A Lie-Group Approach for Nonlinear Dynamic Systems Described by Implicit Ordinary Differential Equations

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Abstract

This contribution presents a Lie-group based approach for the accessibility and the observability problem of dynamic systems described by a set of implicit ordinary differential equations. It is shown that non-accessible or non-observable systems admit Lie-groups acting on their solutions such that distinguished parts of the system remain unchanged. The presented methods use the fact that the dynamic system may be identified with a submanifold in a suitable jet-bundle. Therefore, a short introduction to this theory, as well as its application to systems of differential equations is presented.

1 Introduction

This contribution deals with dynamic systems that are described by a set of n_e nonlinear ordinary implicit differential equations of the type

$$f^{i_e}\left(t, z, \tfrac{\mathrm{d}}{\mathrm{d}t}z\right) = 0 \ , \quad i_e = 1, \ldots, n_e \tag{1.1}$$

in the independent variable t and the dependent variables z^{α_z}, $\alpha_z = 1, \ldots, n_z$. Of course, this class of systems contains explicit systems like

$$\tfrac{\mathrm{d}}{\mathrm{d}t}x^{\alpha_x} = f^{\alpha_x}(t, x, u) \ , \quad \alpha_x = 1, \ldots, n_x \tag{1.2}$$

or DAE-systems (differential algebraic equation)

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}x^{\alpha_x} &= f^{\alpha_x}(t, x, u) \ , \quad \alpha_x = 1, \ldots, n_x \\ 0 &= f^{\alpha_s}(t, x, u) \ , \quad \alpha_s = n_x + 1, \ldots, n_e \ . \end{aligned} \tag{1.3}$$

If we consider the coordinates (t, x) of (1.2) as local coordinates of a smooth manifold $\mathcal{E}$ with $\dim \mathcal{E} = n_x + 1$, then it is easy to see that equation (1.2) defines a submanifold of the tangent bundle $\mathcal{T}(\mathcal{E})$ and that this submanifold is parametrized by the input $u \in U \subset \mathbb{R}^{n_u}$.

This geometric picture is not valid any more for (1.3). Descriptor-systems in the dependent variables z^{α_z}, $\alpha_z = 1, \ldots, n_z$ like

$$\sum\nolimits_{\alpha_z=1}^{n_z} n_{\alpha_z}^{\alpha_e}(t,z)\,\tfrac{\mathrm{d}}{\mathrm{d}t} z^{\alpha_z} = m^{\alpha_e}(t,z) \;\;, \quad \alpha_e = 1, \ldots, n_e \tag{1.4}$$

are a generalization of (1.3). If one takes the coordinates (t, z) of (1.4) again as local coordinates of a smooth manifold $\mathcal{E}$, then (1.4) defines a linear subspace of the cotangent bundle $\mathcal{T}^*(\mathcal{E})$. This geometric picture allows us to extend many design methods, well known for explicit systems (e.g. see [1], [2]) to descriptor-systems like (1.4), e.g. see [7].

Obviously, the system (1.1) is a generalization of (1.4), and its natural geometric picture is that of a submanifold of the first jet bundle $J(\mathcal{E})$ of $\mathcal{E}$. In an exemplary fashion we solve the observability and accessibility problem for systems like (1.1). Of course, the presented mathematical machinery can be applied to other problems like input-to-output, input-to-state linearization, optimal control problems, etc. Several solutions for the descriptor case can be found in [7].

This contribution is organized as follows. The next section summarizes some mathematical facts concerning jet-bundles and Lie-groups, since all investigations of this contribution are based on these concepts. Section 3 presents the application of the Lie-group analysis to explicit systems like (1.3). The implicit case is treated in Section 4. Finally, this contribution closes with some conclusions.

2 Some Mathematical Basics

Lie-groups and their invariants offer well proven methods for the investigation of differential equations. Additionally, many of these methods are implemented in powerful computer algebra systems. Since we consider a dynamic system as a submanifold in a suitable jet-bundle, we give a short introduction to fibered manifolds and jet-bundles in the following subsection. Subsection 2.2 presents some basics of the theory of transformation groups. Formal integrability of systems of differential equations will be treated in Subsection 2.3. Already here the reader is referred to the four books [3], [4], [5], [6] and the references therein, where he will find more details on these topics and many things more. Therefore, we will suppress all citations in the following three subsections.

2.1 Jet-Bundles

A smooth fibered manifold is a triple $(\mathcal{E}, \pi, \mathcal{B})$ with the total manifold $\mathcal{E}$ the base $\mathcal{B}$ and a smooth surjective map $\pi : \mathcal{E} \to \mathcal{B}$. The set $\pi^{-1}(x) = \mathcal{E}_x$ is called the fiber over x. We confine all our considerations to the finite dimensional case with $\dim \mathcal{E} = p + q$, $\dim \mathcal{B} = p$, where we may introduce locally adapted coordinates (x^i, u^α) such that x^i, $i = 1, \ldots, p$ denote the independent variables and u^α, $\alpha = 1, \ldots, q$ denote the dependent variables. We use Latin indices for the independent and Greek indices for the dependent variables.

