Functional Gain Computations for a 1D Parabolic Equation using Non-Uniform Meshes.

John A. Burns* Belinda B. King †
Lizette Zietsman‡

Center for Optimal Design and Control
Interdisciplinary Center for Applied Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061-0531

Abstract

In this paper we consider a numerical algorithm for computing functional gains that define optimal feedback laws for Dirchlét boundary control of parabolic equations. The focus here is on using non-uniform meshes to improve convergence of finite element schemes. Since boundary control problems of this type often lead to functional gains with support near the boundary, uniform meshes are not optimal. Numerical examples are presented to illustrate the effectiveness of using a non-uniform mesh concentrated near the boundary.

1 A Boundary Layer Control Problem

We consider a control problem motivated by the viscous Burgers equation

$$\frac{\partial}{\partial t}w(t,x) = \mu\frac{\partial^2}{\partial x^2}w(t,x) - \frac{\partial}{\partial x}\frac{[w(t,x)]^2}{2}, \quad 0 < x < 1, \quad t > 0, \tag{1.1}$$

with homogenous boundary condition at $x = 0$

$$w(t,0) = 0, \tag{1.2}$$

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and Dirchlét boundary control at $x = 1$

$$w(t, 1) = u(t). \tag{1.3}$$

The initial condition is given by

$$w(0, x) = w_0(x) \tag{1.4}$$

where $w_0(x) \in L^2(0, 1)$ and $0 < \mu << 1$. The problem is motivated by flow control problems where the control action is located on the walls (boundary) of the flow (see [1] and [7]).

If one linearizes the problem and applies LQR theory (LQG, MinMax, etc.), then the optimal controllers have the form

$$u_{opt}(t) = -Kw(t, \cdot) = -\int_0^1 k(x)w(t, x)dx,$$

where $k(x)$ is called the functional gain. In the paper [7] a finite element method on a uniform mesh was used to compute $k(x)$. Typical convergence of these functional gains is illustrated in Figure 1 (here $\mu = \frac{1}{60}$ and a large penalty is placed on the solution near the boundary).

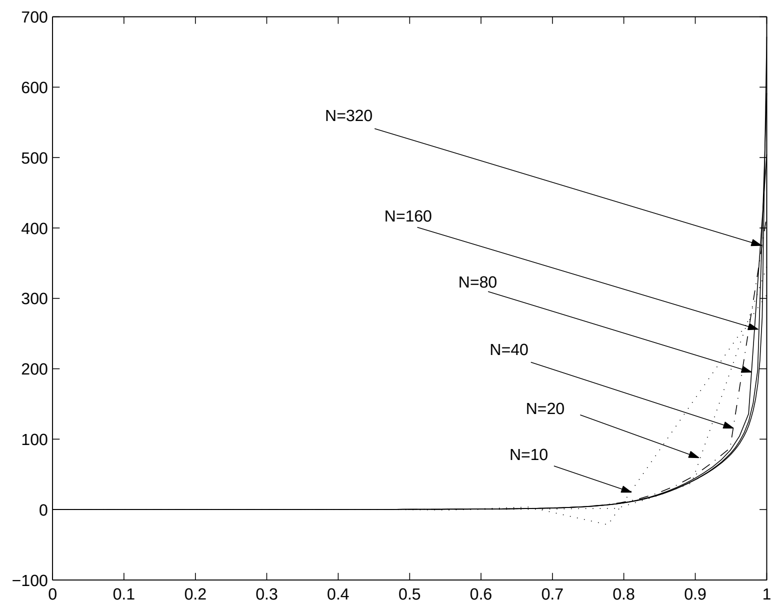

Figure 1: Convergence of the Functional Gains

Observe that the functional gain is supported near the boundary and becomes "singular" at $x = 1$. Also note that convergence is not achieved even when a large number of elements is used. Clearly, the form of the functional gain suggests that one might be able to do better with a non-uniform mesh concentrated near $x = 1$. We investigate this issue in this paper.

