Split Algorithms and ZW-Factorization for Toeplitz and Toeplitz-plus-Hankel Matrices

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Abstract

New algorithms for Toeplitz and Toeplitz-plus-Hankel are presented that are in the spirit of the "split" algorithms of Delsarte/Genin. It is shown that the split algorithms are related to ZW-factorizations like the classical algorithms are related to LU-factorizations. Special attention is paid to skewsymmetric Toeplitz, centrosymmetric Toeplitz-plus-Hankel and general Toeplitz-plus-Hankel matrices.

1 Introduction

In this paper we consider linear systems $M_n\mathbf{f} = \mathbf{b}$, where M_n is a nonsingular Toeplitz matrix $M_n = [\,a_{j-k}\,]_{j,k=1}^n$ or Toeplitz-plus-Hankel matrix $M_n = [\,a_{j-k} + h_{j+k-1}\,]_{j,k=1}^n$. It is well known that linear systems with such a coefficient matrix can be solved with $O(n^2)$ or less computational complexity compared with $O(n^3)$ for a general system. In the corresponding procedures two types of algorithms are used: Levinson-type and Schur-type. The Schur-type algorithms produce, in principle, an LU-factorization of the matrix and the Levinson-type algorithm an LU-factorization (actually UL-factorization) of the inverse.

There are three different possibilities to apply Levinson- and Schur-type algorithms for solving linear systems with a structured coefficient matrix: 1. via inversion formulas, 2. via factorization and back substitution, and 3. by direct recursion.

P. Delsarte and Y. Genin showed in [2] and [3] that in the Levinson and Schur algorithms for the solution of real symmetric Toeplitz systems the number of multiplications can be reduced by about 50% if the symmetry properties of the matrix are exploited properly. The resulting algorithms are referred to as "split" algorithms. The split Levinson algorithm for symmetric Toeplitz matrices has been further studied and improved in [13], [14] and [15]. These improvements are, in principle, also contained in [6], in which a general splitting approach is discussed and applied to centrosymmetric Toeplitz-plus-Hankel matrices.

In the present paper we propose split algorithms for three classes of matrices: 1. skewsymmetric Toeplitz matrices, 2. centrosymmetric Toeplitz-plus-Hankel matrices, and 3. general Toeplitz-plus-Hankel matrices. Besides this we are aiming to show that:

1. There are split algorithms for skewsymmetric Toeplitz matrices that are even more efficient than their symmetric counterparts. This was observed in our paper [12].

2. In a similar way as the classical Schur and Levinson algorithms lead to LU-factorization of the matrix and its inverse, split algorithms lead to a ZW-factorization of the matrix and its inverse. This was first noticed by C.J. Demeure [4].

3. Split algorithms are not only useful for matrices with symmetry properties. In particular, they are convenient to exploit the Toeplitz-plus-Hankel structure. A. E. Yagle [17] mentioned this for the first time.

Let us point out that the algorithms we discuss are not always "split" in the sense that symmetric and skewsymmetric parts are treated separately. Nevertheless, we use this attribute for historical reasons.

Notations. Throughout the paper, let $\mathbb{F}$ be a field and $\mathbf{e}_k$ stands for the kth vector in the standard basis of $\mathbb{F}^n$. J_n denotes the $n \times n$ matrix of the counteridentity,

$$J_n = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}.$$

A vector $\mathbf{u} \in \mathbb{F}^n$ is called symmetric if $J_n\mathbf{u} = \mathbf{u}$ and skewsymmetric if $J_n\mathbf{u} = -\mathbf{u}$. An $n \times n$ matrix A is called centrosymmetric if $J_nAJ_n = A$.

For a matrix $A = [\, a_{ij} \,]$, let $A(t,s)$ denote the bivariate polynomial

$$A(t,s) = \sum_{i,j} a_{ij} t^{i-1} s^{j-1}.$$

$A(t,s)$ is called *generating function of* A. This also applies to column vectors, i.e. if $\mathbf{u} = (u_i)_{i=1}^n$, then $\mathbf{u}(t) = \sum_{i=1}^n u_i t^{i-1}$.

Throughout the paper, T_n will denote a skewsymmetric Toeplitz matrix,

$$T_n = [\, a_{i-j} \,]_{i,j=1}^n, \quad a_{-i} = -a_i,$$

M_n will denote a general nonsingular Toeplitz-plus-Hankel matrix,

$$M_n = [\, a_{i-j} + h_{i+j-1} \,]_{i,j=1}^n,$$

and C_n will denote a centrosymmetric nonsingular Toeplitz-plus-Hankel matrix.

