A New Property of Laguerre Functions

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Abstract

Laguerre filters constitute an orthonormal basis for the Hilbert space, for this they are used in system identification and reduced-order modelling. In this paper a new property of Laguerre filters is introduced: it is shown that the system having as transfer function the sum of the first $n+1$ functions has all the singular values equals each other. On the basis of this property a generalization of Laguerre filters is proposed.

Nomenclature

$\overline{s}$	complex conjugate transpose of complex variable s
A^*	complex conjugate transpose
I	nxn identity matrix
λ_i	ith eigenvalue of A
σ_i	ith Hankel singular value
$\parallel G_0 \parallel_2$	2-norm (spectral norm) of the matrix G_0
$L_2(0,\infty)$	space of squared integrable function on time interval $[0,\infty[$
H_2^{pxm}	set of real pxm matrix functions, analytic for $\lvert z\rvert \geq 1$, that are square integrable on the unit circle.
e_i	ith Euclidian basis vector in R^n

1 Introduction

Recently growing interest has been paid to the use of Laguerre filters in system identification and reduced-order modelling [1, 2, 3]. This is due to the fact that they represent a uniformly bounded orthonormal basis for the Hilbert space [4]. They involve a scalar variable a that can be chosen to match the dominating first-order dynamics of system to be modelled. In this way an high accuracy with very fast convergence rate of the series expansion of the modelled system can be guaranteed.

Many generalizations have been proposed in literature [5]. Some of these are related to the possibility of introducing better models of the dominating dynamics. Kautz filters making use of operator with complex poles allow to handle the resonant part of the system. In fact for moderate damped systems, a large number of terms, in the Laguerre filter based series expansion, is required. While adopting Kautz filters leads to small number of series terms, thus improving the convergence rate of the identification algorithm.

A further generalization is introduced in [6], where orthonormmal basis are generated by using balanced realization of inner transfer functions. They constitute a generalization of the pulse functions, Laguerre functions and Kautz functions and allow an alternative series expansion of rational transfer functions. This approach is characterized by the use a finite number of expansion coefficients with good approximation and speed of convergence.

In this paper we illustrate a new property of Laguerre filters and propose a generalization for which the same property holds. In particular it is shown that for the standard Laguerre functions and for the generalized functions, here introduced, the sum of the first $n+1$ functions has all the singular values equals each other.

The paper is organized as follows: Section 2 briefly reviews basic concepts on singular values and Laguerre filters; Section 3 introduces the main result of the paper; Section 4 presents the conclusions of the paper.

2 Preliminaries

In this section the preliminaries regarding the singular values of a system and the Laguerre filters are introduced.

- Let us consider a stable linear system in a minimal form $S(A,B,C,D)$ with transfer function $G(s)$.

  The controllability and observability Gramians, defined as $W_c = \int_0^\infty e^{At}BB^Te^{A^Tt}dt$ and $W_o = \int_0^\infty e^{A^Tt}C^TCe^{At}dt$ respectively, are symmetric positive semidefinite matrices satisfying the Lyapunov equations:

  $$\begin{aligned} AW_c + W_cA^T &= -BB^T \\ A^TW_o + W_oA &= -C^TC \end{aligned} \tag{2.1}$$

  The Hankel singular values σ_i of the system S are invariant quantities defined as the square root of eigenvalues of the product of the Gramians W_c and W_o [7]:

  $$\sigma_i = \{\lambda_i(W_cW_o)\}^{1/2} \tag{2.2}$$

  and by convention $\sigma_i \geq \sigma_{i+1}$.

- A fundamental result on the singular values of an all-pass function, i.e. G(s) such that $G(s)G^*(-\overline{s}) = I$, is reported in [8]. It states that an all-pass function has all singular values equal to the static gain.

- Laguerre functions are defined in the Laplace domain as it follows [2]:

$$L_k(s) = \frac{\sqrt{2a}}{s+a}(\frac{s-a}{s+a})^k \tag{2.3}$$

where $k = 0, 1, 2, ...$ and a is a positive real number.

These functions constitute an orthogonal complete set in $L_2(0, \infty)$ [3]. The fundamental result on Laguerre functions, reported in [3], states that a function $G(s)$, strictly proper, analytic in Re $s > 0$ and continuous in Re $s \geq 0$ can be expressed as follows:

$$G(s) = \sum_{k=1}^{\infty} g_k \frac{\sqrt{2a}}{s+a}(\frac{s-a}{s+a})^k \tag{2.4}$$

Each Laguerre function is therefore the product of a low-pass term and an all-pass term. In the following it will be shown that this is fundamental to verify the property...

The use of orthogonal functions has been investigated for filter synthesis, system identification, and system approximation in many papers (see [6] and references therein).

3 Main result

In this section we illustrate a new property of Laguerre functions and introduce a generalization of these functions maintaining this property.