1.1 The Control Problem

The details of the problem and the theoretical framework may be found in ([7]). Thus, we briefly describe the problem and outline the results. Let $b_L = 1 - \sqrt{\mu}$ and define $q(\cdot) : [0,1] \longrightarrow \mathbb{R}$ by

$$q(x) = \begin{cases} q_L, & b_L \le x \le 1 \\ q_S, & 0 \le x < b_L \end{cases}, \tag{1.5}$$

where $0 < q_S << q_L$ are positive numbers. For $r > 0$ and $\alpha \ge 0$, we define the cost function by

$$J_\alpha(u(\cdot)) = \int_0^\infty e^{\alpha t} \left\{ \left(\int_0^1 q(x)|w(t,x)|^2 dx \right) + |u(t)|^2 \right\} dt. \tag{1.6}$$

Observe that, since $0 < q_S << q_L$, the cost function places a large penalty on the solution in the "boundary layer" $b_L \le x \le 1$. Also, when $\alpha > 0$ there is an additional performance requirement (see [2], [3] and [11]).

The boundary control problem for the linearized system is the heat equation

$$\frac{\partial}{\partial t} w(t,x) = \mu \frac{\partial^2}{\partial x^2} w(t,x), \quad 0 < x < 1, \quad t > 0, \tag{1.7}$$

with homogenous boundary condition at $x = 0$

$$w(t,0) = 0, \tag{1.8}$$

and Dirchlét boundary control at $x = 1$

$$w(t,1) = u(t). \tag{1.9}$$

One may formulate this problem as a state space system of the form

$$\frac{d}{dt} w(t) = Aw(t) + Bu(t) \tag{1.10}$$

in a very weak sense (see [13]). If one minimizes $J_\alpha(u(\cdot))$ defined by (1.6) subject to (1.10), then it can be shown that the optimal controller has the form

$$u_{opt}(t) = -K_{\mu,\alpha} w(t,\cdot) = -\int_0^1 k_{\mu,\alpha}(x) w(t,x) dx, \tag{1.11}$$

where the kernel $k(\cdot) \in L^2(0,1)$ is called the *functional gain* (see [3] and [4]). We use the formulation (1.10) to guide the construction of finite element approximations of the linear control system (1.7)-(1.9) and of the functional gain $k(\cdot) \in L^2(0,1)$. A convergence theory for these approximations may be found in [11] and [13], and several researchers have used these or similar approximations for a variety of parabolic control problems (see [3], [4], [5], [6], [11] and [13]). However, we shall focus on the application of non-uniform meshes to compute these functional gains.

2 Approximation and Numerical Results

In this section we present a numerical scheme for the boundary control problem (1.7)-(1.9) discussed in Section 1 above. Since the algorithm is similar to the ones given in [3], [5], [6], [7] and [13], we will omit the detailed discussion. The primary difference here is the application of a non-uniform mesh. In particular, we focus on the effect of computing the functional gains by using a non-uniform mesh in the region $b_L < x < 1$. This mesh is constructed in such a way that the mesh is finer near the boundary. The non-uniform mesh produces more accurate approximations for the functional gains while using less elements than required on a uniform mesh.

2.1 The Approximation Scheme

Here we focus on the case where $\alpha = 0$ and start with the equation

$$\int_0^1 w_t(t,x)\phi(x)dx = [\mu\phi_x(1)]\,u(t) + \mu\int_0^1 w(t,x)\phi_{xx}(x)dx. \tag{2.12}$$