2 Inversion Formulas

We present formulas for the inverses of matrices belonging to the classes under investigation. These formulas indicate which vectors have to be computed by a fast algorithm.

2.1. Skewsymmetric Toeplitz Matrices. We consider a nonsingular skewsymmetric Toeplitz matrix T_n, its principal sections $T_k = [\, a_{i-j} \,]_{i,j=1}^k$, and a skewsymmetric extension

$T_{n+1} = [\, a_{i-j} \,]_{i,j=1}^{n+1}$. Clearly, n must be even and T_{n-1} and T_{n+1} have kernel dimension 1. Let $\mathbf{u}_{n-1}$ span the kernel of T_{n-1} and $\mathbf{u}_{n+1}$ the kernel of T_{n+1}. It can be shown that $\mathbf{u}_{n-1}$ and $\mathbf{u}_{n+1}$ are symmetric vectors. We normalize these vectors by assuming that $[\, a_{n-1} \; \ldots \; a_1 \,]\mathbf{u}_{n-1} = 1$ and that the first component of $\mathbf{u}_{n+1}$ equals 1.

Theorem 2.1. *The inverse of T_n can be expressed in terms of $\mathbf{u}_{n-1}$ and $\mathbf{u}_{n+1}$ via*

$$T_n^{-1}(t,s) = \frac{\mathbf{u}_{n+1}(t) s \mathbf{u}_{n-1}(s) - t\mathbf{u}_{n-1}(t)\mathbf{u}_{n+1}(s)}{1 - ts} \,. \tag{2.1}$$

There are many possibilities to transform equality (2.1) into a matrix representation. The simplest way is to multiply T_n^{-1} from the right and the left by a discrete Fourier transformation. This leads to representations of T_n^{-1} that involve only diagonal, permutation, and Fourier matrices (see [10]). The same is true for the forthcoming inversion formulas.

With the help of the matrix representations it is possible to find the solution of a linear system with $O(n \log n)$ operations, since matrix-vector multiplication can be carried out with this complexity if FFT is used.

2.2. General Toeplitz-plus-Hankel Matrices. Let M_n be an $n \times n$ nonsingular Toeplitz-plus-Hankel matrix. Then there exists a nonsingular $(n+2) \times (n+2)$ extension M_{n+2} of M_n such that the $(1,1)$- and $(n+2, n+2)$-entries of M_{n+2}^{-1} are equal to 1, and the $(n+2, 1)$- and $(1, n+2)$-entries are equal to zero. The matrix M_{n+2} will be called *normalized band extension* of M_n. Let $\mathbf{u}_n^-$ be the first and $\mathbf{u}_n^+$ the last columns of M_n^{-1}, and $\mathbf{u}_{n+2}^-$ the first and $\mathbf{u}_{n+2}^+$ the last columns of M_{n+2}^{-1}. Furthermore, let $\widetilde{\mathbf{u}}_n^\pm$ and $\widetilde{\mathbf{u}}_{n+2}^\pm$ denote the corresponding quantities for the transposed matrices.

Theorem 2.2. *The inverse of M_n can be expressed in terms of $\mathbf{u}_{n+2}^\pm$, $\mathbf{u}_n^\pm$, $\widetilde{\mathbf{u}}_{n+2}^\pm$ and $\widetilde{\mathbf{u}}_n^\pm$ via*

$$M_n^{-1}(t,s) = \frac{t\mathbf{u}_n^+(t)\widetilde{\mathbf{u}}_{n+2}^+(s) - \mathbf{u}_{n+2}^+(t) s \widetilde{\mathbf{u}}_n^+(s) + t\mathbf{u}_n^-(t)\widetilde{\mathbf{u}}_{n+2}^-(s) - \mathbf{u}_{n+2}^-(t) s \widetilde{\mathbf{u}}_n^-(s)}{(t-s)(1-ts)} \,.$$

Let us note that an inversion formula of this type was presented in [9] for the first time.

