Synergetic Control of the Unstable Two-Mass System

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Abstract

The paper is devoted to a problem of synergetic regulators synthesis for nonlinear unstable two-mass system "inverted pendulum". The different variants of synergetic regulators synthesis for this system are presented. Synthesized regulators are constructed on basis of nonlinear mathematical models. The synthesized regulators allow not only to solve the pendulum stabilization task at the top unstable position but also form the new modes of dynamic behavior such as auto-oscillations of the pendulum and the cart around the set position

1 Introduction

More than 30 years in the world literature on control theory and systems there are discussions about the inverted pendulum (IP) system control [1–5].

The thing is that the two-mass system "inverted pendulum" to a certain degree reflects various real systems – from a biomechanical system of hand and body motion to the behavior of various manipulating robots and other pendulum systems. Due to its distinctive features this model has became a sort of test problem for control theory methods – from classical linear methods based on PID regulators to the modern ones based on Fuzzy Neural Networks using a certain combination of a PID regulator with a fuzzy one [2–5].

It should be mentioned that most of the works consider linearized IP models. This essentially limits the possibilities of reaching the ultimate characteristics of the IP control systems. The task of controlling the IP described by the full nonlinear model is a complex one. Its solution will allow to reach ultimate recoverable pendulum degrees with an account of limitation on the cart's position and the value of the control force etc.

In all the considered task belongs to the important problem of controlling the nonlinear oscillating objects of various nature. Here for solving this task the principles and the methods of synergetic control theory [6] are used and main research results are presented.

2 Mathematical Model

Let consider the IP shown at figure 1. The pendulum's axis is mounted on the cart that can move horizontally. The cart is driven by a drive that applies a force $\mu(t)$ at time t. This

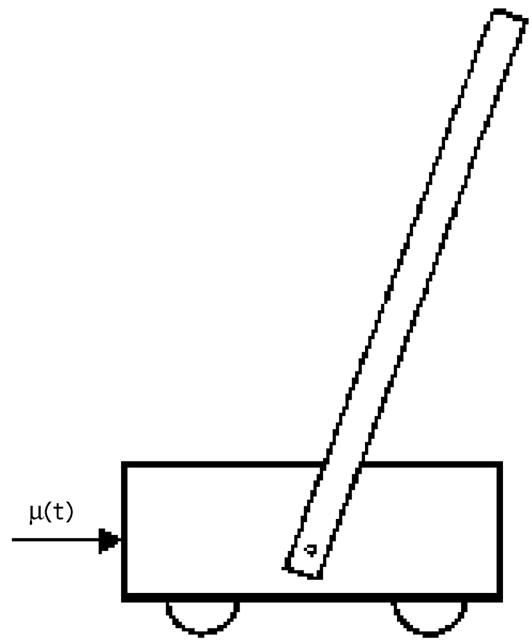

Figure 1: Inverted pendulum on cart

force is the control action for the system.

Analyzing forces and motions we can describe the system dynamics by following nonlinear mathematical model:

$$\begin{aligned}
\dot{x}_1(t) &= x_3;\\
\dot{x}_2(t) &= x_4;\\
\dot{x}_3(t) &= u;\\
\dot{x}_4(t) &= \frac{g}{L'}\sin x_2 - \frac{1}{L'}\cos x_2 \cdot u,
\end{aligned} \tag{2.1}$$

where $x_1 = s$ – cart's horizontal motion; $x_2 = \varphi$ – pendulum's angle, $x_3 = \dot{s}(t)$, $x_4 = \dot{\varphi}(t)$; m, L – pendulum's mass and the distance from axis to the mass center, J– moment of inertia; M – cart's mass, $L' = \frac{J+mL^2}{mL}$ – pendulum's effective length;

$$u = \frac{m\,L\,L'}{L'(M+m) - mL\cos^2 x_2}\left(\frac{\mu}{mL} - \frac{D_s x_3}{mL} - \frac{g\sin 2x_2}{2L'} + x_4^2 \sin x_2\right). \tag{2.2}$$

Nonlinear differential equations (2.1), (2.2) describe the behavior of the system "IP – controlled cart". As we see form these equations, one and the same control $u(t)$ (and therefore $\mu(t)$) is applied to different channels separated by dynamic units. From our point of view this quality lead to many years of not very successful attempts of many specialists to solve the task of synthesis of effective control laws ensuring vertical pendulums position by means of applying force to the cart.

