using a numerical procedure to verify the positivity of the solution $X_0(t)$ through an enough long interval of time.

Proposition 4.4 *The triple $(C_0, \mathbf{A}; \mathbf{Q})$ is observable if and only if does not exist $\tau > 0, i \in \mathcal{D}$ and $x_0 \neq 0$ such that*

$$E\left[|y(t,0,x_0)|^2 | \eta(0) = i\right] = 0$$

$(\forall)$ $t \in [0, \tau]$ where $y(t,0,x_0) = C_0(\eta(t))x(t,0,x_0)$, $x(t,0,x_0)$ being the solution of (2.1) having the initial condition $x(0,0,x_0) = x_0$.

The next result provide a sufficient condition assuring the stochastic observability. Its proof may be found in [10].

Proposition 4.5 *The triple $(C_0, \mathbf{A}; Q)$ is observable if for every $i \in \mathcal{D}$, rank $M(i) = n$, where*

$$\begin{aligned} M(i) = [&C_0^*(i), A_0^*(i)C_0^*(i), \ldots, (A_0^*(i))^{n-1}C_0^*(i), \\ &q_{i1}C_0^*(1), \ldots, q_{id}C_0^*(d), A_1^*(i)C_0^*(i), \ldots, A_r^*(i)C_0^*(i)]\,. \end{aligned}$$

In the following examples, the stochastic observability used in this paper is compared with other types of stochastic observability, for example the one introduced in [6] and [8]. We also show that the stochastic observability used in this paper doesn't imply always the stochastic detectability as we would expected.

Example 1 The case of a system subjected to Markovian jumping with $d = 2, n = 2, p = 1$. Take

$$A_0(1) = A_0(2) = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

$$C_0(1) = [1 \quad 0], C_0(2) = [0 \quad 1], Q = \begin{bmatrix} -q & q \\ q & -q \end{bmatrix}, \alpha \in \mathbf{R}, q > 0.$$

It is obvious that the pairs $(C_0(1), A_0(1)), (C_0(2), A_0(2))$ are not observable in the deterministic sense. Therefore this system is not stochastically observable, in the sense of [8, 6]. We shall show that this system is stochastically observable in the sense of Definition 2.1.

To this end we use the implication $(iii) \Longrightarrow (i)$ in Proposition 4.1. We show that there exists $\tau > 0$ such that $X_1(\tau) > 0, X_2(\tau) > 0$, where $X_i(t), i = 1, 2$ is the solution of the Cauchy problem:

$$\begin{aligned} \frac{d}{dt}X_i(t) &= A_0^*(i)X_i(t) + X_i(t)A_0(i) + \sum_{j=1}^{2} q_{ij}X_j(t) \\ &\quad + C_0^*(i)C_0(i), \\ X_i(0) &= 0, \quad i = 1, 2. \end{aligned} \tag{4.16}$$

From the representation formula

$$(X_1(t), X_2(t)) = \int_0^t e^{\mathcal{L}_0^*(t-s)}\tilde{C}ds$$

it follows that $X_i(t) \geq 0$ for all $t \geq 0$.

Therefore it is sufficient to show that there exists $\tau > 0$ such that $detX_i(\tau) > 0$.

Set $X_i(t) = \begin{pmatrix} x_i(t) & y_i(t) \\ y_i(t) & z_i(t) \end{pmatrix}, i = 1, 2$. After some simple cmputations we have $detX_i(t) = x_i(t)z_i(t) - y_i^2(t) = x_i(t)z_i(t) = \tilde{x}(t)\tilde{z}(t), t \geq 0$ where

$$\tilde{x}(t) = \frac{1}{2}\int_0^t [e^{\alpha s} + e^{(2\alpha - q)s}]ds$$

$$\tilde{z}(t) = \frac{1}{2}\int_0^t [e^{2\alpha s} - e^{2(\alpha - q)s}]ds.$$

For detail see [4].

It is easy to see that for every $\alpha \in \mathbf{R}, q > 0$ we have $\lim_{t\to\infty} \tilde{x}(t)\tilde{z}(t) > 0$.

Remark 4.2 Let us consider the system of type (2.1)-(2.2) with $n = 2, d = 2, p = 1, r = 1$ and $A_0(1) = A_0(2) = \alpha I_2, C_0(1) = (1 \quad 0), C_0(2) = (0 \quad 1), A_1(i)$ a 2×2 arbitrary matrix,

$Q = \begin{bmatrix} -q & q \\ q & -q \end{bmatrix}, \alpha \in \mathbf{R}, q > 0$. Combining the conclusion of Example 1 with Proposition 4.2 (ii) it follows that the system $(C_0,\ (A_0, A_1); Q)$ is observable.