2.3. Centrosymmetric Toeplitz-plus-Hankel Matrices. A centrosymmetric Toeplitz-plus-Hankel matrix C_n can be represented in the form $C_n = T_n^+ P_n^+ + T_n^- P_n^-$, where $T_n^\pm$ are symmetric Toeplitz matrices and $P_n^\pm = \frac{1}{2}(I_n \pm J_n)$. Note that $P_n^\pm$ are projections onto the subspaces of all symmetric or skewsymmetric vectors, respectively. Let $C_{n+2} = T_{n+2}^+ P_{n+2}^+ + T_{n+2}^- P_{n+2}^-$ be a normalized band extension of C_n.

It is remarkable that for the inversion of centrosymmetric Toeplitz-plus-Hankel matrices only solutions of pure symmetric Toeplitz systems are required: Let $\mathbf{x}_n^\pm$ and $\mathbf{x}_{n+2}^\pm$ be the solutions of the equations

$$\begin{aligned} T_n^+ \mathbf{x}_n^+ &= P_n^+ \mathbf{e}_n \,, \quad T_{n+2}^+ \mathbf{x}_{n+2}^+ = P_{n+2}^+ \mathbf{e}_{n+2} \,, \\ T_n^- \mathbf{x}_n^- &= P_n^- \mathbf{e}_n \,, \quad T_{n+2}^- \mathbf{x}_{n+2}^- = P_{n+2}^- \mathbf{e}_{n+2} \,. \end{aligned}$$

Theorem 2.3. *The inverse of C_n can be expressed in terms of $\mathbf{x}_n^\pm$, $\mathbf{x}_{n+2}^\pm$ via*

$$C_n^{-1}(t,s) = 2\left(\frac{\mathbf{x}_{n+2}^+(t)s\mathbf{x}_n^+(s) - t\mathbf{x}_n^+(t)\mathbf{x}_{n+2}^+(s)}{(t-s)(1-ts)} + \frac{\mathbf{x}_{n+2}^-(t)s\mathbf{x}_n^-(s) - t\mathbf{x}_n^-(t)\mathbf{x}_{n+2}^-(s)}{(t-s)(1-ts)}\right).$$

Let point out that in the formula of Theorems 2.3 only four polynomials are involved compared with eight in Theorem 2.2. Furthermore, all vectors in Theorem 2.3 are symmetric or skewsymmetric.

3 Split Levinson-type Algorithms

From now on, we assume that all matrices $A = [\,a_{ij}\,]_{i,j=1}^n$ under consideration are *centro-nonsingular*. This means that all central submatrices $A_{n-2l} = [\,a_{ij}\,]_{i,j=l+1}^{n-l}$ $(l = 1, 2, \dots)$ are nonsingular.

In this section we describe three-term recursions for the quantities occurring in the inversion formulas for the central submatrices. That means that the corresponding algorithms are different to the classical Levinson-type algorithms which are based on two-term recursions for the principal submatrices $[\,a_{ij}\,]_{i,j=1}^k$, $(k = 1, 2, \dots)$.

3.1. Skewsymmetric Toeplitz Matrices. The central submatrices of a nonsingular skewsymmetric Toeplitz matrix T_n are just the matrices T_{2k} $(k = 1, 2, \dots, n/2)$. Let $\mathbf{u}_{2k-1}$ denote the vectors spanning the nullspace of T_{2k-1} with the normalization

$$[\,a_{2k-1}\ \dots\ a_1\,]\mathbf{u}_{2k-1} = 1.$$

Furthermore, we introduce numbers $r_k = [\,a_{2k}\ \dots\ a_2\,]\mathbf{u}_{2k-1}$ and $r'_k = [\,a_{2k+1}\ \dots\ a_3\,]\,\mathbf{u}_{2k-1}$.

Theorem 3.1. *The vectors $\mathbf{u}_{2k-1}$ satisfy the recursion*

$$\mathbf{u}_{2k+3}(t) = \frac{1}{\alpha_k}((t^2 - (r_k - r_{k-1})t + 1)\mathbf{u}_{2k+1}(t) - t^2\mathbf{u}_{2k-1}(t),$$

where $\alpha_k = r'_k - r'_{k-1} - r_k(r_k - r_{k-1})$.

This theorem leads to an algorithm that computes the parameters in the inversion formula with $\frac{7}{8}n^2 + O(n)$ additions and $\frac{1}{2}n^2 + O(n)$ multiplications. This is approximately the same amount as for the improved split Levinson algorithm for symmetric Toeplitz matrices in [15] and for the algorithm in [6]. Note the a comparison with the classical Levinson algorithm is not possible, since this algorithm cannot be applied to skewsymmetric matrices.

