A Comparison of Balanced Truncation Techniques for Reduced Order Controllers

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Abstract

The need for real-time control of a physical system necessitates controllers that are low order. We compare two methods for obtaining such controllers, for systems that are modeled by partial differential equations. The first is the standard technique of balanced realization followed by truncation. The second, LQG balancing, can be thought of as balancing based on the controller, and was discussed for systems modeled by ordinary differential equations by Jonckheere and Silverman. Extensions to PDE systems have been established by Curtain. In this work, we compare the designs for the one-dimensional linear Klein-Gordon Equation.

1 Introduction

The need for practical, robust, real-time controllers for physical systems presents a challenge, especially when the problem at hand is modeled by a partial differential equation (PDE). The standard techniques for robust controller design for PDE systems in the state space yield controllers that are inherently large-scale, and thus a reduction in size must take place at some point. Traditional methods to obtaining lower order controllers involve reducing the model from that for the PDE, and then applying a standard control design technique. We term such approaches as "reduce-then-design". In several recent papers [5, 17, 21, 22, 23], the proper orthogonal decomposition (POD) has been applied to obtain reduced order models. Another commonly used technique is balanced realization and truncation (see for example [24]), which is one of the techniques that we will apply in this paper.

In [3], it was argued that reduce-then-design methods may have an inherent weakness in that some of the essential "physics" in the PDE model may be lost in model reduction. Consequently, more robustness is demanded from the low order controller. In an attempt to capture characteristics of the PDE controller before the reduction step, an alternative to reduce-then-design was suggested in [2, 3] that involved using the POD for controller reduction. Here, we consider a second balancing technique which may accomplish the same effect. This method is called LQG balancing and was established for large-scale systems of ordinary differential equations in [16]. Recently, Curtain has extended these results to PDE systems [7].

In this paper, we compare balanced realization and truncation with LQG balanced realization and truncation, by formally applying both methods to the Klein-Gordon equation. In Section 2, we present an overview of the linear-quadratic-Gaussian (LQG) compensator-based feedback controller design. In Section 3, we outline the two balancing techniques. In Section 4, we present our

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example problem and numerical results which include a comparison of performance and robustness properties of the closed loop systems. We conclude with some future directions in Section 5.

2 LQG Feedback Control Design

Assume a PDE model for a physical system of interest, given in abstract form as

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \tag{1}$$

where $x(t)$ is the state of the system in Hilbert space X, and $u(t)$ is the control in Hilbert space U. In addition, we assume a state measurement, $y(t)$ in Hilbert space Y, of the form

$$y(t) = Cx(t). \tag{2}$$

We shall denote the state space system given by (1) and (2) as $\Sigma(A, B, C)$. Associated with this system is the transfer function given by $G(s) = C(sI - A)^{-1}B$ where $G(s) : U \to Y$. We note that although $G(s)$ is the unique transfer function for $\Sigma(A, B, C)$, there are many state space systems that would give rise to $G(s)$. This observation provides a rationale for balancing, and will be discussed further in the next section.

One method to derive a compensator-based controller is via the solution to the LQG control problem. This solution provides a state estimate, $x_c(t)$, and control, $u(t)$, that are given by the equations

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + Fy(t) \qquad x_c(0) = x_{c_0} && (3)\\ u(t) &= -Kx_c(t). && (4) \end{aligned}$$

Design of a controller entails determining A_c, F and K that produce a stable closed loop system

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A & -BK \\ FC & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ x_c(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ x_{c_0} \end{bmatrix}. \tag{5}$$

Under conditions of stabilizability of (A, B) and detectability of (A, C) (see for example, [6, 9, 12]), the operators A_c, F and K can be obtained by the solution of two algebraic Riccati equations

$$\begin{aligned} A^*\Pi + \Pi A - \Pi BB^*\Pi + C^*C = 0 && (6)\\ AP + PA^* - PC^*CP + BB^* = 0. && (7) \end{aligned}$$

