Modeling distributed parameter systems with discrete element networks

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1 Introduction

In this paper, we compare three general methods aimed to obtain reduced-order models of simple distributed parameter systems. Reduced-order models are particularly useful in the case of real-time applications for which simplicity and accuracy are the main points.

We heavily use electrical analogies therefore these models are presented as equivalent electric networks of discrete elements. As a consequence, they can easily be used with dedicated network simulation programs.

Eventually, the example of a thermic system puts in evidence the efficiency of the continued fractions as a modeling tool.

2 Infinite order Impedances

A large class of physical phenomena is said to be *diffusive* or *propagative*. They have in common the same mathematical formalism coupling two variables from the local point of view. If we limit the problem to the linear case, we can describe the system of coupled variables $(u,\, i)$ by the means of electrical analogies and the Laplace transform.

$$\left\{ \begin{aligned} \nabla u &= -zi \\ \nabla i &= -yu \end{aligned} \right. \tag{2.1}$$

Then, z and y can be seen respectively as an impedance and an admittance per unit of length, representing the *distributed parameters*.

When the geometry of the system is simple enough to integrate the local equations, we can obtain global models composed of formal impedances. These impedances are in particular expressed with hyperbolic trigonometry functions or Bessel functions and can be considered as infinite order transfer functions.

For example, with $Z = zx$ and $Y = yx$ the impedance[1]

$$\mathcal{Z} = \sqrt{\frac{Z}{Y}} \tanh(\sqrt{ZY}) \tag{2.2}$$

represents the transfer function between the temperature and the flux at the extremity of a thermal wall, the other being maintained at a constant temperature. But it is also the impedance of a transmission line with the far end short-circuited.

Nevertheless, this kind of infinite order models are sometimes inefficient, particularly for real-time applications. Thus, we present three methods in order to obtain models of reduced orders, represented by electric networks.

3 Reduced order models

The first two methods are based on the *Mittag-Leffler theorem* whereas the third relies on the *continued fraction* theory.

Let us consider the hyperbolic tangent that appeared previously in the expression of a particular infinite order impedance. This function can be developed thanks to the Mittag-Leffler theorem [6].

$$\tanh x = \sum_{n=1}^{\infty} \frac{2x}{x^2 + (n\pi - \frac{\pi}{2})^2} \tag{3.3}$$

As well as its inverse,

$$\coth(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 + (n\pi)^2} \tag{3.4}$$

On the other hand, the same function admits the continued fraction representation [4, 3]

$$\tanh(x) = \cfrac{1}{\cfrac{1}{x} + \cfrac{1}{\cfrac{3}{x} + \cfrac{1}{\cfrac{5}{x} + \ddots}}} \tag{3.5}$$

Continued fractions are well-known in the number theory to provide the *best* rational approximation. In the case of the hyperbolic tangent, the *convergents* of the continued fraction are the *Padé approximants* of the function. That is to say the optimal approximations when x tends to zero.

$$\tanh(x) \approx \cfrac{1}{\cfrac{1}{x}} = x, \ \tanh(x) \approx \cfrac{1}{\cfrac{1}{x} + \cfrac{1}{\cfrac{3}{x}}} = \frac{3x}{3+x^2}, \ \tanh(x) \approx \cfrac{1}{\cfrac{1}{x} + \cfrac{1}{\cfrac{3}{x} + \cfrac{1}{\cfrac{5}{x}}}} = \frac{15x + x^3}{15 + 6x^2} \tag{3.6}$$

[1]The variable x denotes the spatial dimension for these unidimensionnal problems.

Using the three different developments, we can obtain three kinds of approximate models of the impedance $\mathcal{Z}$ (2.2). With the first development (3.3), we can write:

$$\mathcal{Z} = \sqrt{\frac{Z}{Y}} \sum_{n=1}^{\infty} \frac{2\sqrt{ZY}}{ZY + (n\pi - \frac{\pi}{2})^2} = \sum_{n=1}^{\infty} \frac{1}{\frac{Y}{2} + \frac{(n\pi - \frac{\pi}{2})^2}{2Z}} \tag{3.7}$$

Keeping only the first $2N$ poles of the impedance brings

$$\mathcal{Z} \approx \sum_{n=1}^{N} \frac{1}{\frac{Y}{2} + \frac{(n\pi - \frac{\pi}{2})^2}{2Z}} \tag{3.8}$$

That is the impedance of the network of the fig. 1, provided we have

$$Z_n = \frac{2Z}{(n\pi - \frac{\pi}{2})^2} \qquad Y_n = \frac{Y}{2} \tag{3.9}$$

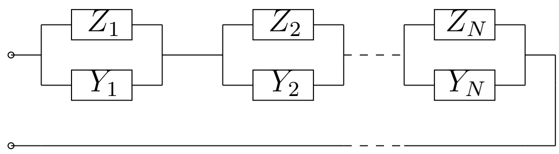


Figure 1: Reduced model of the first kind.