3.2. Centrosymmetric Toeplitz-plus-Hankel Matrices. Now we consider a centro-nonsingular centrosymmetric Toeplitz-plus-Hankel matrix $C_n = T_n^+P_n^+ + T_n^-P_n^-$, where $T_n^\pm = [\,c_{|i-j|}^\pm\,]_{i,j=1}^n$. For the sake of simplicity of notation we assume that n is even. Then the central submatrices are given by $C_k = T_k^+P_k^+ + T_k^-P_k^-$ for $k = 2, 4, \dots, n/2$.

Let k be even and $\mathbf{x}_k^\pm$ denote the solutions of the equation $T_k^\pm\mathbf{x}_k^\pm = P_k^\pm\mathbf{e}_k$. We introduce the numbers $r_{jk}^\pm = \left[\,c_{j+k-1}^\pm\ \dots\ c_j^\pm\right]\mathbf{x}_k^\pm$ for $j = 1, 2$.

Theorem 3.2. *The solutions* $\mathbf{x}_k^{\pm}$ *with even* k *satisfy the recursion*

$$\mathbf{x}_{k+2}^{\pm}(t) = \frac{1}{2\,\alpha_k^{\pm}}\Big((t^2 - 2(r_{1,k}^{\pm} - r_{1,k-2}^{\pm})t + 1)\mathbf{x}_k^{\pm}(t) - t^2\mathbf{x}_{k-2}^{\pm}(t)\Big)$$

where $\alpha_k^{\pm} = r_{2,k}^{\pm} - r_{2,k-2}^{\pm} - 2r_{1,k}^{\pm}(r_{1,k}^{\pm} - r_{1,k-2}^{\pm}) + \frac{1}{2}$.

The algorithm emerging from this theorem requires $\frac{7}{4}n^2 + O(n)$ additions and $n^2 + O(n)$ multiplications in order to compute the quantities occurring in the inversion formula. Let us point out that for this kind of matrices the algorithm is preferable compared with classical ones, because it fully utilizes the centrosymmetry of the matrix.

3.3. General Toeplitz-plus-Hankel Matrices. Now we consider an $n \times n$ Toeplitz-plus-Hankel matrix M_n. Again we assume that n is even. We represent M_n in the form $M_n = T_n(\mathbf{a}) + T_n(\mathbf{b})J_n$, where $T_n(\mathbf{a}) = [\, a_{i-j}\,]_{i,j=1}^n$ and $T_n(\mathbf{b}) = [\, b_{i-j}\,]_{i,j=1}^n$. Then the central submatrices of M_n are $M_k := T_k(\mathbf{a}) + T_k(\mathbf{b})J_k$, where $k = 2, 4, \ldots, n/2$.

For even k, let $\mathbf{u}_k^-$ denote the first and $\mathbf{u}_k^+$ the last column of M_k^{-1} and

$$\mathbf{u}_k(t) = [\, \mathbf{u}_k^-(t) \;\; \mathbf{u}_k^+(t)\,].$$

We introduce the notation $\mathbf{c}(i:j) = [\, a_i + b_j \;\ldots\; a_j + b_i\,]$ and numbers

$$\begin{aligned} r_{-1,k}^{\pm} &= \mathbf{c}(-1:-k)\,\mathbf{u}_k^{\pm}, & r_{-2,k}^{\pm} &= \mathbf{c}(-2:-k-1)\,\mathbf{u}_k^{\pm}, \\ r_{1,k}^{\pm} &= \mathbf{c}(k:1)\,\mathbf{u}_k^{\pm}, & r_{2,k}^{\pm} &= \mathbf{c}(k+1:2)\,\mathbf{u}_k^{\pm}\,, \end{aligned}$$

$\alpha_{j,k}^{\pm} = r_{j,k}^{\pm} - r_{j,k-2}^{\pm}$ $(j = -2, -1, 1, 2)$, and

$$\begin{aligned} \gamma_k^{+\pm} &= \alpha_{2,k}^{\pm} + \tau_{\pm} - \alpha_{1,k}^{\pm}r_{1,k}^{+} - \alpha_{-1,k}^{\pm}r_{1,k}^{-}\,, \\ \gamma_k^{-\pm} &= \alpha_{-2,k}^{\pm} + \tau_{\mp} - \alpha_{1,k}^{\pm}r_{-1,k}^{+} - \alpha_{-1,k}^{\pm}r_{-1,k}^{-}\,, \end{aligned}$$

where $\tau_+ = 1$ and $\tau_- = 0$. Finally, we introduce matrices

$$A_k = \begin{bmatrix} \alpha_{-1,k}^{-} & \alpha_{-1,k}^{+} \\ \alpha_{1,k}^{-} & \alpha_{1,k}^{+} \end{bmatrix}, \quad \Gamma_k = \begin{bmatrix} \gamma_k^{--} & \gamma_k^{-+} \\ \gamma_k^{+-} & \gamma_k^{++} \end{bmatrix}.$$