A section σ is a map $\sigma : \mathcal{B} \to \mathcal{E}$ such that $\pi \circ \sigma = \mathrm{id}_{\mathcal{B}}$ is met for points, where σ is defined. The identity map on $\mathcal{B}$ is denoted by $\mathrm{id}_{\mathcal{B}}$. The tangent-bundle $(\mathcal{T}(\mathcal{M}), \tau_{\mathcal{T}}, \mathcal{M})$ or the cotangent-bundle $(\mathcal{T}^*(\mathcal{M}), \tau_{\mathcal{T}}^*, \mathcal{M})$ of a manifold $\mathcal{M}$ are well known examples of fibered manifolds. We write

$$\sigma = a^i \partial_i \quad \text{and} \quad \omega = a_i \mathrm{d}x^i$$

for sections of $\mathcal{T}(\mathcal{M})$ and $\mathcal{T}^*(\mathcal{M})$ with functions a^i, a_i defined on $\mathcal{M}$. If the map π and the base-manifold $\mathcal{B}$ are clear, we write $\mathcal{E}$ instead of $(\mathcal{E}, \pi, \mathcal{B})$. Furthermore, we use the Einstein convention for sums, whenever the range of the index i follows from the context.

Let f be a smooth section of $(\mathcal{E}, \pi, \mathcal{B})$. We write

$$\frac{\partial^k}{\partial_1^{j_1} \cdots \partial_p^{j_p}} f^\alpha = \partial_J f^\alpha \ , \quad \partial_i = \frac{\partial}{\partial x^i}$$

for a k-th order partial derivative of f^α with respect to the independent coordinates with the ordered multi-index $J = j_1, \ldots, j_p$ and $k = \#J = \sum_{i=1}^p j_i$. We will use the abbreviations $1_k = j_1, \ldots, j_p$, $j_i = \delta_{ik}$ and $J + \bar{J} = j_1 + \bar{j}_1, \ldots j_p + \bar{j}_p$. Let (x^i, u^α) be adapted coordinates of $\mathcal{E}$. The first prolongation $j^1(f) = j(f)$ of f is the map $j(f) : x \to (x^i, f^\alpha(x), \partial_i f^\alpha(x))$. One can show that the first jet-bundle $J^1(\mathcal{E}) = J(\mathcal{E})$ of $\mathcal{E}$ is that manifold that contains all first prolongations of sections of $\mathcal{E}$ and that respects the transition rules for first order derivatives. An adapted coordinate system (x^i, u^α) of $\mathcal{E}$ induces the adapted system $\left(x^i, u^\alpha, u^\alpha_{1_i}\right)$ for $J(\mathcal{E})$ with the pq new coordinates $u^\alpha_{1_i}$. Now, $J(\mathcal{E})$ admits the two maps $\pi : J(\mathcal{E}) \to \mathcal{B}$ and $\pi_0^1 : J(\mathcal{E}) \to \mathcal{E}$ with $\pi\left(x^i, f^\alpha(x), f^\alpha_{1_i}(x)\right) = x$ and $\pi_0^1\left(x^i, f^\alpha(x), f^\alpha_{1_i}(x)\right) = (x^i, f^\alpha(x))$ such that $(J(\mathcal{E}), \pi, \mathcal{B})$, as well as $(J(\mathcal{E}), \pi_0^1, \mathcal{E})$ are fibered manifolds. Let us consider a section f of $(J(\mathcal{E}), \pi_0^1, \mathcal{E})$. It is easy to see that f is the prolongation of a section σ of $\mathcal{E}$, iff $j(\pi_0^1(f)) = f$ or in coordinates $\partial_i f^\alpha - f^\alpha_{1_i} = 0$ is met. Therefore, $(J(\mathcal{E}), \pi_0^1, \mathcal{E})$ contains sections that are not the prolongation of sections of $\mathcal{E}$.

Analogously to the first jet-bundle $J(\mathcal{E})$ of a manifold $\mathcal{E}$, we define the n^{th} jet-bundle $J^n(\mathcal{E})$, the manifold that contains the n^{th} prolongation of sections f of $\mathcal{E}$ given by $j^n(f)(x) = (x, f(x), \partial_J f(x))$, $\#J = 1, \ldots, n$ and that respects the transition rules for partial derivatives up to the order n. The adapted coordinates (x, u) induce the system $\left(x^i, u^{(n)}\right)$ and $u^{(n)} = u^\alpha_J$, $\alpha = 1, \ldots, q$, $\#J = 0, \ldots, n$ for $J^n(\mathcal{E})$. Again, one can define the maps $\pi : J^n(\mathcal{E}) \to \mathcal{B}$ and $\pi^n_m : J^n(\mathcal{E}) \to J^m(\mathcal{E})$, $m = 1, \ldots, n-1$ with $\pi(j^n(f)(x)) = x$ and $\pi^n_m(j^n(f)(x)) = j^m(f)(x)$ such that $(J^n(\mathcal{E}), \pi, \mathcal{B})$ and $(J^n(\mathcal{E}), \pi^n_m, J^m(\mathcal{E}))$ are fibered manifolds. To simplify certain formulas later on, we set $J^0(\mathcal{E}) = \mathcal{E}$.