Once the solutions of the control Riccati equation, Π, and the filter Riccati equation, P, are obtained, then

$$\begin{aligned} K &= B^*\Pi \\ F &= PC \\ A_c &= A - BK - FC. \end{aligned} \tag{8}$$

For certain PDEs, the control law can be written in integral form as

$$u(t) = -Kx_c(t) = \int_\Omega k(s)x_c(t,s)ds, \tag{9}$$

for spatial variable $s \in \Omega$ (see for example [19]). The kernel of the integral is called a functional gain and is important for several reasons. First, gains can be computed off-line and stored, so that in computation of the control, the gain is multiplied by the state estimate and numerically integrated. In addition, research has been done involving reduced order controllers and sensor design based on information in the functional gains in [4, 3, 11, 18, 20]. It was noted in [1, 4, 3] that a reduce-then-design approach based on POD reduction of the model followed by control design can yield finite dimensional approximations of the functional gains that do not converge. In our numerical example, we will examine gains from the two balancing methods.

In order to compute the PDE controller, an approximation scheme for which convergence is known is applied to (1), (2). In the case of a finite element scheme with basis of dimension N (where $N \to \infty$ yields the PDE system), the approximating system is given by

$$\dot{x}^N(t) = A^N x^N(t) + B^N u^N(t), \qquad x^N(0) = x_0^N \tag{10}$$

$$y^N(t) = C^N x^N(t). \tag{11}$$

A finite dimensional compensator for the approximating system can then be obtained by solving the Riccati equations (see [12])

$$\dot{x}_c^N(t) = A_c^N x_c^N(t) + F^N y(t), \qquad x_c^N(0) = x_{c_0}^N \tag{12}$$

$$u^N(t) = -K^N x_c^N(t). \tag{13}$$

Then combining equations (10), (11), (12) and (13) yields the finite dimensional approximation to the closed loop system given by

$$\begin{bmatrix} \dot{x}^N \\ \dot{x}_c^N \end{bmatrix} = \begin{bmatrix} A^N & -B^N K^N \\ F^N C^N & A_c^N \end{bmatrix} \begin{bmatrix} x^N \\ x_c^N \end{bmatrix} \qquad \begin{bmatrix} x^N(0) \\ x_c^N(0) \end{bmatrix} = \begin{bmatrix} x_0^N \\ x_{c_0}^N \end{bmatrix}. \tag{14}$$

Throughout this paper, we assume that N is large enough so that the behavior of the approximating system has converged to the behavior of the PDE model. We refer to this large-scale finite dimensional approximation as the *full-order system* and use the notation $\Sigma(A^N, B^N, C^N)$ to denote corresponding approximation in (10), (11). We denote the closed loop system matrix for the full order system as

$$\mathcal{A} = \begin{bmatrix} A^N & -B^N K^N \\ F^N C^N & A_c^N \end{bmatrix}.$$

Note that an implementation problem exists at this point, in that a full order compensator will not offer real time control for most physical problems. For this reason, reduced order compensators need to be considered. As discussed in the introduction, one way to obtain a low order compensator is to perform model reduction followed by control design. We can think of the reduced order model as

$$\dot{x}^q(t) = A^q x^q(t) + B^q u^q(t), \qquad x^q(0) = x_0^q \tag{15}$$

$$y^q(t) = C^q x^q(t). \tag{16}$$

where $q << N$. By solving (6), (7) with A^q, B^q, and C^q, we obtain the low order compensator and control law

$$\begin{aligned} \dot{x}_c^q(t) &= A_c^q x_c^q(t) + F^q y(t), \qquad x_c^q(0) = x_{c0}^q \\ u^q(t) &= -K^q x_c^q(t). \end{aligned} \tag{17}$$