Theorem 3.3. *For even* k*, the vector polynomials* $\mathbf{u}_k(t)$ *satisfy the recursion*

$$\mathbf{u}_{k+2}(t) = (\mathbf{u}_k(t)((t^2+1)I_2 - tA_k) - t^2\mathbf{u}_{k-2}(t))\Gamma_k^{-1}\,.$$

The algorithm emerging from this theorem computes the vectors $\mathbf{u}_n^{\pm}$ and $\mathbf{u}_{n+2}^{\pm}$ with $\frac{9}{2}n^2 + O(n)$ additions and $4n^2 + O(n)$ multiplications. This is less than the cheapest algorithm in [8], which requires $5n^2 + O(n)$ additions and $\frac{11}{2}n^2 + O(n)$ multiplications.

4 Schur-type algorithm

One of several motivations to consider Schur-type algorithms is that the Levinson-type recursions cannot be completely parallelized, since they include inner product calculations. But the corresponding parameters can be precomputed by a Schur-type algorithm. The combination of the Levinson-type and Schur-type recursions leads to an algorithm with a parallel complexity of $O(n)$.

A second, possibly still more important motivation to consider Schur-type recursions is that they produce a factorization of the matrix. This will be discussed in Section 5. Note that for ill-conditioned matrices Schur-type recursions behave, as a rule, more stable than Levinson-type recursions.

4.1. Skewsymmetric Toeplitz Matrices. For $k = 1, 2, \ldots, n/2$, we introduce the "residual" vectors $\mathbf{r}_k = (r_{j,k})_{j=1}^{n-2k+2}$, where

$$r_{j,k} = [\, a_{j+2k-2} \ \ldots \ a_j \,]\, \mathbf{u}_{2k-1}.$$

Theorem 4.1. *The vectors $\mathbf{r}_k$ satisfy the recursion*

$$\mathbf{r}_{k+1}(t) = \frac{1}{\alpha_k}((t^{-2} - (r_{1,k} - r_{1,k-1})t^{-1} + 1)\mathbf{r}_k(t) - t^{-2}\mathbf{r}_{k-1}(t),$$

where α_k is as in Theorem 3.1.

4.2. Centrosymmetric Toeplitz-plus-Hankel Matrices. We extend the definition of the residuals $r_{jk}^{\pm} = [\, c_{j+k-1}^{\pm} \ \ldots \ c_j^{\pm} \,]\, \mathbf{x}_k^{\pm}$ to all $j = 1, \ldots, n+2-2k$ and define

$$\mathbf{r}_k^{\pm}(t) = \sum_{j=1}^{n+2-2k} r_{jk}^{\pm} t^j.$$

Theorem 3.2 leads to the following.

Theorem 4.2. *The polynomials $\mathbf{x}_k^{\pm}$ with even k satisfy the recursion*

$$\mathbf{r}_{k+2}^{\pm}(t) = \frac{1}{2\,\alpha_k^{\pm}}\Big((t^{-2} - 2(r_{1,k}^{\pm} - r_{1,k-2}^{\pm})t^{-1} + 1)\mathbf{r}_k^{\pm}(t) - t^{-2}\mathbf{r}_{k-2}^{\pm}(t)\Big)$$

where $\alpha_k^{\pm}$ is as in Theorem 3.2.