Let $(\mathcal{E}, \pi, \mathcal{B})$ be a fibered manifold with adapted coordinates (x^i, u^α), then we may introduce special vector fields d_i on $\mathcal{T}(J^\infty(\mathcal{E}))$, the so called total derivatives with respect to the independent coordinates x^i, given by

$$d_i = \partial_i + u^\alpha_{J+1_i} \partial^J_\alpha \ , \quad \partial^J_\alpha = \frac{\partial}{\partial u^\alpha_J} \ . \tag{2.5}$$

Let $f : J^k(\mathcal{E}) \to \mathbb{R}$ be a real valued smooth function and let σ be a section of $\mathcal{E}$, then we have

$$d_i f \circ j^{k+1}\sigma = \partial_i f\left(j^k \sigma\right) \ . \tag{2.6}$$

The dual objects to the fields $\pi^{\infty}_{n,*}d_i \in J^{n+1}(\mathcal{E})$ are called contact forms. A basis of these 1-forms $\omega^{\alpha}_J \in \mathcal{T}^* J^{n+1}(\mathcal{E})$ is given by

$$\omega^{\alpha}_J = \mathrm{d}u^{\alpha}_J - u^{\alpha}_{J+1_i}\mathrm{d}x^i \ , \quad \#J \leq n \ . \tag{2.7}$$

Let us consider the two fibered manifolds $(\mathcal{E}, \pi, \mathcal{B})$, $\left(\bar{\mathcal{E}}, \bar{\pi}, \bar{\mathcal{B}}\right)$ with adapted coordinates (x, u), $(\bar{x}, \bar{u})$. A bundle-morphism is a map $(\psi, \Psi) : \mathcal{E} \to \bar{\mathcal{E}}$ that respects the bundle structure or meets $\bar{\pi} \circ \Psi = \psi \circ \pi$ with $\psi : \mathcal{B} \to \bar{\mathcal{B}}$, $\Psi : \mathcal{E} \to \bar{\mathcal{E}}$. If the inverse of (ψ, Ψ) is smooth, then we call (ψ, Ψ) a bundle-diffeomorphism. If ψ is a diffeomorphism, then we can prolong (ψ, Ψ) to a map $j(\psi, \Psi) : J(\mathcal{E}) \to J\left(\bar{\mathcal{E}}\right)$ that is given in adapted coordinates by

$$\bar{u}^{\bar{\alpha}}_{1_{\bar{j}}} = (d_i \Psi^{\alpha}) \left(\bar{\partial}_{\bar{j}} \left(\psi^{-1}\right)^i \circ \psi\right) \quad \text{with} \quad \bar{\partial}_{\bar{j}} = \frac{\partial}{\partial \bar{x}_{\bar{j}}} \quad \text{and} \quad \psi^{-1} \circ \psi = \mathrm{id}_{\mathcal{B}} \ . \tag{2.8}$$

Now, it is straightforward to see that $j_*(\psi, \Psi) : \mathcal{T}(J(\mathcal{E})) \to \mathcal{T}\left(J\left(\bar{\mathcal{E}}\right)\right)$ maps the fields $\pi^{\infty}_0 d_i$ to span $\left\{\pi^{\infty}_0 \bar{d}_i\right\}$ and that $j^*(\psi, \Psi) : \mathcal{T}^* J\left(\bar{\mathcal{E}}\right) \to \mathcal{T}^* J(\mathcal{E})$ maps contact forms ω^{α} to span $\{\bar{\omega}^{\alpha}\}$. These properties single out maps $J(\mathcal{E}) \to J\left(\bar{\mathcal{E}}\right)$ that are prolongation of bundle-morphisms (ψ, Ψ).

2.2 Invariants and Lie-Groups

A Lie-group is a smooth manifold $\mathcal{G}$ that is also a group. Its members meet the laws of a group such that the composition and inversion map are smooth. A transformation group, which acts on a manifold $\mathcal{M}$ with local coordinates x, is a Lie-group $\mathcal{G}$ together with a smooth map $\Phi : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ such that

$$\Phi_e(x) = x \ , \quad \Phi_{g \circ h}(x) = \Phi_g \circ \Phi_h(x) \ , \quad x \in \mathcal{M}$$

is met with the neutral element e of $\mathcal{G}$ and the composition $g \circ h$, $g, h \in \mathcal{G}$. A function $I : \mathcal{M} \to \mathbb{R}$ is called an invariant I of the transformation group Φ, iff

$$I(x) = I(\Phi_g(x)) \ , \quad \forall g \in \mathcal{G} \tag{2.9}$$

is met. Obviously, the sets $\mathcal{N}_c = \{x \in \mathcal{M} \mid I(x) = c\}$, $c \in \mathbb{R}$ fulfill the relations $\Phi(\mathcal{N}_c) \subset \mathcal{N}_c$. If the relations above are met only in a neighborhood of e, then we call Φ a local transformation group.