We had an idea to apply synergetic method of analytical design of aggregated regulators (ADAR) [6] and to search for such a nonlinear transform that would allow to extend the allowed angle range to the ultimate values: $-\frac{\pi}{2} < x_2 < \frac{\pi}{2}$. We also wanted to take out the limitations on the cart's position. This allows to obtain an exhaustive solution of set control task. Let's proceed to consideration of such a transform of coordinates.

3 Nonlinear Transform of Coordinates

Let's introduce the following macrovariable:

$$\psi = x_1 - \rho \ln \left| \tan \left(\frac{\pi}{4} - \frac{x_2}{2} \right) \right| - \gamma \int \ln \left| \tan \left(\frac{\pi}{4} - \frac{x_2}{2} \right) \right| dt \tag{3.1}$$

then we get

$$\dot{\psi}(t) = \dot{x}_1(t) + \frac{\rho \dot{x}_2(t)}{\cos x_2} - \gamma \ln \left| \tan \left(\frac{\pi}{4} - \frac{x_2}{2} \right) \right| \tag{3.2}$$

and

$$\ddot{\psi}(t) = \ddot{x}_1(t) + \frac{\rho}{\cos x_2} \ddot{x}_2(t) + \frac{\rho \dot{x}_2^2(t)}{\cos x_2} \tan x_2 + \frac{\gamma \dot{x}_2(t)}{\cos x_2}. \tag{3.3}$$

Substitute the derivatives $\ddot{x}_1(t)$ and $\ddot{x}_2(t)$ into (3.3) from the initial equations i.e.

$$\ddot{x}_1(t) = u, \tag{3.4}$$

$$\ddot{x}_2(t) = \frac{g}{L'} \sin x_2 - \frac{1}{L'} \cos x_2 \cdot u. \tag{3.5}$$

As a result we get

$$\ddot{\psi}(t) = u + \frac{\rho}{\cos x_2} \left[\frac{g}{L'} \sin x_2 - \frac{1}{L'} \cos x_2 \cdot u \right] + \frac{\rho \dot{x}_2^2(t)}{\cos x_2} \tan x_2 + \frac{\gamma \dot{x}_2(t)}{\cos x_2} \tag{3.6}$$

or

$$\ddot{\psi}(t) = Bu + \frac{\rho g}{L'} \tan x_2 + \frac{\rho \dot{x}_2^2(t)}{\cos x_2} \tan x_2 + \frac{\gamma \dot{x}_2(t)}{\cos x_2}, \tag{3.7}$$

where $B = 1 - \frac{\rho}{L'}$.

According to the ADAR method we form the functional equation

$$\ddot{\psi}(t) + \alpha_1 \dot{\psi}(t) + \alpha_2 \psi = 0, \quad \alpha_1 > 0, \quad \alpha_2 > 0. \tag{3.8}$$

Substituting $\ddot{\psi}(t)$ (3.7) into (3.8) we get the control

$$u = -\frac{\rho g}{B\, L'} \tan x_2 - \frac{\rho \dot{x}_2^2(t)}{B \cos x_2} \tan x_2 - \frac{\gamma \dot{x}_2(t)}{B \cos x_2} - \frac{F(\dot{x}_2)}{B \cos x_2} - \frac{\alpha_1}{B} \dot{\psi}(t) - \frac{\alpha_2}{B} \psi. \tag{3.9}$$

4 Motion on the Manifolds

According to the ADAR method, control u (3.9) moves the system (3.4), (3.5) to the manifolds $\psi = 0$ (3.1) and $\dot{\psi}(t) = 0$ (3.2) from the arbitrary initial conditions. Motion of the coordinate x_2 on the manifolds $\psi = \dot{\psi}(t) = 0$ is described by the equations (3.5), (3.9), i.e.

