Quotients of Fully Nonlinear Control Systems

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Abstract

In this paper, we define and study quotients for fully nonlinear control systems. Our definition is inspired by categorical definitions of quotients as well as recent work on abstractions of affine control systems. We show that quotients always exist under mild regularity assumptions, and characterize the structure of the quotient control bundle. We also introduce a notion of projectability which turns out to be equivalent to controlled invariance. This allows to regard previous work on symmetries, partial symmetries, and controlled invariance as leading to special types of quotients. We also show the existence of quotients that are not induced by symmetries or controlled invariance.

1 Introduction

The analysis and synthesis problems for nonlinear control systems are often very difficult due to the the size and the complicated nature of the equations describing the processes to be controlled. It is therefore desirable to have a methodology that decomposes control systems into smaller subsystems while preserving the properties relevant for analysis or synthesis. From a theoretical point of view, the problem of decomposing control systems is also extremely interesting since it reveals system structure that must be understood and exploited.

In the study of control systems structure by several authors we implicitly encounter notions of quotients. When symmetries for control systems exist, one of the blocks of the decompositions introduced in [4] is simply the original control system factored by the action of a Lie group representing the symmetry. If a control system admits a controlled invariant distribution, it is shown in [10] that it has a simpler local representation. This simpler representation can be obtained by factoring the original control system by an equivalence relation induced by the controlled invariant distribution. The notion of abstraction introduced in [12] can also be seen as a quotient since the abstraction is a control system on a quotient state space. These facts motivate fundamental questions such as existence and characterization of quotient systems.

The study of quotients has also important consequences for hierarchical control, since the construction of quotients proposed in [12] implicitly indicates that certain states of the original system may become inputs on the quotient control system. We can, therefore, regard a control design performed on a quotient system as a design specification for the original system. A complete and thorough understanding of how the states and inputs propagate from control systems to their quotients will enable such a hierarchical design scheme.

In this paper, we take a new approach to the study of quotients by introducing the category of control systems as the natural setting for such problems in systems theory. The use of category theory for the study of problems in system theory also has a long history which can be traced back to the works of Arbib [2]. More recently several authors have also adopted a categorical approach as in [6] where the category of affine control system is investigated or [13], where a categorical approach has been used to provide a general theory of systems. We define the category of control systems whose objects are fully (non-affine) nonlinear control systems, and morphisms map trajectories between objects. In this categorical setting we formulate the notion of quotient control systems, and show that under mild regularity assumptions quotients always exist. We introduce the notion of projectable control sections, which will be a fundamental ingredient to characterize the structure of quotients. This notion is in fact equivalent to controlled invariance, and this allows to regard quotients based on symmetries or controlled invariance as a special type of quotients. General quotients, however, are not necessarily induced by symmetries or controlled invariance and have the property that some of their inputs are related to states of the original model. This fact, implicit in [12], is explicitly characterized in this paper by understanding, how the state and input space of the quotient is related to the state and input space of the original control system.

2 Control Systems

In this section we review the relevant notions from differential geometry [1] and control systems [10] necessary for the remaining paper.

2.1 Fiber Bundles

A fiber bundle is a tuple $(B, M, \pi_B, \mathcal{F}, \{O_i\}_{i\in I})$, where B, M and $\mathcal{F}$ are manifolds called the *total space*, the *base space* and *standard fiber* respectively. The map $\pi_B : B \longrightarrow M$ is a surjective submersion and $\{O_i\}_{i\in I}$ is an open cover of M such that for every $i \in I$ there exists a diffeomorphism $\Psi_i : \pi_B^{-1}(O_i) \longrightarrow O_i \times \mathcal{F}$ satisfying $\pi_{o_i} \circ \Psi_i = \pi_B$, where π_{o_i} is the projection from $O_i \times \mathcal{F}$ to O_i. The submanifold $\pi_B^{-1}(x)$ is called the fiber at $x \in M$ and is diffeomorphic to $\mathcal{F}$. We will usually denote a fiber bundle simply by $\pi_B : B \longrightarrow M$. Since a fiber bundle is locally a product, we can always find local coordinates, which we shall call trivializing coordinates, of the form (x, b), where x are coordinates for the base space and b are coordinates for the local representative of the standard fiber. A map $\varphi : B_1 \longrightarrow B_2$ between two fiber bundles is fiber preserving iff there exists a map $\phi : M_1 \longrightarrow M_2$ between the base spaces such that the following diagram commutes:

$$\begin{CD} B_1 @>{\varphi}>> B_2 \\ @V{\pi_{B_1}}VV @VV{\pi_{B_2}}V \\ M_1 @>>{\phi}> M_2 \end{CD} \tag{2.1}$$

that is to say, iff $\pi_{B_2} \circ \varphi = \phi \circ \pi_{B_1}$. In such a case we also refer to φ as a fiber preserving lift of ϕ. Given fiber bundles B_1 and B_2 we will say that B_1 is a subbundle of B_2 if the inclusion map $i : B_1 \hookrightarrow B_2$ is fiber preserving.