4.3. General Toeplitz-plus-Hankel Matrices. We introduce the numbers $r_{j,k}^{\pm}$ for $j = \pm 1, \pm 2, \ldots, \pm(n+2-k)$ and $k = 2, 4, \ldots, n$ by

$$r_{jk}^{\pm} = \begin{cases} c(j+k-1 : j)\mathbf{x}_k^{\pm} & : \quad j > 0\,, \\ c(j : j-k+1)\mathbf{x}_k^{\pm} & : \quad j < 0\,. \end{cases} \tag{4.2}$$

With these numbers we form polynomials

$$\mathbf{r}_k^{+\pm}(t) = \sum_{j=1}^{n+2-k} r_{j,k}^{\pm} t^j, \quad \mathbf{r}_k^{-\pm}(t) = \sum_{j=1}^{n+2-k} r_{-j,k}^{\pm} t^j,$$

and 2×2 matrix polynomials

$$\mathbf{r}_k(t) = \begin{bmatrix} \mathbf{r}_k^{--}(t) & \mathbf{r}_k^{-+}(t) \\ \mathbf{r}_k^{+-}(t) & \mathbf{r}_k^{++}(t) \end{bmatrix}.$$

From Theorem 3.3 one can deduce the following.

Theorem 4.3. *The matrix polynomials of the residuals $\mathbf{r}_k(t)$ satisfy the recursion*

$$\mathbf{r}_{k+2}(t) = (\mathbf{r}_k(t)((t^{-2}+1)I_2 - t^{-1}A_k) - t^{-2}\mathbf{r}_{k-2}(t))\Gamma_k^{-1}\ ,$$

where A_k and Γ_k are as in Theorem 3.3.

The amount for computing all $\mathbf{r}_k$ for even k is $5n^2 + O(n)$ additions and $4n^2 + O(n)$ multiplications. The combination of the recursions in Theorem 3.3 and Theorem 4.3 requires $\frac{15}{2}\,n^2 + O(n)$ additions and $6n^2 + O(n)$ multiplications.

5 ZW-Factorizations

Classical Schur-type algorithms for Toeplitz and Toeplitz-plus-Hankel matrices produce an LU-factorization of the matrix whereas Levinson-type algorithms produce a UL-factorization of the inverse. We show now that the split Schur-type algorithms described in the previous section produce a ZW-factorization of the matrix and the split Levinson-type algorithms discussed in Section 3 a WZ-factorization of the inverse.

Let us recall some concepts. A matrix $A = [\,a_{ij}\,]_{i,j=1}^n$ is called a *W-matrix* if $a_{ij} = 0$ for all (i,j) for which $i > j$ and $i + j > n$ or $i < j$ and $i + j \le n$. The matrix A will be called a *unit W-matrix* if in addition $a_{ii} = 1$ for $i = 1, \dots, n$ and $a_{i,n+1-i} = 0$ for $i \neq (n+1)/2$. The transpose of a W-matrix is called a *Z-matrix.* A matrix which is both a Z- and a W-matrix will be called an *X-matrix.* All these names come from the shapes of the set of all possible positions for nonzero entries, which are as follows:

$$W = \begin{bmatrix} \bullet & & & & & \bullet \\ \bullet & \circ & & & \circ & \bullet \\ \bullet & \circ & \circ & \circ & \circ & \bullet \\ \bullet & \circ & \bullet & \bullet & \circ & \bullet \\ \bullet & \bullet & & & \bullet & \bullet \\ \bullet & & & & & \bullet \end{bmatrix}, \ Z = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \circ & \circ & \circ & \bullet & \\ & & \circ & \bullet & & \\ & & \bullet & \circ & & \\ & \bullet & \circ & \circ & \circ & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix}, \ X = \begin{bmatrix} \bullet & & & & & \bullet \\ & \bullet & & & \bullet & \\ & & \bullet & \bullet & & \\ & & \bullet & \bullet & & \\ & \bullet & & & \bullet & \\ \bullet & & & & & \bullet \end{bmatrix}.$$

A centro-nonsingular matrix A admits a unique factorization $A = ZXW$ in which Z is a unit Z-matrix, W is a unit W-matrix and X is an X-matrix. Speaking about ZW-factorization we always have in mind this unique representation. The WZ-factorization was introduced by D. J. Evans and his coauthors in connection with the parallel solution of tridiagonal systems (see [5], [16]). It turned out that this kind of factorization is in particular appropriate for centrosymmetric and centroskewsymmetric matrices, since the symmetry properties are inherited in the factors.