Observe that if $v(\cdot) \in H_0^1(0,1))$ and $\phi(\cdot) \in H^2(0,1) \cap H_0^1(0,1)$, then

$$\int_0^1 v(x)\phi_{xx}(x)dx = -\int_0^1 v_x(x)\phi_x(x)dx.$$

Therefore, by projecting (2.12) onto any finite element subspace $V_h \subset H_0^1(0,1)$ (see [14], page 126) one obtains the system

$$\int_0^1 w_t^h(t,x)\phi(x)dx = [\mu\phi_x(1)]\,u(t) - \mu\int_0^1 w_x^h(t,x)\phi_x(x)dx. \tag{2.13}$$

We seek a $w^h(t,\cdot) \in V_h \subset H_0^1(0,1))$ such that (2.13) holds for all $\phi^h(\cdot) \in V_h \subset H_0^1(0,1)$ and construct a new scheme that is a variation of the schemes found in [3], [13] and [14]. Let $0 = x_0 < x_1 < x_2 < \ldots < x_{N+1} = 1$ be a partition of $[0,1]$ and let x_L correspond to the smallest x_i in the partition such that $b_L \le x_i$. Let $\delta x_i = x_i - x_{i-1}$ for $i = 1,\ 2,\ \ldots,\ N$ where $N+1$ is the number of elements in the partition of $[0,1]$. The refinement is done in such a way that a fine mesh is obtained near the boundary. The mesh generation algorithm is as follows:

- Start with a uniform mesh with interval width δx. Choose this partition so that there are an odd number of nodes greater or equal to b_L. Set $x_L^1 = (1 - x_L)/2$.
- The interval $[x_L^1, 1]$ is partitioned into subintervals of length $\delta x/2$ and the remaining part of the partition remains unchanged. Set $x_L^2 = (1 - x_L^1)/2$.
- For $k \ge 2$, the interval $[x_L^k, 1]$ is partitioned into subintervals of length $\delta x/2^k$ and the remaining part of the partition remains unchanged. Set $x_L^{k+1} = (1 - x_L^k)/2$.

In this short paper we refine the mesh as above manually. However, the goal is to develop an adaptive algorithm based on automatic mesh refinement.

Let $H_0^N \subset H_0^1(0,1)$ be the N-dimensional finite element space given by

$$H_0^N = \left\{ \sum_{i=1}^{N} w_i h_i^N(x) : w_i \in \mathbb{R},\ i = 1,\ 2,\ \ldots,\ N \right\},$$

where the basis functions $h_i^N(x)$ are defined by

$$h_i^N(x) = \begin{cases} (x - x_{i-1})/\delta x_i, & x_{i-1} \le x \le x_i \\ -(x - x_{i+1})/\delta x_{i+1}, & x_i \le x \le x_{i+1} \\ 0, & \text{otherwise} \end{cases}.$$

Projecting (2.13) onto H_0^N produces the approximate solution $w^N(t,x)$ given by

$$w^N(t,x) = \sum_{i=1}^{N} w_i^N(t) h_i^N(x)$$

where $w_i^N(t) \in \mathbb{R},\ i = 1,\ 2,\ \ldots,\ N$. Thus, the system (2.13) can be represented by the following finite dimensional system

$$\frac{d}{dt}\hat{w}^N(t) = \left[A^N\right]\hat{w}^N(t) + B^N u^N(t), \tag{2.14}$$

where $\hat{w}^N(t) = [w_1^N(t),\ w_2^N(t),\ \ldots,\ w_N^N(t)]^{\mathrm{T}}$. Here, for a uniform mesh, $[A^N] = [G^N]^{-1}[\tilde{A}^N]$, $B^N = [G^N]^{-1}\tilde{B}^N$ are the usual finite element matrices with mass matrix $[G^N]$ and stiffness matrix $[\tilde{A}^N]$. The column vector $\tilde{B}^N$ has only one non-zero entry.