Let $F(s)$ be the transfer function obtained by summing $n+1$ Laguerre functions as it follows:

$$F(s) = \sum_{k=0}^{n} L_k(s) \tag{3.5}$$

It is easy to prove that each of the $n+1$ poles of the transfer function $F(s)$ are different from its zeros. The singular values of the system with transfer function $F(s)$ have the property expressed by the following Proposition.

Proposition 3.1 *The singular values σ_i of the systems ($n = 0, 1, ..$) with transfer function $F(s)$ are equal to $\sigma_i = \frac{1}{\sqrt{2a}}$ with $i = 1, 2, ...n + 1$.*

Proof. Let us compute the function $F(s)$ as follows:

$$F(s) = \sum_{k=0}^{n} L_k(s) = \sum_{k=0}^{n} \frac{\sqrt{2a}}{s+a}(\frac{s-a}{s+a})^k = \frac{\sqrt{2a}}{s+a} \sum_{k=0}^{n} (\frac{s-a}{s+a})^k \tag{3.6}$$

The sum in (3.6), being the finite sum of n terms of a geometric series with ratio $\frac{s-a}{s+a}$, can be explicitly computed as follows:

$$\sum_{k=0}^{n} (\frac{s-a}{s+a})^k = \frac{1 - (\frac{s-a}{s+a})^{n+1}}{1 - \frac{s-a}{s+a}} = \frac{1 - (\frac{s-a}{s+a})^{n+1}}{\frac{2a}{s+a}} \tag{3.7}$$

Therefore, $F(s)$ in equation (3.5) can be written as follows:

$$F(s) = \frac{\sqrt{2a}}{s+a} \frac{1-(\frac{s-a}{s+a})^{n+1}}{\frac{2a}{s+a}} = \frac{1}{\sqrt{2a}}(1-(\frac{s-a}{s+a})^{n+1}) \tag{3.8}$$

The function $F(s)$ is the sum of an all-pass term and a constant. Since the singular values of an all-pass function are equal to the static gain, the singular values of $F(s)$ are $\sigma_i = \frac{1}{\sqrt{2a}}$. $\diamond$

Now we propose a generalization of Laguerre filters, based on Proposition 3.1. In other words, a new set of filters, for which a property similar to that of Proposition 3.1 holds, is generated.

Definition 3.1 *(Generalized Laguerre Filters) Let $Q(s)$ be an all-pass function, the generalized Laguerre filters $LQ_k(s)$ are defined as follows:*

$$LQ_k(s) = (1-Q(s))(Q(s))^k \tag{3.9}$$

Proposition 3.2 *The singular values σ_i of the system with transfer function:*

$$FQ(s) = \sum_{k=0}^{n} LQ_k(s) \tag{3.10}$$

are equal to $\sigma_i = 1$.

Proof. The proof is similar to that one of Proposition 3.1. It is based on writing $FQ(s)$ as sum of two terms, a constant and an all-pass function, and concluding that $\sigma_i = 1$ with $i = 1, 2, ...n+1$.

In fact, $FQ(s)$ can be written as follows:

$$\begin{aligned} FQ(s) &= \sum_{k=0}^{n} LQ_k(s) = \sum_{k=0}^{n} (1-Q(s))(Q(s))^k = \\ &= (1-Q(s)) \sum_{k=0}^{n} (Q(s))^k = 1-(Q(s))^{n+1} \diamond . \end{aligned} \tag{3.11}$$

As example let us consider a $Q(s)$ defined as it follows:

$$Q(s) = \frac{(s-a)(s-b)}{(s+a)(s+b)} \tag{3.12}$$

where a and b are real positive values.

With this choice the generalized Laguerre functions (3.9) are:

$$LQ_k(s) = \frac{2s(a+b)}{(s+a)(s+b)} \left(\frac{(s-a)(s-b)}{(s+a)(s+b)} \right)^k \tag{3.13}$$

Clearly these functions respect the previous introduced property.

Remark 3.1 *In [6], a generalization of Laguerre function by using a balanced realization of a generic all-pass function is proposed. In particular, given* G *an* mxn *inner transfer function with McMillan degree* $n \geq 1$*, having a Laurent expansion* $G(z) = \sum_{k=0}^{\infty} G_k z^{-k}$ *and satisfying* $\| G_0 \|_2 < 1$*, and a balanced realization of* $G(z)$ (A, B, C, D)*, and denoting* $V_k(z) = z(zI - A)^{-1} B G^k(z)$*, then the set of functions* $\{e_i^T V_k(z)\}_{i=1,\ldots n; k=0,\ldots\infty}$ *constitute an orthonormal basis of the function space* H_2^{1xm}*. It should be noted that for this generalization Proposition 3.2 does not hold.*

4 Conclusions

In this paper a new property of Laguerre filters has been introduced. It has been shown that the system having as transfer function the sum of the first $n+1$ Laguerre functions has all the singular values equals each other. On the basis of this property a generalization of Laguerre filters has been proposed. A new set of functions has been introduced. Each function is the product of an all-pass term and a low-pass term. Moreover, it has been shown that the system having as transfer function the sum of these generalized Laguerre filters has all singular values equals to the unit.

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