With the second development (3.4), the inverse of (Z) becomes

$$\frac{1}{\mathcal{Z}} = \sqrt{\frac{Y}{Z}} \coth(\sqrt{ZY}) = \frac{1}{Z} + \sum_{n=1}^{\infty} \frac{1}{\frac{Z}{2} + \frac{(n\pi)^2}{2Y}} \tag{3.10}$$

This time we keep the first $2N$ zeros of the impedance.

$$\frac{1}{\mathcal{Z}} \approx \frac{1}{Z} + \sum_{n=1}^{N} \frac{1}{\frac{Z}{2} + \frac{(n\pi)^2}{2Y}} \tag{3.11}$$

Thus, we can identify the network of the fig. 2 with the following values :

$$\begin{array}{l} Z_1 = Z \quad Y_1 \equiv \infty \\ Z_n = \frac{Z}{2} \quad Y_n = \frac{2Y}{((n-1)\pi)^2} \text{ for } n > 1 \end{array} \tag{3.12}$$

The last development (3.5) allows us to express the impedance with a continued fraction

$$\mathcal{Z} = \sqrt{\frac{Z}{Y}} \frac{1}{\frac{1}{\sqrt{ZY}} + \frac{1}{\frac{3}{\sqrt{ZY}} + \frac{1}{\frac{5}{\sqrt{ZY}} + \ddots}}} = \frac{1}{\frac{1}{Z} + \frac{1}{\frac{3}{Y} + \frac{1}{\frac{5}{Z} + \ddots}}} \tag{3.13}$$

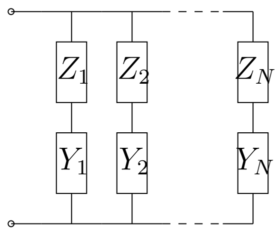


Figure 2: Reduced model of the second kind.

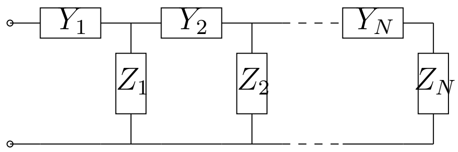


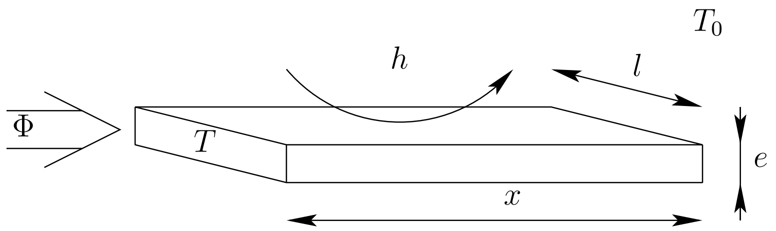
Figure 3: Reduced model of the third kind.

whose N-th convergent is the impedance of the network fig. 3 composed of the elements

$$Y_1 \equiv \infty \quad Z_n = \frac{Z}{4n-3} \quad Y_n = \frac{Y}{4n-5} \text{ for } n \geq 2 \tag{3.14}$$

Thus, the Mittag-Leffler theorem and the continued fraction theory ensure the convergence of the approximate models to the infinite-order model while $N \to \infty$. These techniques can also be seen as an extension of the Foster and Cauer syntheses [1, 2] from the rational functions to the meromorphic ones.

4 The modeling of a cooling fin



Symbol	**Quantity**
ρ	density
λ	thermal conductivity
c	specific heat
h	convective exchange coefficient

Figure 4: A simple distibuted parameter system: a cooling fin.

To illustrate the presented modeling techniques we have taken the example of a cooling fin (fig. 4). Given the flux Φ, we study the temperature at first end of the fin, the entire system being cooled by convection by the ambient fluid, characterized by the coefficient h. Let us adopt the following notations for the thermic constants

$$R = \frac{x}{\lambda le}, \quad C = \rho cxle, \quad G = h(l+e)x \tag{4.15}$$

and the electrical analogy

$$Z = R = \frac{x}{\lambda le}, \quad Y = G + Cs = (h(l+e) + \rho cles)x \tag{4.16}$$

We assume, without any lost of generality that $T_0 = 0$. Then, using the *thin fin approximation*, we have the relation

$$T = \sqrt{\frac{Z}{Y}} \tanh(\sqrt{ZY})\Phi = \mathcal{Z}\Phi \tag{4.17}$$

that is used to calculate the reference response by an inverse Fourier transform (dashed line on the figure) [5].

The responses T of the models relating to the presented techniques are shown on the fig. 5. They are obtained with a network simulation program by time step integration.