For the non-uniform mesh, the matrices remain tridiagonal and symmetric and are constructed in the following way. Let M^{k-1} denote that part of the matrix (mass or stiffness matrix) computed in iteration $(k-1)$ that correspond to those elements that remained *unchanged* in iteration k. Let M^k denote the matrix (mass or stiffness matrix) computed in iteration k for those elements that have been refined in iteration k. The matrices for the partition associated with the k^{th} iteration are of the form

$$M = \left[\begin{array}{ccc|c} & M^{k-1} & \vdots & 0 \\ & & \cdots\vdots & \cdots\quad\cdots\quad\cdots \\ \cdots\quad\cdots & \cdots & \vdots\ M_{ii}\ \vdots & \\ & & \vdots & \\ & 0 & \vdots & M^k \end{array}\right].$$

Note that the entry where the two matrices overlap, say M_{ii}, must be re-evaluated. This entry is associated with the basis function over two non-uniform elements. That is, the last element of the previous partition that remains unchanged and the first element that has been refined in the current iteration. In particular, the corresponding global basis function has the form

$$h_i^N(x) = \begin{cases} (x - x_{i-1})/\delta x_i, & x_{i-1} \le x \le x_i \\ -(x - x_{i+1})/\delta x_{i+1}, & x_i \le x \le x_{i+1} \\ 0, & \text{otherwise} \end{cases},$$

where $\delta x_{i+1} = \delta x_i/2$.

This scheme was used to compute the functional gains of the systems presented below. Convergence of the feedback gain operators is provided by the theory in [13] and [14]. This theory applies to the non-uniform mesh. However, it would be valuable to obtain error estimates for these methods. This will be the subject of a future paper.

2.2 Numerical Examples

We present three examples to illustrate the benefits of using non-uniform grids to solve for the functional gains.

Example 1

This is the example illustrated in Figure 1 above. A uniform mesh was used to compute $\left[k_{\frac{1}{60},0}\right]^N(\cdot)$ where

$$u_{opt}(t) = -K_{\frac{1}{60},0}w(t,\cdot) = -\int_0^1 k_{\frac{1}{60},0}(x)w(t,x)dx.$$

Here $\alpha = 0$, $\mu = \frac{1}{60}$, $r = .25$, $q_S = 1$ and $q_L = \frac{50}{\sqrt{\mu}} = 50\sqrt{60} = 387.298$. The boundary layer thickness is $\sqrt{\mu} = \frac{1}{\sqrt{60}} = .1291$.

It is important to note that it was necessary to use $N = 320$ elements in order to produce "convergent functional gains". This convergence rate is unlike previous cases (see [3], [5], [6] and [11]) where convergence usually occurred at a much smaller N. The reason for the "slow" convergence is twofold. First, the heavy weight placed in the boundary layer causes the functional gain to be very large near the boundary. Second, since $\mu = \frac{1}{60}$ is small for this problem, convergence near the boundary $x = 1$ requires a fine mesh. As $\mu \longrightarrow 0$ this problem becomes even more difficult.

In Figures 2 and 3 we illustrate how a non-uniform mesh can improve convergence.

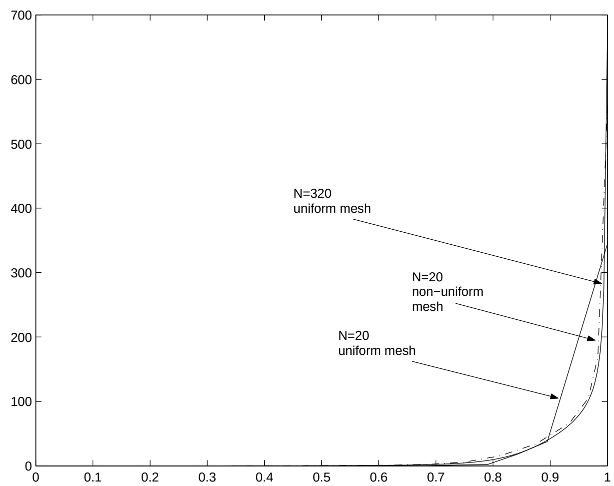

Figure 2: Functional gains using uniform and non-uniform meshes: $\delta x = .0625$