To simulate the performance of the low order compensator, we apply it to the full order system in (10), (11); that is, we simulate the closed loop system

$$\begin{bmatrix} \dot{x}^N(t) \\ \dot{x}_c^q(t) \end{bmatrix} = \begin{bmatrix} A^N & -B^N K^q \\ F^q C^N & A_c^q \end{bmatrix} \begin{bmatrix} x^N(t) \\ x_c^q(t) \end{bmatrix}, \qquad \begin{bmatrix} x^N(0) \\ x_c^q(0) \end{bmatrix} = \begin{bmatrix} x_0^N \\ x_{c0}^q \end{bmatrix}. \tag{18}$$

In this paper, to form the low order system in (15), (16), we will use balanced realization and truncation, and LQG balanced realization and truncation. For notational purposes, let the closed loop system matrix

$$\begin{bmatrix} A^N & -B^N K^q \\ F^q C^N & A_c^q \end{bmatrix} \tag{19}$$

be denoted by A_T for balanced truncation and $ricA_T$ for LQG balanced truncation. We discuss these truncation methods in the following section.

3 Balanced Realizations and Truncation

Balanced realization and truncation is a common procedure that can be found in standard references on control, e.g., [9] for PDE systems. It is based on the premise that a low order approximation to $\Sigma(A, B, C)$ could be obtained by eliminating any states that are difficult to control and to observe. Then, the reduced order system, denoted by $\Sigma(A^q, B^q, C^q)$ where $q << N$, is used as described above to obtain the approximate solutions Π^q, P^q to the algebraic Riccati equations (6), (7), and the corresponding K^q, F^q, A_c^q.

To identify controllable and observable states, we consider the controllability Gramian

$$L_B = \int_0^\infty e^{At} B B^* e^{A^* t} dt \tag{20}$$

and the observability Gramian

$$L_C = \int_0^\infty e^{A^* t} C^* C e^{At} dt. \tag{21}$$

The reachable states are reflected by L_B and the observable states by L_C. A difficulty in eliminating states based upon L_B and L_C is that these Gramians are realization dependent. That is, one can choose two systems $\Sigma(A_1, B_1, C_1)$ and $\Sigma(A_2, B_2, C_2)$ that both give rise to transfer function $G(s)$, but have different Gramians. Thus, if a realization was sought which led to few controllable states (therefore many states to truncate), that realization may have many observable states (therefore few states to truncate) and vice versa. This apparent dilemma is addressed by the *balanced realization*, in which states that are difficult to control coincide with states that are difficult to observe. In this particular realization,

$$L_B = L_C = diag(\sigma_1, \sigma_1, \dots, \sigma_n, \dots), \qquad \sigma_1 \geq \sigma_2 \geq, \cdots, \geq \sigma_n \geq \cdots \geq 0$$

where $\sigma_i = \sqrt{\lambda_i(L_B L_C)}$ are the Hankel singular values and are realization invariant.

The following theorem from [8, 13, 14] and others gives conditions for existence of the Gramians.

Theorem: If $\Sigma(A, B, C)$ is exponentially stable, then the controllability and observability Gramians, L_B, L_C, exist and are the unique positive definite solutions to the Lyapunov equations

$$AL_B + L_B A^* 0 - BB^*$$

$$A^* L_C + L_C A = -C^* C.$$

If L_B, L_C exist, then there exists a similarity transformation T that gives rise to the balanced state space system which we denote as $\Sigma(A_{bal}, B_{bal}, C_{bal}) = \Sigma(TAT^{-1}, TB, CT^{-1})$. The ordering of the Hankel singular values gives information on the states that are insignificant with regard to both controllability and observability of the system. In particular, we write

$$A_{bal} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B_{bal} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \qquad C_{bal} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

where the $A_{bal}, B_{bal}, C_{bal}$ are partitioned so that the states corresponding to the q most “significant” Hankel singular values are given by the truncated system $\Sigma(A_{11}, B_1, C_1)$. The corresponding reduced transfer function is $G_r(s) = C_1(sI - A_{11})^{-1}B_1$, and the truncation error can be represented as

$$||G(s) - G_r(s)||_\infty \leq 2 \sum_{i=q+1}^{\infty} \sigma_i.$$

Note that for infinite dimensional (PDE) systems, convergence of this quantity is an issue that must be considered (see [7]).

