LINEAR TIME-VARYING DARLINGTON SYNTHESIS

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Abstract. Given a contractive linear system T which is represented as a causal linear operator on a Hilbert space $\mathcal{H}$. We study the question: when does there exist an isometric or unitary causal extension of T to $\mathcal{H} \bigoplus \mathcal{H}$?

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and T a linear contraction operator, $\|T\| \leq 1$, on $\mathcal{H}$. More than half a century ago P. Halmos showed [Ha] that there exists a unitary operator U which extends T to a unitary operator on $\mathcal{H} \bigoplus \mathcal{H}$. U has the matrix representation

$$U = \begin{bmatrix} T & (I - TT^*)^{\frac{1}{2}} \\ (I - T^*T)^{\frac{1}{2}} & T \end{bmatrix}.$$

In fact a related question had been considered a decade earlier in a classic paper of Darlington ([Dar]). Let $T(s)$ be a bounded real rational transfer function whose absolute values are bounded by one. Such a function defines a contraction operator on a Hilbert space of functions. Darlington asked: can $T(s)$ be realised as a part of "lossless" transfer matrix $\Sigma(s)$ of the form:

$$\Sigma(s) = \begin{bmatrix} T(s) & \Sigma_{12}(s) \\ \Sigma_{21}(s) & \Sigma_{22}(s) \end{bmatrix}?$$

In operator theoretic terms, "lossless" means isometric. However, unlike in the solution of Halmos, the context requires that Σ_{12}, Σ_{21}, Σ_{22} fulfill analyticity conditions which reflect the physical realizability of Σ, namely that Σ be a *causal* transfer functions. This problem, called "Darlington Synthesis", was solved in the more general context of transfer functions which are not necessarily rational but belong to the Hardy space H^∞ ([Ar1], [Ar2], [Do-H], [De1], [E-P]). These papers settled the Darlington Synthesis problem for time-invariant systems.

In this paper we consider the time-varying analogue of this problem: Given a causal contraction T on $\mathcal{H}$, when does there exist a causal unitary extension on U of T to $\mathcal{H} \bigoplus \mathcal{H}$? This problem for isometric extensions was studied in [Fe-M1], [Fe-M2] and for unitary extensions in [Pi]. A related (though not identical) problem was studied in [De2], [De-V], [V-De].

2. Preliminaries

Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{N}$ be a complete nest of subspaces on $\mathcal{H}$ (see [Dav]). The standard examples of such nests which appear in control theory ([Fe]) are the truncated spaces for the discrete time signal spaces $l^2[0,\infty),\ l^2(-\infty,\infty)$ and the continuous time signal spaces $L^2[a,b],\ L^2[0,\infty), L^2(-\infty,\infty)$ with Lebesgue measure. The nest $\mathcal{N}$ defines the notion of causality for operators on $\mathcal{H}$. If P_N denotes the orthogonal projection onto $N \in \mathcal{N}$, then the bounded linear operator T on $\mathcal{H}$ is causal if $TP_N = P_N T P_N$ for each $N \in \mathcal{N}$. In terms of the discrete time signal spaces and the nest $\{l^2[k,\infty)\}$ this means that the matrix representation of T with respect to the standard orthonormal basis is lower triangular.

The problem considered in this paper can now be stated: Suppose T is a causal contraction on $\mathcal{H}$. When do there exist $T_{12},\ T_{21},\ T_{22}$ causal such that $\Sigma = \begin{bmatrix} T & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ is unitary?

It was shown in [Fe-M1] that for $\mathcal{H} = l^2[0,\infty)$ no non-diagonal causal contraction has a unitary causal extensions. On the other hand isometric extensions can exist. To understand this fact is necessary to introduce the following terminology. For $N \in \mathcal{N},\ N_+ = \bigcap\{N' \in \mathcal{N},\ N' \supset N,\ N' \neq N\}$ is called the successor of N and $N_- = \bigvee\{N' \in \mathcal{N} :\ N' \subset N,\ N' \neq N\}$ is the predecessor of N. It was shown in [Fe-M2] that one of the necessary conditions for the existence of a isometric causal extension is that $\{0\}_+ = \{0\}$. This holds for the nest $\{l^2[k,\infty)\}$ but not for the dual nest $\{l^2[0,k]\}$ on $l^2[0,\infty)$. Since a causal unitary extension is a isometric extension for T with respect to $\{l^2[k,\infty)\}$ as well as for T^* with respect to $\{l^2[0,k]\}$, a unitary causal extension doesn't exist on that space. On the other hand, on $l^2(-\infty,\infty)$ both nests satisfy the required property and this presents a proper framework for the question. The same is true for the continuous time signal space.