Of special interest are groups Φ_{ε} with one real parameter $\varepsilon \in \mathbb{R}$ and the commutative composition law $+$. From (2.9) it follows

$$v(I) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(I(\Phi_{\varepsilon}(x)) - I(x)\right) \tag{2.10a}$$

$$v = v^i \partial_i \ , \quad v^i(x) = \partial_{\varepsilon} \Phi_{\varepsilon}(x)\big|_{\varepsilon=0} \ , \tag{2.10b}$$

i.e. Φ_{ε} generates the vector field $v \in \mathcal{T}(\mathcal{M})$ and $v(I) = 0$ is the infinitesimal condition for I to be an invariant of Φ_{ε}. Of course, it is well known that the field v generates the one-parameter Lie-group Φ_{ε} at least locally.

Let $(\mathcal{E}, \pi, \mathcal{B})$ be a fibered manifold with adapted coordinates (x^i, u^α) and let Φ_ε,

$$\Phi_\varepsilon : \mathcal{E} \times \mathbb{R} \to \mathcal{E} \tag{2.11}$$

be a one-parameter Lie-group with the real parameter ε. The group (2.11) generates the field v,

$$v = X^i (x, u) \partial_i + U^\alpha (x, u) \partial_\alpha \ , \tag{2.12}$$

see (2.10b). One can prolong (2.11) to a map $j(\Phi_\varepsilon) : J(\mathcal{E}) \times \mathbb{R} \to J(\mathcal{E})$ by (2.8), which is a very laborious task in general. Fortunately, the first prolongation $j(v) \in \mathcal{T}(J(\mathcal{E}))$ of the field v of (2.12) can be determined in a straightforward way, since $j(v)$ is given by

$$j(v) = v + U^\alpha_J \partial^J_\alpha \ , \quad \#J = 1 \ , \tag{2.13}$$

where the functions U^α_J follow from

$$j(v)(\pi^\infty_0(d_i)) \in \operatorname{span}\{\pi^\infty_0(d_i)\} \quad \text{or} \quad j(v)(\omega^\alpha) \in \operatorname{span}\{\omega^\alpha\} \tag{2.14}$$

with d_i from (2.5) and ω^α from (2.7).

2.3 Formal Integrability

Let us consider the fibered manifold $(\mathcal{E}, \pi, \mathcal{B})$ with adapted coordinates (x^i, u^α), $i = 1, \ldots, p$, $\alpha = 1, \ldots, q$ and the equations

$$f^r\left(x, u^{(n)}\right) = 0, \quad r = 1, \ldots, l \tag{2.15}$$

in the adapted coordinates $\left(x, u^{(n)}\right)$ of $J^n(\mathcal{E})$. We assume that the set (2.15) defines a regular submanifold

$$\mathcal{S}_n \subset J^n(\mathcal{E}) \ . \tag{2.16}$$

Obviously, a section σ of $\mathcal{E}$ is a solution of (2.15), iff $j^n(\sigma)(x) \in \mathcal{S}_n$ or $f^r(j^n\sigma(x)) = 0$ is met. The manifold $\mathcal{S}_n$ gives a true geometric picture of the system (2.15).

Again, we may define the two maps, prolongation $j^n : J^m(\mathcal{E}) \to J^{n+m}(\mathcal{E})$ and projection $\pi^n_m : J^n(\mathcal{E}) \to J^m(\mathcal{E})$ for systems like (2.16). One obtains the projection $\pi^n_m(\mathcal{S}_n)$ of $\mathcal{S}_n$ in adapted coordinates simply by elimination of the dependent variables u^α_J, $m < \#J \leq n$ from (2.15). We assume that $\pi^n_m(\mathcal{S}_n)$ describes a regular submanifold at least locally. The first prolongation $j(\mathcal{S}_n) \subset J^{n+1}(\mathcal{E})$ of $\mathcal{S}_n$ denoted by $\mathcal{S}^1_n$ is simply the solution set of

$$f^r\left(x, u^{(n)}\right) = 0 \ , \quad d_i f^r\left(x, u^{(n)}\right) = 0$$

with the total derivatives d_i from (2.5). The r-times repeated prolongation will be denoted by $\mathcal{S}^r_n$.

Repeated prolongation and projection of the system (2.16) leads to $\pi^{n+r+s}_{n+r}\left(\mathcal{S}^{r+s}_n\right) \subseteq \mathcal{S}^r_n$ with $r, s \geq 0$ in general. The system (2.16) is called formally integrable, iff $\mathcal{S}^r_n$ is a regular submanifold of $J^{n+r}(\mathcal{E})$ and $\pi^{n+r+s}_{n+r}(\mathcal{S}^{r+s}_n) = \mathcal{S}^r_n$ is met for all $r, s \geq 0$. Here, three facts are worth mentioning: 1) There exists no test for general systems (2.15) with respect to formal integrability that terminates reliable after a finite number of steps; 2) The determination of the formally integrable system for systems of the Frobenius type (linear PDEs) is straightforward. 3) The Lie-group method requires formally integrable systems.