Besides, the widely used nodal modeling provides a supplementary model in form of a network. It is simply a spatial discretization (or *sampling*) of the continuous model. We have chosen the order of approximation so that all the models have the same (three) number of nodes.

5 Conclusion

The main advantage of the presented techniques is that they preserve the independance of the physical constants of the system, like does the nodal method. Thus, whatever the complexity of the reduced models, they only depend on very few parameters.

Although the Mittag-Leffler theorem ensure the rigth values for either the poles or the zeros of the models, the continued fraction method is the most efficient for the simulation when the number of nodes is constant. That can be both explained by its fast convergence and by its links with the Padé approximants.

Eventually, the network representation of the models allows easy simulations and associations with standard programs avaliable in industrial environments.

References

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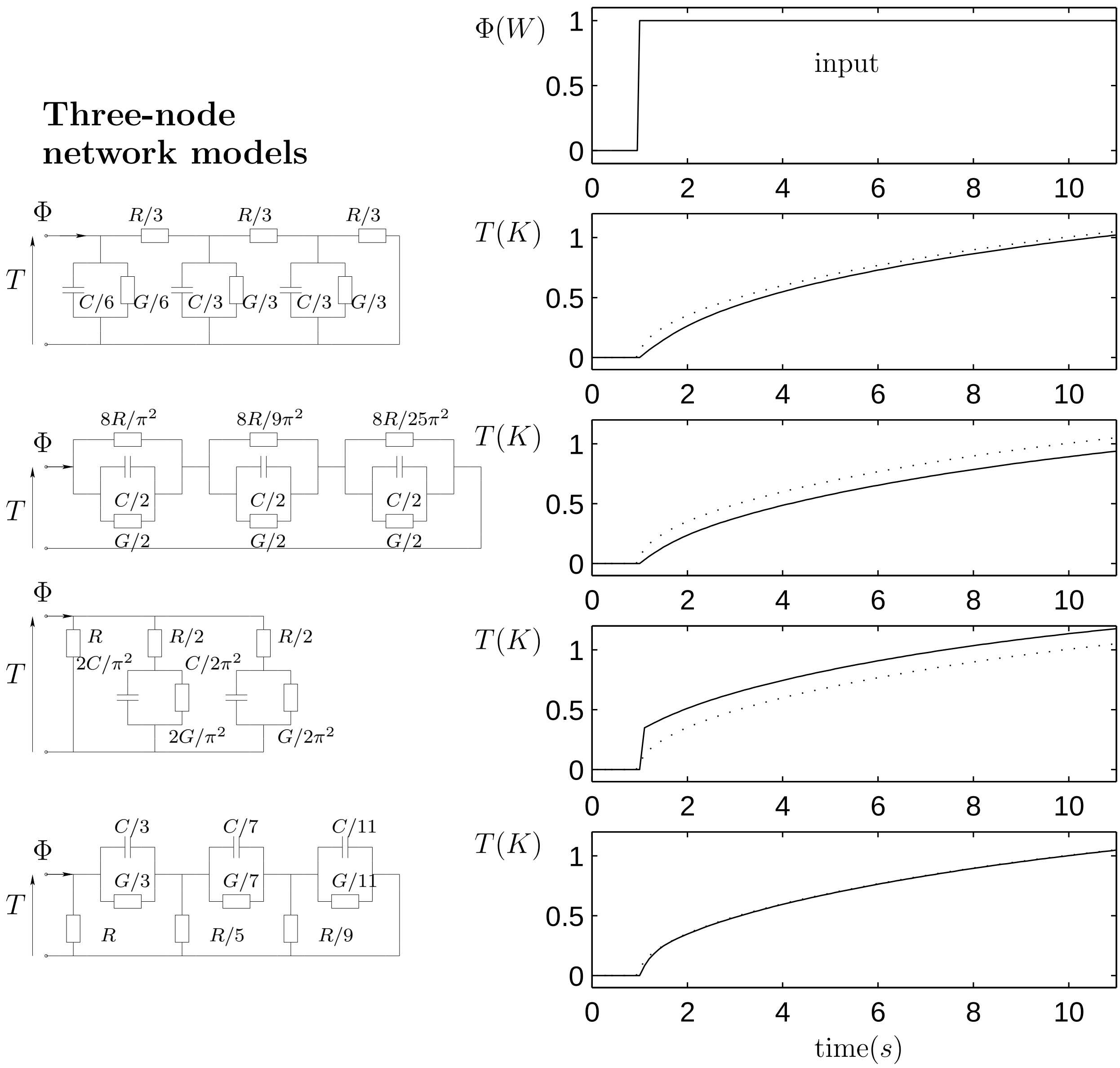

Figure 5: The comparison of the responses of the network models limited to three nodes. From top to bottom: the nodal method, the first Mittag-Leffler method, the second Mittag-Leffler method and the continued fraction method.

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