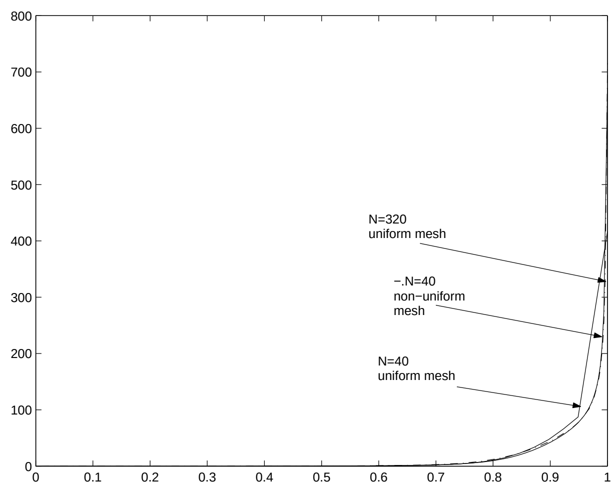

Figure 3: Functional gains using uniform and non-uniform meshes: $\delta x = .0323$

To understand the plots we note that N denotes the size of the final system used to compute $[k_{\mu,\alpha}]^N(\cdot)$. In particular, in Figure 3 one can see that a non-uniform mesh using 40 elements produces the same accuracy as with 320 elements on a uniform mesh.

Example 2

Here we again use $\alpha = 0$, $\mu = \frac{1}{60}$, $r = .25$ but set $q_S = q_L = 1$. The boundary layer thickness is $\sqrt{\mu} = \frac{1}{\sqrt{60}} = .1291$. As seen in Figure 4 this problem produces a functional gain with support on the entire domain $[0, 1]$. However, the gain has a sharp slope at $x = 1$ and as illustrated in Figure 5, non-uniform meshes again enhance convergence near the controlling boundary.

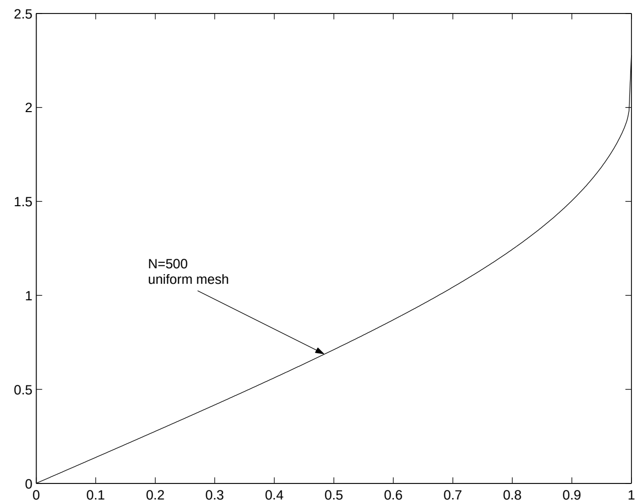

Figure 4: Functional gains with $q_S = q_L = 1$ and a uniform mesh

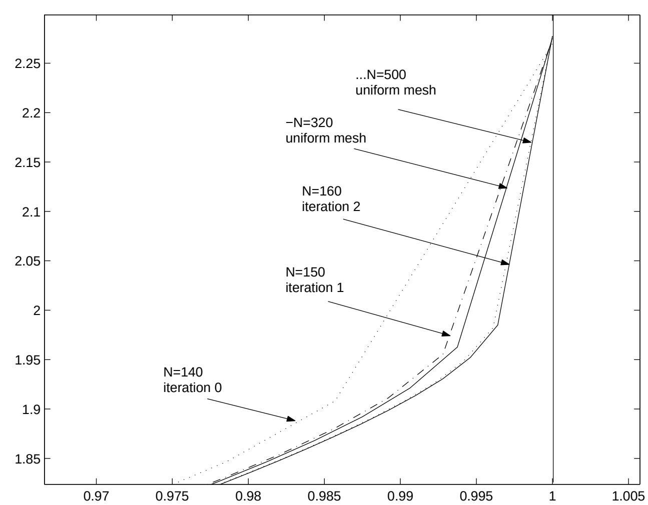

Figure 5: Functional gains with $q_S = q_L = 1$ and non-uniform meshes: $\delta x = .00714$