3 Explicit Systems

Let us consider the system

$$x^{\alpha_x}_1 = f^{\alpha_x}(t, x, u) \ , \quad \alpha_x = 1, \ldots, n_x \tag{3.17a}$$

$$y^{\alpha_y} = c^{\alpha_y}(t, x, u) \ , \quad \alpha_y = 1, \ldots, n_y \tag{3.17b}$$

with the state x, the input $u = (u^{\alpha_u}) \in \mathcal{U} \subseteq R^{n_u}$ and the output $y \in \mathcal{Y} \subseteq R^{n_y}$. The variables (t, x, u) are locally coordinates of a fibered manifold $\mathcal{E}$ with t as the local coordinate of the base $\mathcal{B}$. Obviously, the equation (3.17a) defines a regular submanifold $\mathcal{S}_1 \subset J^1(\mathcal{E})$. The explicit system (3.17a) may also be considered as the field f_e,

$$f_e = \partial_1 + f^{\alpha_x}\partial_{\alpha_x} + u^{\alpha_u}_{k+1}\partial^k_{\alpha_u} \ , \quad k = 0, \ldots, \infty \tag{3.18}$$

on $J^\infty(\mathcal{E})$.

3.1 Observability

Let us assume, we can find a one-parameter Lie-group $\Phi_\varepsilon : \mathcal{E} \to \mathcal{E}$ with parameter ε such that it acts only on the state x of (3.17a) and that the functions c of (3.17b) are invariants of Φ_ε or

$$(t, \bar{x}, u) = \Phi_\varepsilon(t, x, u) \ , \quad c^{\alpha_y} = c^{\alpha_y} \circ \Phi_\varepsilon \tag{3.19}$$

is met. Then the system (3.17a, 3.17b) is not observable. Because of (2.12) and (2.13) we get for the infinitesimal generator of $v \in \mathcal{T}(\mathcal{E})$ of Φ_ε and its first prolongation $j(v) \in \mathcal{T}(J^1(\mathcal{E}))$,

$$v = X^{\alpha_x}\partial_{\alpha_x} \tag{3.20a}$$

$$j(v) = v + d_1(X^{\alpha_x})\partial^1_{\alpha_x} \ , \tag{3.20b}$$

see (2.14). Applying the fields (3.20a, 3.20b) to (3.17a, 3.17b) we get

$$j(v)(x^{\alpha_x}_1 - f^{\alpha_x}) = d_1(X^{\alpha_x}) - X^{\beta_x}\partial_{\beta_x}f^{\alpha_x} = 0 \tag{3.21a}$$

$$v(c^{\alpha_y}) = X^{\beta_x}\partial_{\beta_x}c^{\alpha_y} = \langle \mathrm{d}c^{\alpha_y}, X \rangle = 0 \ . \tag{3.21b}$$

Now, we have only to check, whether the system (3.17a) and (3.21a, 3.21b) admits a non trivial solution for v. The first prolongation and projection to $\mathcal{E}$ yields

$$d_1(\langle \mathrm{d}c, X \rangle)|_{\mathcal{E}} = \langle f_e(\mathrm{d}c), X \rangle = \langle \mathrm{d}f_e(c), X \rangle \ .$$

The reiteration leads to the well known criteria for observability, see e.g. [1], [2].

3.2 Accessibility

Let us consider the set of one-parameter Lie-groups Φ_ε acting on the dependent variables (x, u) and let Φ_ε be a member of this set, then we find for its infinitesimal generator v the expressions

$$v = X^{\alpha_x}\partial_{\alpha_x} + U^{\alpha_u}\partial_{\alpha_u} \tag{3.22a}$$

$$j(v) = v + d_1(X^{\alpha_x})\partial^1_{\alpha_x} + d_1(U^{\alpha_u})\partial^1_{\alpha_u} \tag{3.22b}$$

because of (2.12, 2.13, 2.14). Let us assume that the set above contains for any choice of U a subset of groups that own a common invariant $I(t, x)$, then obviously, the system (3.17a) is not strongly accessible. Because of (2.10a, 2.10b) I and v must meet the relations

$$v(I) = \langle \omega, v\rangle = 0\ , \quad \mathrm{d}I = \omega + \partial_1 I \mathrm{d}t\ , \quad \omega = \omega_{\beta_x}\mathrm{d}x^{\beta_x}\ . \tag{3.23}$$

Applying the field (3.22b) to (3.17a) we get

$$j(v)(x_1^{\alpha_x} - f^{\alpha_x}) = d_1(X^{\alpha_x}) - X^{\beta_x}\partial_{\beta_x}f^{\alpha_x} - U^{\alpha_u}\partial_{\alpha_u}f^{\alpha_x} = 0\ .$$

From $d_1\langle\omega, v\rangle = \langle d_1\omega, v\rangle + \langle\omega, d_1 v\rangle$ and (3.23) we derive

$$(d_1(\omega_{\beta_x}) + \omega_{\alpha_x}\partial_{\beta_x}f^{\alpha_x})X^{\beta_x} + \omega_{\alpha_x}\partial_{\alpha_u}f^{\alpha_x}U^{\alpha_u} = 0\ .$$

Now, the conditions above imply that this relation holds for every choice of U and X. Therefore, we must have

$$d_1(\omega_{\beta_x}) = -\omega_{\alpha_x}\partial_{\beta_x}f^{\alpha_x} \tag{3.24a}$$

$$0 = \langle\omega, b_{\alpha_u}\rangle\ , \quad b^{\alpha_x}_{\alpha_u} = \partial_{\alpha_u}f^{\alpha_x} \tag{3.24b}$$