The following algorithm can be used to find the balancing transformation T.

- Compute L_B, L_C from $\Sigma(A, B, C)$.
- Since L_B is symmetric positive definite, the Cholesky decomposition yields $L_B = R^* R$.
- $R^* L_C R = U\Sigma^2 U^*$ where $\Sigma^2 = diag(\sigma_i)_{i=1}^{\infty}$ and U is unitary.
- Define $T = \Sigma^{\frac{1}{2}} U^* R$. It follows that $(T^{-1})^* L_C T^{-1} = \Sigma = T L_B T^*$.

Balanced realization and truncation is a rather standard technique and has been used with favorable results. However, our goal is to utilize information regarding the closed loop design for the PDE system (which we determine through computation of the full order system with large N) in our reduced order controller. To do this, we turn to LQG balancing.

Just as the Hankel singular values are invariant with respect to system realization, so are the *Riccati singular values,* given by $\mu_i = \sqrt{\lambda_i(\Pi P)}$, [16]. As Π, P are based on the PDE system which includes the physics before model reduction takes place, it is reasonable to conjecture that that reduction based on the Riccati singular values would result in a low order model that contains information that is important for the PDE controller. To accomplish LQG balancing, the balancing algorithm described above is applied to Π, P, and truncation is based upon the Riccati singular values, keeping the states which correspond to the q largest values, $\{\mu_i\}_{i=1}^q$.

4 Klein-Gordon Equation and Numerical Results

To study the effects of regular and LQG balancing, the linear Klein Gordon equation (KGL) is used as an example. Taking the constants velocity of light in a vacuum and $\hbar$ to be one and adding a damping term, $\gamma\omega_t$, to ensure stabilizability, the KGL equation can be written as

$$\begin{gathered} \omega_{tt}(t,x) + \gamma\omega_t(t,x) - \omega_{xx}(t,x) + m^2\omega(t,x) = \textstyle\sum_{i=1}^r b_i(x)u_i(t), \\ \omega(0,x) = \omega_{1_0}(x) \qquad\qquad \dot{\omega}(0,x) = \omega_{2_0}(x), \\ \omega(t,0) = 0 = \omega(t,L) \end{gathered} \tag{22}$$

where m is the rest mass, L is the length of the spatial domain, t is time, $\{u_i(t)\}_{i=1}^r$ are controls, and $\{b_i(x)\}_{i=1}^r$ are control input functions. The KGL equation is a relativistic wave equation which arises in quantum mechanics; it is a momentum equation used when there is a need to describe phenomena at high energies [15].

Using PDE theory, the KGL equation can be written as a system in abstract form as in (1), (2). To formulate the full order approximation in (10), (11), a Galerkin finite element scheme is applied with linear splines. The weak form of the KGL equation in (22) is given by

$$\begin{aligned} \int_0^L \ddot{\omega}(t,x)\psi(x)dx &= -\int_0^L \omega'(t,x)\psi'(x)dx - m^2\int_0^L \omega(t,x)\psi(x)dx \\ &\quad -\gamma\int_0^L \dot{\omega}(t,x)\psi(x)dx + \int_0^L \sum_{i=1}^r u_i(t)b_i(x)\psi(x)dx \end{aligned}$$

for all $\omega(t), \psi \subseteq H_0^1(0,L)$. We approximate $\omega(t,x)$ by $\omega^N(t,x) = \sum_{i=1}^{N-1}\omega_i^N(t)\varphi_i^N(x)$, where $\{\varphi_i^N\}_{i=1}^{N-1}$ are piecewise linear basis functions; ψ ranges over $\{\varphi_i^N\}_{i=1}^{N-1}$. For the purpose of defining $\{b_i\}_{i=1}^r$, we create a partition of $[0,L]$ as $\{x_i\}_{i=1}^r$ where $x_i = i * L/r$. We then choose the functions b_i to be defined by

$$b_i(x) = e^{-(x-x_i^*)^2} \quad \text{for} \quad x_{i-1} \le x \le x_i, \quad \text{where} \quad x_i^* = \frac{x_{i-1}+x_i}{2}.$$