Example 2 The stochastic observability does not imply always stochastic detectability. Let us consider the system subjected to Markovian jumping with $d = 2,\ n = 2,\ p = 1$,

$$A_0(1) = A_0(2) = \frac{q}{2} I_2,\ C_0(1) = [1 \quad 0]\,,\ C_0(2) = [0 \quad 1]\,,\ Q = \begin{bmatrix} -q & q \\ q & -q \end{bmatrix}. \tag{4.17}$$

From the previous example we conclude that the system $(C_0, A_0; Q)$ is observable. Invoking $(i) \Leftrightarrow (ii)$ from Proposition 3.2 we deduce that if the system (4.4) would be stochastically detectable, then would exist the matrices $X(i) > 0$ and $\Gamma(i) = \begin{bmatrix} \gamma_1(i) \\ \gamma_2(i) \end{bmatrix}, i = 1, 2$ which verify the following system of linear equations:

$$A_0^*(i)X(i) + X(i)A_0(i) + \Gamma(i)C_0(i) + C_0^*(i)\Gamma^*(i) + \sum_{j=1}^{2} q_{ij}X(j) + I_2 = 0,\ i = 1, 2$$

which implies $I_2 + \begin{bmatrix} 2\gamma_1(1) & \gamma_2(1) \\ \gamma_2(1) & 0 \end{bmatrix} < 0$ which is a contradiction.

Example 3 Let us consider the stochastic system (2.1)-(2.2) with $n = 2,\ d = 2,\ r = 1,\ p = 1,\ A_0(1) = A_0(2) = \alpha I_2,\ C_0(1) = [1 \quad 0]\,,\ C_0(2) = [0 \quad 1]\,,\ A_1(1) = \beta I_2,\ A_1(2)$ is a 2×2 arbitrary matrix, $Q = \begin{bmatrix} -q & q \\ q & -q \end{bmatrix},\ \alpha \in \mathbf{R},\ \beta \in \mathbf{R},\ q > 0$ which satisfy $2\alpha - q + \beta^2 = 0$.

¿From Remark 4.2 it follows that the above system is stochastically observable. As in the previous example based on equivalence $(i) \leftrightarrow (ii)$ from Proposition 3.2 we may show that it is not stochastically detectable (see citepreprint01 for details).

The following two results show that the stochastic observability defined in this paper guarantees the positivity of the observability gramian and the fact that all semipositive solutions of the Riccati equations are stabilizing solutions as in the deterministic case. The proofs are omitted for shortness and may be found in [10].

Proposition 4.6 *Assume that* $(C_0, A_0, \ldots, A_r; Q)$ *is observable and the algebraic equation on* $\mathcal{S}_n^d$

$$\mathcal{L}^* X + \tilde{C} = 0 \tag{4.18}$$

has a solution $\tilde{X} \geq 0$.

Then :

(i) The system $(A_0, A_1, \ldots, A_r; Q)$ *is stable.*

(ii) $\tilde{X} > 0$.

(iii) The equation (4.5) has a unique positive semidefinite solution.

Let us consider the following system of general algebraic Riccati equations:

$$A_0^*(i)X(i)+X(i)A_0(i)+\sum_{k=1}^{r} A_K^*(i)X(i)A_k(i)+\sum_{j=1} dq_{ij}X(j)-(X(i)B_0(i)+\sum_{k=1}^{r} A_k^*(i)X(i)B_k(i))(D^*(i)D(i)+\sum_{k=1}^{r} B_k^*(i)X(i)B_k(i))^{-1}(B_0^*(i)X(i)+\sum_{k=1}^{r} B_k^*(i)X(i)A_k(i))+C_0^*(i)C_0(i)=0 \tag{4.19}$$

where $A_k(i), C_0(i)$ are as in (2.1)-(2.2) and $B_k(i), D(i)$ are $n \times m$ and $p \times m$ given matrices. Such algebraic equations are closly related to the linear quadratic optimization problem for a linear stochastic system subjected both to Markovian jumping and multiplicative white noise (see [4, 10] for details).

The following result is proved in [10].

Proposition 4.7 *If* $(C_0, \mathbf{A}; Q)$ *is observable then any semipositive solution of the system (4.6) is a stabilizing solution.*

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