Whether the equations (3.17a), (3.24a, 3.24b) admit a nontrivial solution, requires the determination of the formally integrabel systems. The first prolongation and projection of (3.24b) to $\mathcal{E}$ leads to

$$d_1(\langle\omega, b_{\alpha_u}\rangle)|_{\mathcal{E}} = \langle\omega, f_e(b_{\alpha_u})\rangle\ .$$

because of (3.18) and (3.24a). Again, reiteration yields the well known results concerning strong accessibility, e.g. see [1], [2]. Of course, we have also to meet

$$\mathrm{d}\omega \wedge \mathrm{d}t = 0\ . \tag{3.25}$$

If the system (3.17a) is weakly accessible, then there exists a group with infinitesimal generator w such that

$$\langle \mathrm{d}x^{\alpha_x} - f^{\alpha_x}\mathrm{d}t, v\rangle = 0 \tag{3.26}$$

is met. A special solution is $v = f_e$ with f_e from (3.18). Therefore, the invariant I from (3.23) must meet also the condition

$$v(I) = 0\ .$$

4 Implicit Systems

Let us consider the system

$$0 = f^{i_e}(t, z, z_1) \ , \quad i_e = 1, \ldots, n_e \tag{4.27a}$$

$$y^{\alpha_y} = c^{\alpha_y}(t, z) \ , \quad \alpha_y = 1, \ldots, n_y \tag{4.27b}$$

of n_e nonlinear implicit ordinary differential equations of first order with the output y. The variables t, z^{α_z}, $\alpha_z = 1, \ldots, q$, $n_e \leq q$ are local coordinates of a fibered manifold $(\mathcal{E}, \pi, \mathcal{B})$, where t denotes the coordinate of the base. We assume that (4.27a) defines a regular submanifold $\mathcal{S}_1 \subset J(\mathcal{E})$. It is worth mentioning that $\mathcal{S}_1$ has an intrinsic meaning opposite to the equations (4.27a). Under some conditions, we may assume that also the system

$$f^{\alpha_x}(t, z, z_1) = 0 \ , \quad \alpha_x = 1, \ldots, n_x \tag{4.28a}$$

$$f^{\alpha_s}(t, z) = 0 \ , \quad \alpha_s = n_x + 1, \ldots, n_e \tag{4.28b}$$

defines $\mathcal{S}_1$ and additionally that the functions of the set $\{d_1 f^{\alpha_s}, f^{\alpha_x}\}$ are functionally independent with respect to z_1 on $\mathcal{S}_1$. It is easy to see that the set $\{f^{\alpha_s}, d_1 f^{\alpha_s}, f^{\alpha_x}\}$ is formally integrable. It is shown in [8], how one determines the system (4.28a), if it is possible, corresponding to (4.27a, 4.27b). Additionally, one can prove (see [8]) that there exists a bundle diffeomorphism $\varphi : \mathcal{E} \to \bar{\mathcal{E}}$,

$$\begin{aligned} \bar{z}^{\alpha_x} &= \varphi^{\alpha_x}(t, z) \ , & \alpha_x &= 1, \ldots, n_x \\ \bar{z}^{\alpha_s} &= \varphi^{\alpha_s}(t, z) = f^{\alpha_s} \ , & \alpha_s &= n_x + 1, \ldots, n_e \\ \bar{z}^{\alpha_u} &= \varphi^{\alpha_u}(t, z) \ , & \alpha_u &= n_e + 1, \ldots, q \end{aligned} \tag{4.29a}$$

$$z^{\alpha_z} = \psi^{\alpha_z}(t, \bar{z}) \ , \tag{4.29b}$$

such that the system (4.28a, 4.28b) can be rewritten as

$$\bar{f}^{\alpha_x}(t, \bar{z}^x, \bar{z}^x_1, \bar{z}^s, \bar{z}^u, \bar{z}^u_1) = 0 \ , \quad \bar{z}^s = \bar{z}^s_1 = 0 \tag{4.30}$$

$$\bar{f}^{\alpha_x} \circ j^1\psi = 0 \ , \quad \bar{f}^{\alpha_x} = f^{\alpha_x} - e^{\alpha_x}_{\alpha_s} d_1 f^{\alpha_s} \tag{4.31}$$

and such that the functions of the set $\left\{d_1 \bar{f}^{\alpha_x}\right\}$ are functionally independent with respect to z^x_1. Here, the abbreviations $\bar{z}^x$, $\bar{z}^s$, $\bar{z}^u$, etc. indicate the set of variables $\bar{z}^{\alpha_x}$, $\bar{z}^{\alpha_s}$, $\bar{z}^{\alpha_u}$, etc. By construction, the functions $\bar{f}^{\alpha_x}$ fulfill the conditions $\partial^1_{\alpha_s} \bar{f}^{\alpha_x} = 0$ where (4.28b) is met. Now, one solves (4.30) with respect to z_1 and gets finally the system

$$\bar{z}^{\alpha_x}_1 = g^{\alpha_x}(t, \bar{z}^x, \bar{z}^u, \bar{z}^u_1) \ , \quad \bar{z}^s = \bar{z}^{\alpha_s}_1 = 0 \ . \tag{4.32}$$

It is worth mentioning that the first derivatives of the input may appear.