We define $B_i = \int_0^L b_i(x)\varphi_j(x)dx$ for $i = 1, \ldots, r$, $j = 1, \ldots, N-1$. Additionally, we assume there are four averaged measurements—two each of position and velocity. This creates the $4 \times$

$(N-2)$ dimensional matrix C^N given by

$$C^N = \begin{bmatrix} \left[2/L\int_0^{L/2}\varphi_i^N(t,x)dx\right]_{i=1}^{N-1} & 0 \\ \left[2/L\int_{L/2}^{L}\varphi_i^N(t,x)dx\right]_{i=1}^{N-1} & 0 \\ 0 & \left[2/L\int_0^{L/2}\varphi_i^N(t,x)dx\right]_{i=1}^{N-1} \\ 0 & \left[2/L\int_{L/2}^{L}\varphi_i^N(t,x)dx\right]_{i=1}^{N-1} \end{bmatrix}$$

Defining $v^N(t,x) = \dot{\omega}^N(t,x)$, we obtain the system corresponding to (10), (11)

$$\begin{bmatrix} \dot{\omega}^N(t) \\ \dot{v}^N(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K - m^2 I & -\gamma I \end{bmatrix} \begin{bmatrix} \omega^N(t) \\ v^N(t) \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 \\ M^{-1}B_1 & \cdots & M^{-1}B_r \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_r(t) \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = C^N \begin{bmatrix} \omega^N(t) \\ v^N(t) \end{bmatrix}, \tag{24}$$

where

$$M = \left[\int_0^L \varphi_i^N(x)\varphi_j^N(x)dx\right]_{i,j=1}^{N-1} \qquad \text{and} \qquad K = \left[\int_0^L \varphi_i^{N'}(x)\varphi_j^{N'}(x)dx\right]_{i,j=1}^{N-1}$$

are the mass and stiffness matrices, respectively. We now apply the ideas presented above for balanced realizations and truncation to design a reduced order controller.

4.1 Numerical Results

In the KGL equation (22), the mass of the particle is specified to be .01 while the damping constant is set at .1. For the finite element approximation, the number of subintervals is chosen to be $N = 100$, on a spatial interval of length $L = 10$. Four control input functions are specified, that is, $r = 4$. There are two functional gains for this control problem, one for position and one for velocity. They are shown in Figure 1.

We now apply both balancing techniques to the system given in (23), (24). Recall that for balancing, this requires computing the Gramians, and for LQG balancing, computing the solutions to the Riccati equations. Figure 2 shows the Hankel singular values found through balancing around the Gramians on the left and the Riccati singular values found through balancing around the Riccati operators on the right.

To form the reduced order systems, we chose to truncate the full order systems with $q = 10$. With these two reduced systems—one from balancing and one from LQG balancing (see equations (15), (16))—we compute the two corresponding LQG compensators (in (17)) by solving the Riccati equations. We can then form the two closed loop systems, according to (18) using A_T from balancing and $ricA_T$ from Riccati balancing.