So lets assume that our nest satisfies $\{0\}_+ = \{0\}$ and (the dual condition) $\mathcal{H}_- = \mathcal{H}$. The following theorem is from [Fe-M2].

Theorem 2.1. *Let T be a causal contraction on $\mathcal{H}$ and let $D_T = (I - T^*T)^{\frac{1}{2}}$. Then T has a causal isometric extension to $\mathcal{H} \bigoplus \mathcal{H} \Longleftrightarrow \bigcap\{[D_T N] :\ \{0\} \neq N \in \mathcal{N}\} = \{0\}$.*

Thus for a causal unitary extension it is also necessary that the dual condition $\bigcap\{[D_{T^*} N^\perp] :\ \{0\} \neq N \in \mathcal{N}\} = \{0\}$ holds. It is possible to show that these two conditions are in fact independent and that, while necessary, are not sufficient.

3. Unitary Causal Extensions

In this section we summarize the main results of this paper.

Theorem 3.1. *If* $\Sigma = \begin{bmatrix} T & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ *is a causal unitary extension for* T *on* $\mathcal{H} \bigoplus \mathcal{H}$ *then there exist partial isometries* $V_1,\ V_2$ *such that* $T_{21} = V_1 D_T,\ D_{T^*} V_2^* = T_{12}$ *and* $(T_{22} - V_1 T^* V_2^*) V_2 = 0,\ (T_{22}^* - V_2 T V_1^*) V_1 = 0.$

Example 3.2. Let $\mathcal{H} = l^2(-\infty, \infty)$ and let T be the causal contraction

$$I \oplus \begin{bmatrix} 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \oplus I \ \text{ on } \ l^2(-\infty, 1] \oplus l^2[0,1] \oplus l^2[2, \infty).$$

T satisfies the necessary conditions of Theorem 2.1 but doesn't have a causal unitary extension. Thus the necessary condition given above is not sufficient.

Theorem 3.3. *Suppose* T *is a causal contraction on* $\mathcal{H}$*. If there exist unitary operators* $V_1,\ V_2$ *such that* $V_1 D_T,\ D_{T^*} V_2^*,\ V_1 T^* V_2^*$ *are causal then*

$$\Sigma = \begin{bmatrix} I & 0 \\ 0 & V_1 \end{bmatrix} \begin{bmatrix} T & D_{T^*} \\ D_T & -T^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & V_2^* \end{bmatrix}$$

is a causal unitary extension for T*.*

Example 3.4. Every finite rank causal contraction satisfies this property.

The gap between the necessary condition in Theorem 3.1 and the sufficient condition in Theorem 3.3 can be closed for a class of embeddings considered in [Pi]. The results there are given for discrete time and in terms of inner-outer factorizations. We consider arbitrary nests.

Definition 3.5. If T is a causal contraction then the unitary embedding

$$\Sigma = \begin{bmatrix} T & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

is a D-embedding if T_{21} and T_{12}^* have dense range.

Theorem 3.6. *Suppose* T *is a causal contraction. Then* T *has a D-embedding if and only if*

(1) $D_T,\ D_{T^*}$ *have dense range*
(2) *there exist unitary operators* $U_1,\ U_2$ *such that* $U_1 D_T,\ D_{T^*} U_2^*,\ U_1 T^* U_2^*$ *are causal.*

A unitary extension is then given by

$$\begin{bmatrix} I & 0 \\ 0 & U_1 \end{bmatrix} \begin{bmatrix} T & D_{T^*} \\ D_T & -T^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & U_2^* \end{bmatrix}.$$

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