Since one can transform the formally integrable system (4.28a, 4.28b) into (4.32), it is left to extend the well known test for accessibility and observability to systems like (4.32). That can be done in a straightforward manner. But the construction of (4.32) requires numerical methods in general. Although the map φ of (4.29a) can be found by symbolic methods, the determination of its inverse ψ (4.29a) requires numerical methods in general. Therefore, the next two subsections show, how one can perform the test with respect to observability and accessibility without the use of (4.29b).

4.1 Observability

Let us take a look back at the construction of the Lie-group (3.19). The whole approach depends only on the construction of its infinitesimal generator v. Let us consider the field v and its first prolongation $j(v)$,

$$v = Z^{\alpha_z}\partial_{\alpha_z} \ , \quad j(v) = v + d_1(Z^{\alpha_z})\,\partial^1_{\alpha_z} \tag{4.33}$$

that acts on the dependent variables only. To restrict the action of v to the state, we simply have to add the relations

$$\begin{array}{rcccl} v(\varphi^{\alpha_s}) & = & \langle \mathrm{d}\varphi^{\alpha_s}, v\rangle & = & 0 \\ v(\varphi^{\alpha_u}) & = & \langle \mathrm{d}\varphi^{\alpha_u}, v\rangle & = & 0 \end{array} \tag{4.34}$$

with the functions φ^{α_s}, φ^{α_u} from (4.29a). Now, it is straightforward to transfer the construction of the Lie-group (3.19) from the explicit case to the implicit one, we simply have to add the relations

$$j(v)\left(f^{\alpha_x}(t, z, z_1)\right) \;=\; 0 \tag{4.35}$$

$$v(c^{\alpha_y}) \;=\; \langle \mathrm{d}c^{\alpha_y}, v\rangle = 0 \tag{4.36}$$

with the fields from (4.33). Finally, one has to check by repeated prolongation of (4.36) and projection by (4.28a, 4.28b, 4.35), whether there exists a non trivial solution for v. Here, one has to deal with nonlinear equations in contrast to the explicit case. Nevertheless, as a result the observability problem for implicit systems of the type (4.28a, 4.28b) is solved.

4.2 Accessibility

A short look back at the accessibility analysis shows that the approach depends on the construction of the invariants I of (3.23) of a suitable set of Lie-groups with infinitesimal generators v from (3.22a). Since the functions φ^{α_s} of (4.29a) must be invariants of the group action, we get the relations

$$\begin{array}{c} v(\varphi^{\alpha_s}) = 0 \ , \quad j(v)(d_1\varphi^{\alpha_s}) = d_1\langle \mathrm{d}\varphi^{\alpha_s}, v\rangle = 0 \\ v = Z^{\alpha_z}\partial_{\alpha_z} \ , \quad j(v) = v + d_1(Z^{\alpha_z})\,\partial^1_{\alpha_z} \ . \end{array} \tag{4.37a}$$

Now, we must look for additional invariants I that meet

$$v(I) = \langle \omega, v\rangle = 0 \ , \quad \mathrm{d}I = \omega + \partial_1 I \mathrm{d}t \ , \quad \omega = \omega_{\beta_x}\mathrm{d}\varphi^{\beta_x} + \omega_{\beta_s}\mathrm{d}\varphi^{\beta_s} \tag{4.38}$$

for any admissible field v. The application of the field $j(v)$ to (4.28a) yields

$$\begin{array}{rl} j(v)(f^{\alpha_x}) = & e^{\beta_x}_{\gamma_x} d_1\langle \mathrm{d}\varphi^{\gamma_x}, v\rangle + e^{\beta_x}_{\gamma_s} d_1\langle \mathrm{d}\varphi^{\gamma_s}, v\rangle + e^{\beta_x}_{\gamma_u} d_1\langle \mathrm{d}\varphi^{\gamma_u}, v\rangle \\ & +d^{\beta_x}_{\gamma_x}\langle \mathrm{d}\varphi^{\gamma_x}, v\rangle + d^{\beta_x}_{\gamma_s}\langle \mathrm{d}\varphi^{\gamma_s}, v\rangle + d^{\beta_x}_{\gamma_u}\langle \mathrm{d}\varphi^{\gamma_u}, v\rangle \quad = 0 \end{array} \tag{4.39}$$

for suitable functions $e^{\beta_x}_{\gamma_x}, e^{\beta_x}_{\gamma_s}, e^{\beta_x}_{\gamma_u}, d^{\beta_x}_{\gamma_x}, d^{\beta_x}_{\gamma_s}, d^{\beta_x}_{\gamma_u}$ defined on $J(\mathcal{E})$. Using the same technique like in the explicit case, we derive the relations