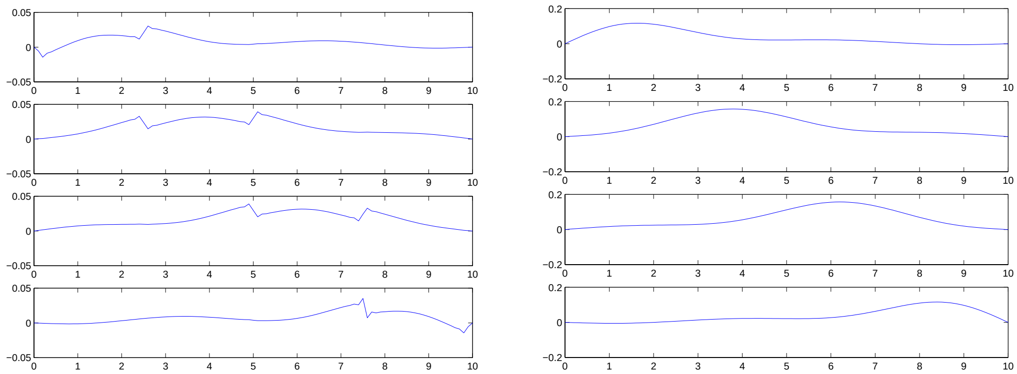

Figure 1: Full Order Functional Gains for Position (left) and Velocity (right), (ordered top to bottom within each plot)

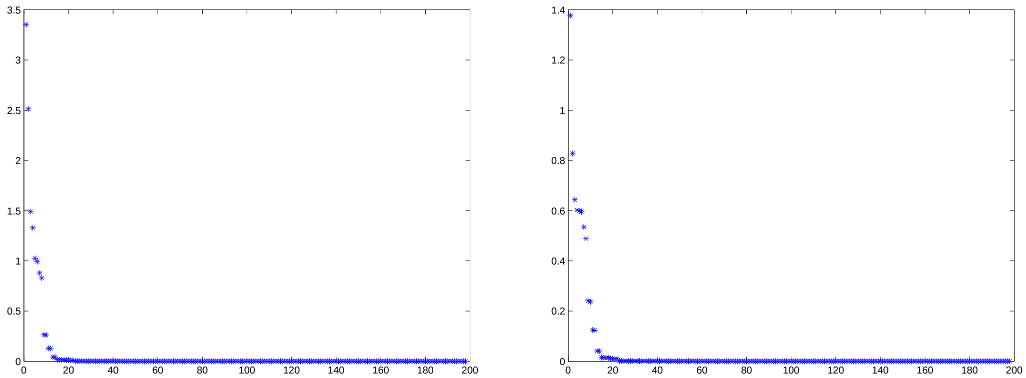

Figure 2: Hankel Singular Values (left) and Riccati Values (right)

As a first step in our comparison of these two closed loop systems with the full order closed loop system in (14), we compute the functional gains corresponding to the reduced compensator designs. The gains for the controller found through regular balancing can be found in Figure 3; those for the controller found through LQG balancing are in Figure 4. Again the position gains are on the left, and the velocity gains are on the right.

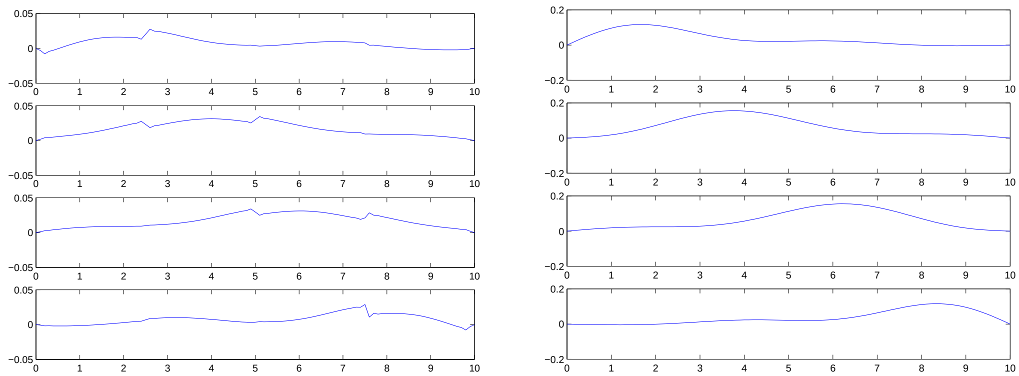

Figure 3: Reduced Order Functional Gains for Position (left) and Velocity (right) using Regular Balancing, (ordered top to bottom within each plot)