$$\ddot{x}_{2\psi}(t) = \frac{g}{L'}\sin x_{2\psi} + \frac{\rho g}{B(L')^2}\sin x_{2\psi} + \frac{\rho}{B\,L'}\dot{x}_{2\psi}^2\tan x_{2\psi} + \frac{\gamma}{B\,L'}\dot{x}_{2\psi}(t). \tag{4.1}$$

To make the solutions of the equation (4.1) stable we'll assign $B = -\lambda < 0$, then

$$\frac{\rho g}{B(L')^2} = -\frac{1+\lambda}{\lambda L'}g,\ \frac{\rho}{B\,L'} = -\frac{1+\lambda}{\lambda},\quad \rho = (\lambda+1)L'. \tag{4.2}$$

Accounting for (4.2), equation (4.1) takes the following form:

$$\ddot{x}_{2\psi}(t) + a_1\sin x_{2\psi} + a_2\dot{x}_{2\psi}^2(t)\tan x_{2\psi} + a_3\dot{x}_{2\psi}(t) = 0, \tag{4.3}$$

where

$$a_1 = \frac{g}{\lambda L'},\ a_2 = \frac{\lambda+1}{\lambda},\ a_3 = \frac{\gamma}{\lambda L'}. \tag{4.4}$$

The equation (4.3) with $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ is asymptotically stable with respect to $x_{2\psi} = 0$ in the range $-\frac{\pi}{2} < x_{2\psi} < \frac{\pi}{2}$. Depending on the values of the coefficients a_1, a_2, a_3, the equation (4.3) can have various character of the transients. The most influence is caused by the coefficient γ.

The equation (4.3) appeared because of introduction of the nonlinear transform (3.1), which allowed to ensure asymptotic stability of motion on the manifolds $\psi = 0$ and $\dot{\psi}(t) = 0$ and, therefore, stability of the initial system with a control law (3.9). Indeed, according to (4.1) with $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ the angle x_2 always approaches zero, which means stabilization of the pendulum at the top position. *This phenomenon can be interpreted as an effect of generation of the internal stabilizing controls resulting from nonlinear transform of coordinates.* It should be underlined that this quality of internal controls generation is a result of synergetic control theory laws.

5 Control Laws and Modeling Results

Let's write the control law (3.9) using the notation (4.2) in the following form:

$$u = -\frac{1+\lambda}{\lambda}\tan x_2\left(g - \frac{\dot{x}_2^2(t)}{\cos x_2}L'\right) + \frac{\gamma}{\lambda}\frac{\dot{x}_2(t)}{\cos x_2} + \frac{\alpha_1}{\lambda}\dot{\psi}(t) + \frac{\alpha_2}{\lambda}\psi, \tag{5.1}$$

where:

$$\psi = x_1 + (1+\lambda)L'\ln\left|\tan\left(\frac{\pi}{4} - \frac{x_2}{2}\right)\right| - \gamma\int\ln\left|\tan\left(\frac{\pi}{4} - \frac{x_2}{2}\right)\right|dt. \tag{5.2}$$

$$\dot{\psi}(t) = \dot{x}_1(t) + \frac{(1+\lambda)L'}{\cos x_2}\dot{x}_2(t) - \gamma\ln\left|\tan\left(\frac{\pi}{4} - \frac{x_2}{2}\right)\right|. \tag{5.3}$$