$$\begin{aligned} d_1\omega_{\gamma_x} &= \omega_{\alpha_x}\hat{e}^{\alpha_x}_{\beta_x}d^{\beta_x}_{\gamma_x} && \text{(4.40a)}\\ 0 &= \langle\omega, b_{\gamma_u}\rangle\ ,\quad b_{\gamma_u} = \hat{e}^{\alpha_x}_{\beta_x}e^{\beta_x}_{\gamma_u}\partial_{\alpha_x} && \text{(4.40b)}\\ 0 &= \left\langle\omega, b^1_{\gamma_u}\right\rangle\ ,\quad b^1_{\gamma_u} = \hat{e}^{\alpha_x}_{\beta_x}d^{\beta_x}_{\gamma_u}\partial_{\alpha_x} && \text{(4.40c)}\end{aligned}$$

analogously to (3.24a, 3.24b) with $\left\langle \mathrm{d}\varphi^{\beta_x}, \partial_{\alpha_x}\right\rangle = \delta^{\beta_x}_{\alpha_x}$, $\left\langle \mathrm{d}\varphi^{\beta_s}, \partial_{\alpha_x}\right\rangle = \left\langle \mathrm{d}\varphi^{\beta_u}, \partial_{\alpha_x}\right\rangle = 0$ and $\hat{e}^{\alpha_x}_{\beta_x}e^{\beta_x}_{\gamma_x} = \delta^{\alpha_x}_{\gamma_x}$. Finally, one has to check by repeated prolongation of (4.40b, 4.40c) and projection by (4.28a, 4.28b, 4.40a), whether there exists a non trivial solution for ω. Also here, one has to deal with nonlinear equations. It is worth mentioning that the invariants of (4.38) are not the only possible choice. Since also the first time derivatives of the input appear, one may ask for invariants that depend also on the input u. Therefore, we rewrite (4.39) as

$$\begin{aligned} d_1\left(\langle \mathrm{d}\varphi^{\alpha_x}, v\rangle + \hat{e}^{\alpha_x}_{\beta_x}e^{\beta_x}_{\gamma_u}\langle \mathrm{d}\varphi^{\gamma_u}, v\rangle\right) + \hat{e}^{\alpha_x}_{\beta_x}e^{\beta_x}_{\gamma_s}d_1\langle \mathrm{d}\varphi^{\gamma_s}, v\rangle - d_1\left(\hat{e}^{\alpha_x}_{\beta_x}e^{\beta_x}_{\gamma_u}\right)\langle \mathrm{d}\varphi^{\gamma_u}, v\rangle \\ +\hat{e}^{\alpha_x}_{\beta_x}d^{\beta_x}_{\gamma_x}\left(\langle \mathrm{d}\varphi^{\gamma_x}, v\rangle + \hat{e}^{\gamma_x}_{\eta_x}e^{\eta_x}_{\gamma_u}\langle \mathrm{d}\varphi^{\gamma_u}, v\rangle\right) + \hat{e}^{\alpha_x}_{\beta_x}d^{\beta_x}_{\gamma_s}\langle \mathrm{d}\varphi^{\gamma_s}, v\rangle \\ +\left(\hat{e}^{\alpha_x}_{\beta_x}d^{\beta_x}_{\gamma_u} - \hat{e}^{\alpha_x}_{\beta_x}d^{\beta_x}_{\gamma_x}\hat{e}^{\gamma_x}_{\eta_x}e^{\eta_x}_{\gamma_u}\right)\langle \mathrm{d}\varphi^{\gamma_u}, v\rangle = 0\end{aligned}$$

and apply the same procedure like above to derive the relations

$$\begin{aligned} d_1\omega_{\gamma_x} &= \omega_{\alpha_x}\hat{e}^{\alpha_x}_{\beta_x}d^{\beta_x}_{\gamma_x} && \text{(4.42a)}\\ 0 &= \langle\omega, b_{\gamma_u}\rangle\ ,\quad b_{\gamma_u} = \left(\hat{e}^{\alpha_x}_{\beta_x}e^{\beta_x}_{\gamma_u} - \hat{e}^{\alpha_x}_{\beta_x}d^{\beta_x}_{\gamma_x}\hat{e}^{\gamma_x}_{\eta_x}e^{\eta_x}_{\gamma_u} - d_1\left(\hat{e}^{\alpha_x}_{\beta_x}e^{\beta_x}_{\gamma_u}\right)\right)\partial_{\alpha_x}\ . && \text{(4.42b)}\end{aligned}$$

Again, one has to to check by repeated prolongation of (4.42b) and projection by (4.28a, 4.28b, 4.42a), whether one can find a non trivial solution for ω. Also here, one has to deal with nonlinear equations. Nevertheless, this approach solves the problem of strong accessibility for implicit systems.

5 Conclusions

The goal of this contribution is to demonstrate that Lie-groups are an efficient tool for the investigation of dynamic systems described by nonlinear ordinary differential equations. It was shown that this approach reproduces the well known results concerning accessibility and observability of explicit systems in a straightforward manner, but additionally, it works also for implicit formally integrable systems. It turns out that the prolongation and projection of systems are the main operation for performing the tests. In the case of DAEs or descriptor-systems linear algebra is sufficient, whereas for general implicit systems one has to deal with nonlinear equations. This fact requires new symbolic or numerical algorithms to overcome this problem, at least for problems of practical interest.

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