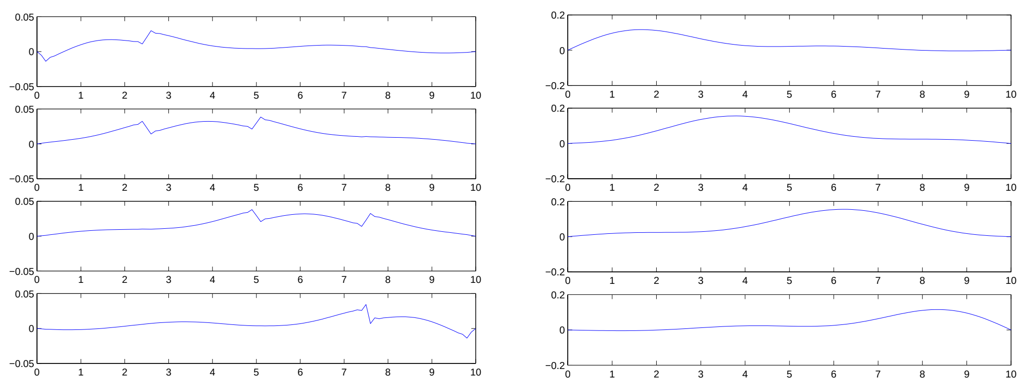

Figure 4: Reduced Order Functional Gains for Position (left) and Velocity (right) using LQG Balancing, (ordered top to bottom within each plot)

We observe that there is little difference in the three sets of gains, which leads us to conclude that for this example, both balancing methods can preserve characteristics of the functional gains from the full order compensator. This is an improvement over previously reported results for some reduce-then-design methods [1, 3, 4].

To simulate the performance of the closed loop systems, the initial conditions are taken to be

$$\omega(0,x) = \sin(x) \qquad \dot{\omega}(0,x) = \cos(x). \tag{25}$$

The result of simulating the uncontrolled system ($u_i(t) = 0$), is depicted in Figure 5. The solution displays the type of wave behavior with damping effects as would be expected. Simulating the full order closed loop system requires an initial condition for the state estimate. To represent an error in the initial estimate, $x_c^N(0)$ is taken to be $0.75 * x^N(0)$. Figure 5 shows the behavior of the state when the full order controller is applied. A plot of the four components of the controller $u(t) = -Kx_c(t)$ is given in Figure 6.

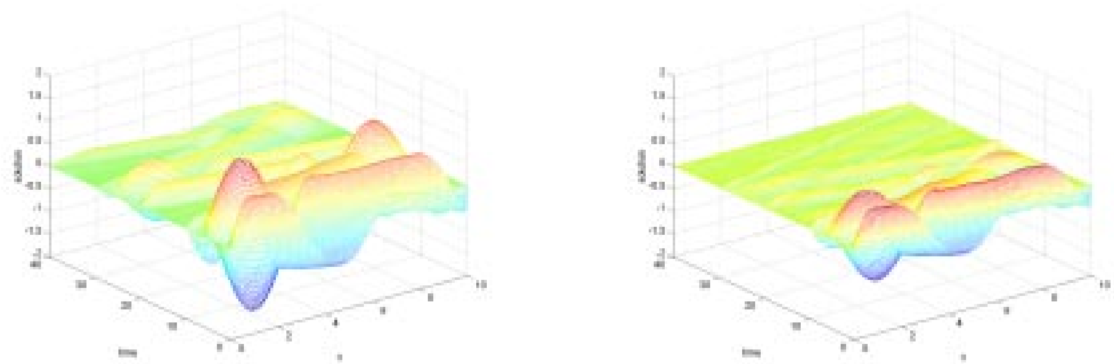

Figure 5: Uncontrolled State (left) and State with Full Order Compensator (right)