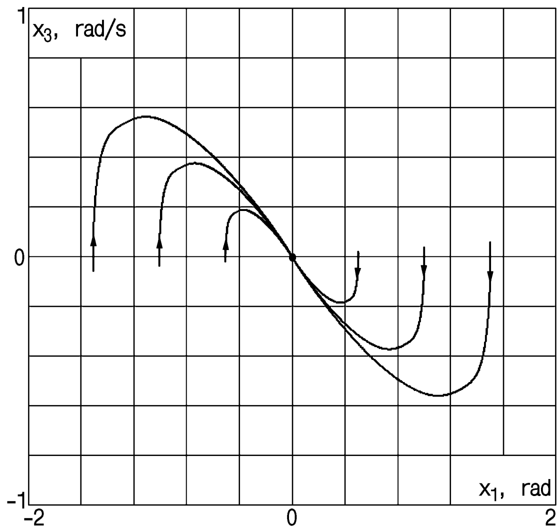

Figure 2: Pendulum's Subsystem Phase portrait

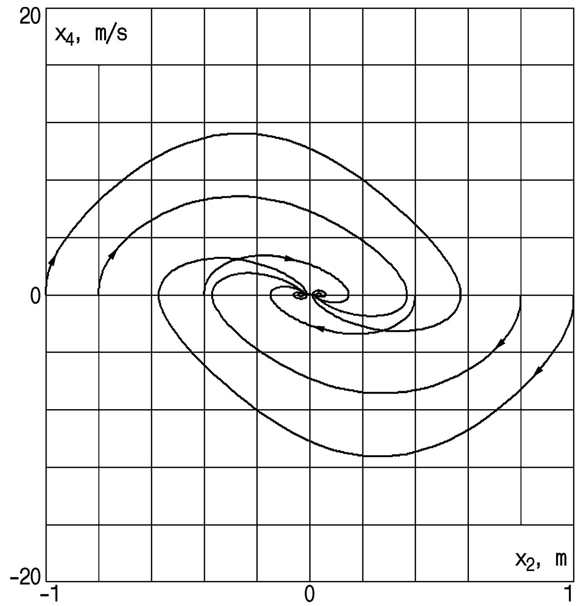

Figure 3: Cart's Subsystem Phase portrait

Now using the expressions (2.1) and (2.2) basing on (5.1)–(5.3) we find the control $\mu(x_1, x_2, x_3, x_4)$:

$$\mu = -mL\left(x_4^2 \sin x_2 + \frac{g \sin 2x_2}{2L'}\right) + D_s x_3 + \frac{1}{\lambda}\left(M + m - \frac{mL\cos^2 x_2}{L'}\right)$$
$$\left[(1+\lambda)\left(\frac{L'}{\cos x_2}x_4^2 - g\right)\tan x_2 + \alpha_1\left(x_2 + \frac{(1+\lambda)L'}{\cos x_2}x_4 - \gamma \ln\left|\tan\left(\frac{\pi}{4} - \frac{x_2}{2}\right)\right|\right) + \right. \tag{5.4}$$
$$\left. + \frac{\gamma x_4}{\cos x_2} + \alpha_2\left(x_1 + (1+\lambda)L' \ln\left|\tan\left(\frac{\pi}{4} - \frac{x_2}{2}\right)\right| - \gamma \int \ln\left|\tan\left(\frac{\pi}{4} - \frac{x_2}{2}\right)\right| dt\right)\right].$$

In (4.1) we should select such α_i that $\psi(t)$ has the desired character of changing.
So during the aperiodic process:

$$\alpha_2 = \frac{1}{T_1 T_2}; \quad \frac{\alpha_1}{\alpha_2} = T_1 + T_2, \tag{5.5}$$

then

$$\psi(t) = c_1 e^{-\frac{t}{T_1}} + c_2 e^{-\frac{t}{T_2}}, \tag{5.6}$$

which is ensured by the proper selection of T_1 and T_2. If the selection of α_1, α_2 is different from (5.5) results in a damping oscillating transient. Figures 2 and 3 present the modeling results for the control law (5.4) for the subsystems of the pendulum and the cart. Fig 4 presents transients for $x_{10} = 0,5$, $x_{20} = -0,5$.

We can use the following control instead of (3.8)