Example 3

Here we set $\mu = \frac{1}{600}$, $\alpha = 0$, $r = .25$ and again place $q_S = 1$ and $q_L = \frac{50}{\sqrt{\mu}} = 50\sqrt{600} = 1224.7$. The boundary layer thickness is $\sqrt{\mu} = \frac{1}{\sqrt{600}} = .0408$. Thus we do the same problem as in Example 1 but with a lower value for μ. The results show that, as expected, the non-uniform mesh produces improvements on uniform mesh solutions.

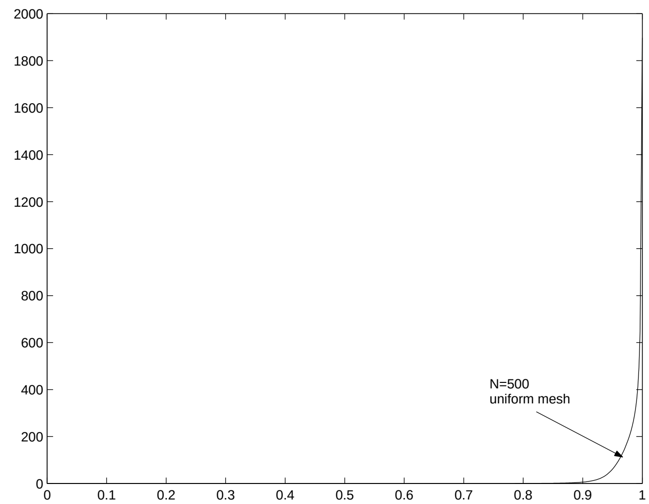

Figure 6: Functional gains with $\mu = \frac{1}{\sqrt{600}}$ and a uniform mesh

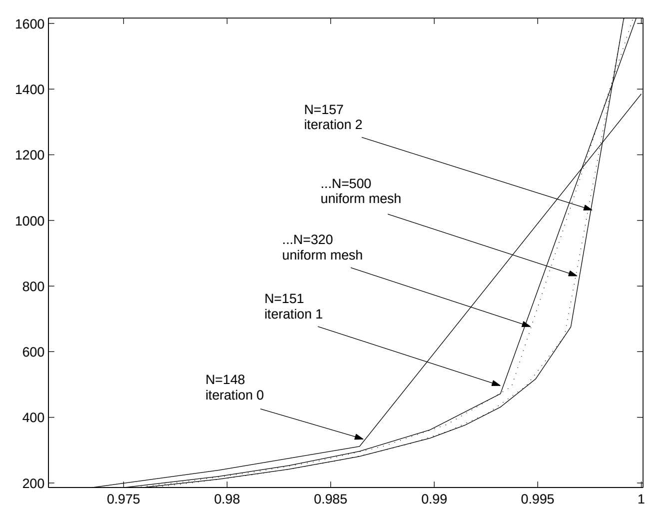

Figure 7: Functional gains with $\mu = \frac{1}{\sqrt{600}}$ and non-uniform meshes: $\delta x = .00676$

3 Conclusion

The previous numerical examples serve to demonstrate the potential benefits of employing non-uniform meshes in solving for functional gains. The observations is based on a priori knowledge that the functional gains are “singular” near the boundary. These facts point to the need to develop an adaptive algorithm based on automatic mesh generators. The development of such adapted algorithms will depend on ones ability to obtain practical error estimates. This is the subject of on-going research and will appear in a future paper.

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