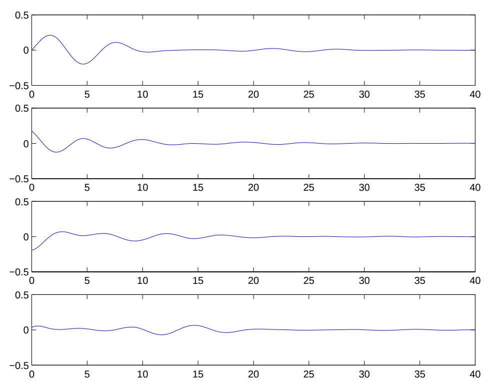

Figure 6: Controller (ordered top to bottom)

The reduced order compensator performances from the two balancing techniques are shown in Figure 7. It is difficult to make any conclusive statements about the two behaviors, other than they do show more rapid damping than does the uncontrolled system. They do not show, however, as rapid a damping in the oscillations as does the system with the full order compensator.

To investigate robustness of the closed loop systems, we compute the stability radii of $\mathcal{A}$, A_T and $ricA_T$. The stability radius gives the smalles perturbation that destabilizes the closed loop systems, [10].The results are summarized in Table 1. We note that the closed loop system with the largest stability radius, i.e., the most robust closed loop system, is that given by $ricA_T$.

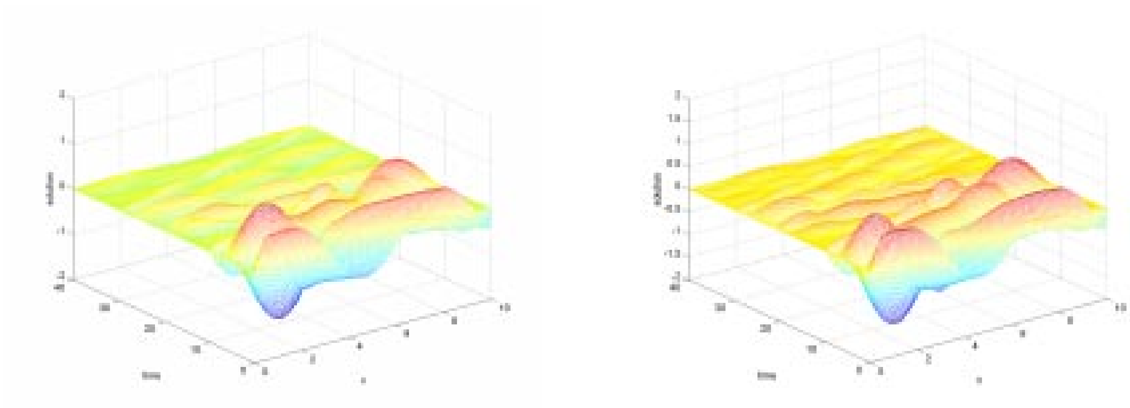

Figure 7: Reduced Order Compensator Performance via Balanced Truncation (left) LQG Balanced Truncation (right)

System Matrix	Stability Radius
$\mathcal{A}$	1.198360×10^{-9}
A_T	4.529261×10^{-7}
$ricA_T$	7.008909×10^{-5}

Table 1: Stability Radii

5 Future Work

This paper contains preliminary numerical investigations into the application of LQG balancing and truncation to a PDE system for low-order compensator design. Research into the method continues on many fronts. On the theoretical side, Curtain and others are extending the LQG balancing technique to Min-Max control design which will allow more flexibility in choice of controlled outputs and disturbance inputs. Camp and King are investigating other PDE systems for which the two approaches may show more pronounced differences in the results. At this point, we remain optimistic that the LQG balancing provides a way to utilize information from the infinite dimensional controller in the reduction steps.

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