$$\ddot{\psi}(t) - \xi(1 - \beta\psi^2)\dot{\psi}(t) + \psi = 0. \tag{5.7}$$

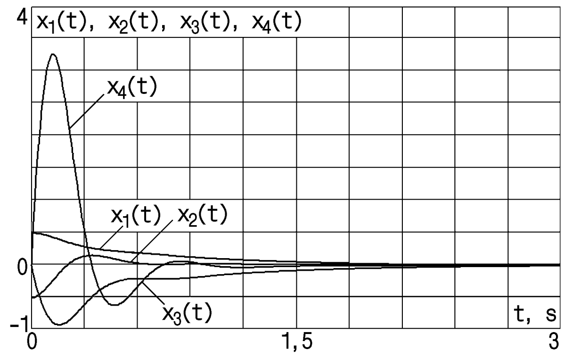

Figure 4: Transients

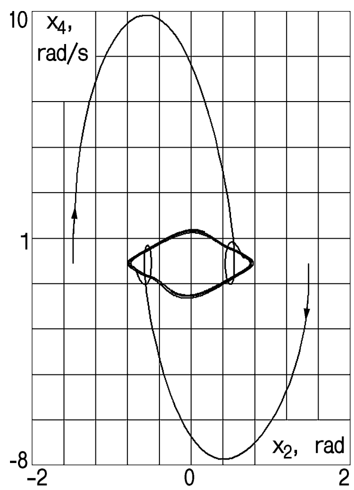

Figure 5: Pendulum's Subsystem Phase portrait

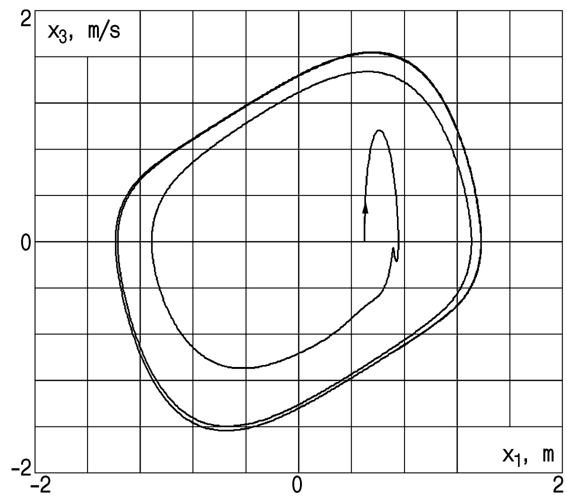

Figure 6: Cart's Subsystem Phase portrait

The equation (5.7) is the Van-der-Paul Equation, that describe the auto-oscillations mode. The control law takes the following form:

$$u = \frac{1+\lambda}{\lambda}\left(g + \frac{\dot{x}_2(t)}{\cos x_2}L'\right)\tan x_2 + \frac{\gamma}{\lambda}\frac{\dot{x}_2(t)}{\cos x_2} - \xi(1-\beta\psi^2)\dot{\psi}(t) + \psi. \tag{5.8}$$

The phase portraits for the pendulum's and cart's subsystems are presented in Figures 5 and 6 for u (5.8). Modeling was performed with the following parameters: $L = 0,1$, $L' = 0,1$, $M = 1$, $m = 0,1$, $g = 9,81$, $D_s = 100$, $\gamma = 1$, $\lambda = 1$, $\beta = 2$, $\xi = 0,5$.

It should be mentioned that control laws $\mu(\mathbf{x})$ acquired as a result of application of the synergetic synthesis methods are actually the torques created by cart electric drives. According to the ADAR method, knowing $\mu(x_1, \ldots, x_4)$ it is very easy to get the corresponding closed loop laws controlling the voltage on the drive's input. To do this, $\mu(x_1, \ldots, x_4)$ should be presented as internal controls and then basing on the drive's model we should synthesize corresponding control laws for electric drive control.

6 Conclusion

The performed research allowed to achieve the following scientific results:

- an absolutely new synergetic control method was developed. It is based on the proposed nonlinear transform of coordinates and uses a full nonlinear system's model. This allowed to obtain exhaustive solution of the control problem comparing to the known results. The two-mass system ”IP-cart” is theoretically controllable in the maximal range $(-\frac{\pi}{2} < x_2 < \frac{\pi}{2})$ without any limitations on the cart's motion. The developed new synergetic control method has an important individual meaning. It allows to solve a wide range of control tasks for various oscillating mechanical systems that were unsolvable by the known methods of control theory;

- the developed methods of synergetic control allow not only to solve the pendulum stabilization task at the top unstable position but also form the new modes of dynamic behavior such as auto-oscillations of the pendulum and the cart around the set position.

So application of the synergetic approach to the control task for the unstable nonlinear system ”IP-cart” allowed to obtain exhaustive solution of the control task for a full nonlinear model. This indicates the outstanding capabilities of the synergetic approach to solution the complex control tasks arising in various classed of nonlinear objects.
Acknowledgement: This work was supported by the US Office of Naval Research under Grant N00014–00–1–031.

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