5.1. Skewsymmetric Toeplitz Matrices. The ZW-factorization is in particular nicely structured for skewsymmetric Toeplitz matrices T_n. Such a matrix admits a factorization $T_n = ZXZ^T$, in which Z is a centrosymmetric unit Z-matrix and X is a skewsymmetric antidiagonal matrix. Another specific property is that

$$T_{2k} \begin{bmatrix} 0 & \mathbf{u}_{2k-1} \\ \mathbf{u}_{2k-1} & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ \mathbf{0} & \mathbf{0} \\ 0 & 1 \end{bmatrix}.$$

Hence the Z-factor has some additional symmetry properties. We illustrate this for the case $n = 6$:

$$Z = \begin{bmatrix} 1 & r_{22} & r_{31} & -r_{21} & -1 & 0 \\ 0 & 1 & r_{21} & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & r_{21} & 1 & 0 \\ 0 & -1 & -r_{21} & r_{31} & r_{22} & 1 \end{bmatrix}.$$

The Schur-type algorithm in 4.1 produce the ZW-factorization of T_n with $\frac{3}{4}n^2 + O(n)$ additions and $\frac{1}{2}n^2 + O(n)$ multiplications. Note that there is (so far) no analogous algorithm for the ZW-factorization of a symmetric Toeplitz matrices with this low complexity.

Dividing the vectors $\mathbf{u}_{2k-1}$ by their last component and building up a W-matrix W with them in an appropriate way we obtain a factorization $T_n^{-1} = W\Xi W^T$ in which W is a centrosymmetric unit W-matrix (possessing additional symmetry properties like Z) and Ξ is a skewsymmetric antidiagonal matrix. That means a WZ-factorization of T_n^{-1} can be obtained with $\frac{7}{8}n^2 + O(n)$ additions and $\frac{1}{2}n^2 + O(n)$ multiplications.

5.2. Centrosymmetric Toeplitz-plus-Hankel Matrices. Since every centrosymmetric Toeplitz-plus-Hankel matrix C_n is also symmetric, it admits a ZW-factorization $C_n = ZXZ^T$, in which all factors, due to central symmetry, are centrosymmetric.

If we build up a W-matrix $\widetilde{V}$ from the solutions $\mathbf{x}_{2k}^{\pm}$ then $C_n\widetilde{V} = \widetilde{Z}$ will be a Z-matrix with symmetric or skewsymmetric columns consisting of the residuals. The unit Z-factor is now easily obtained by pairwise addition and subtraction of columns. In general, $\frac{3}{2}n^2 + O(n)$ additions and $n^2 + O(n)$ multiplications are needed.

From the matrix $\widetilde{V}$ one can easily obtain the factors of the WZ-factorization of C_n^{-1}.

5.3. General Toeplitz-plus-Hankel Matrices. For general Toeplitz-plus-Hankel matrices, of course, no special symmetry of the factors of the ZW-factorization can be observed.

The factor Z in the factorization $M_n = ZXW$ is simply obtained from the first components of the residual vectors. In order to obtain W^T one has to run the split Schur algorithm a second time for the transpose matrix. The factor X is obtained by an $O(n)$ complexity recursion.

More efficient from the complexity point of view is to consider a factorization $M_n U = Z$ in which Z is the Z-factor of the ZW-factorization and U is a W-matrix. In fact, this factorization appears as the result of the application of both the split Levinson and split Schur algorithm. The solution of a linear system $M_n \mathbf{f} = \mathbf{b}$ can now be obtained by solving the system $Z\mathbf{g} = \mathbf{b}$ and the multiplication of this solution by the W-matrix U, i.e. $\mathbf{f} = U\mathbf{g}$.

6 Final Remarks

- The combination of split Levinson-type and Schur-type algorithms can be speeded up to a "superfast" algorithm with complexity $O(n \log^2 n)$ if a divide-and-conquer strategy and FFT is employed. For skewsymmetric Toeplitz matrices this might be of practical interest. For more involved structures the practical efficiency is still uncertain.

- The split algorithm for skewsymmetric Toeplitz matrices can be extended to the general case, i.e. without the condition of centro-nonsingularity. There are two ways of doing this. The first is to compute recursively the fundamental system (see [11]), the second is a look-ahead approach. These results will be published elsewhere. For general Toeplitz-plus-Hankel matrices algorithms working without additional conditions are proposed in the paper [7].

- It is also possible to design split algorithms for general Toeplitz matrices. The computational gain is, however, small. Therefore, we refrained from presenting it here.

- There exist split algorithms for Hermitian Toeplitz matrices in the literature (see [1] and references therein). However, these algorithms do not provide a ZW-factorization. We realized that split algorithms for ZW-factorization are possible, but with higher complexity.

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