Given a map $h : M \longrightarrow N$ defined on the base space of a fiber bundle we denote its extension to all of the bundle B by h^e, defined by $h^e = h \circ \pi_B$. We now consider the extension of a map $H : B \longrightarrow TM$ to a vector field in B. Globally, we define H^e as the set of all vector fields $X \in TB$ such that:

$$\begin{array}{ccc} & & TB \\ & \overset{X}{\nearrow} & \downarrow T\pi_B \\ B & \xrightarrow[H]{} & TM \end{array} \tag{2.2}$$

commutes, that is $T\pi_B(X) = H$. When working locally, one can be more specific and select a distinguished element of H^e, denoted by H^l, which satisfies in trivializing coordinates $T\pi_{\mathcal{F}}(H^l) = 0$, where $\pi_{\mathcal{F}}$ is the projection from $O_i \times \mathcal{F}$ to $\mathcal{F}$. Using trivializing coordinates (x, b) this simply means that $H^l = H\frac{\partial}{\partial x} + 0\frac{\partial}{\partial b}$. A vector field $Y : M \longrightarrow TM$ on the base space M of a fiber bundle can also be extended to a vector field on the whole bundle. It suffices to compose Y with the projection $\pi_B : B \longrightarrow M$ and recover the previous situation since $Y \circ \pi_B$ is a map from B to TM. Given a distribution $\mathcal{D}$ on M, we define[1] its extension, $\mathcal{D}^e$, as:

$$\mathcal{D}^e = \bigcup_{X \in \mathcal{D}} X^e \tag{2.3}$$

2.2 Control Systems

Since the early days of control theory it was clear that in order to give a global definition of control systems the notion of input could not be decoupled from the notion of state [17].

Definition 2.1 (Control System) *A control system $\Sigma_M = (U_M, F_M)$ consists of a fiber bundle $\pi_{U_M} : U_M \longrightarrow M$ called the control bundle and a map $F_M : U_M \longrightarrow TM$ making the following diagram commutative:*

$$\begin{array}{ccc} U_M & \xrightarrow{F_M} & TM \\ \downarrow \pi_{U_M} & \overset{\pi_M}{\swarrow} & \\ M & & \end{array} \tag{2.4}$$

that is, $\pi_M \circ F_M = \pi_{U_M}$, where $\pi_M : TM \longrightarrow M$ is the tangent bundle projection.

[1]Note that this definition implies the equality $Ker(Th^e) = (Ker(Th))^e$.

The input space U_M is modeled as a fiber bundle since in general the control inputs available may depend on the current state of the system. Closely related to control systems is the notion of control section which will be fundamental in our study of quotients:

Definition 2.2 (Control Section) *Given a manifold M, a control section on M is a subbundle $\pi_{\mathcal{S}_M} : \mathcal{S}_M \longrightarrow M$ of TM.*

We denote by $\mathcal{S}_M(x)$ the set of vectors $X \in T_xM$ such that $X \in \pi^{-1}_{\mathcal{S}_M}(x)$, which allows to show that any control system (U_M, F_M) defines a control section by the pointwise assignment $\mathcal{S}_M(x) = F_M(\pi^{-1}_{U_M}(x))$. We shall call a control system, control affine, when the control section defines an affine distribution and fully nonlinear, otherwise. Having defined control systems the concept of trajectories or solutions of a control system is naturally expressed as:

Definition 2.3 (Trajectories of Control Systems) *A curve $c : I \longrightarrow M$, $I \subseteq \mathbb{R}^+_0$ is called a trajectory of control system $\Sigma_M = (U_M, F_M)$, if there exists a curve $c^U : I \longrightarrow U_M$ making the following diagrams commutative:*

$$\begin{array}{ccc} & & U_M \\ & \nearrow^{c^U} & \downarrow \pi_{U_M} \\ I & \xrightarrow{c} & M \end{array} \qquad \begin{array}{ccc} & & U_M \\ & \nearrow^{c^U} & \downarrow F_M \\ I & \xrightarrow{Tc} & TM \end{array} \tag{2.5}$$

where we have identified I with TI.

3 The Category of Control Systems

We start by reviewing the notion of ϕ-related control systems introduced in [11] and which motivates the construction of the category of control systems to be later presented.

Definition 3.1 (ϕ-related Control Systems) *Let Σ_M and Σ_N be two control systems defined on manifolds M and N, respectively. Given a map $\phi : M \longrightarrow N$ we say that Σ_N is ϕ-related to Σ_M iff for every $x \in M$:*

$$T_x\phi(\mathcal{S}_M(x)) \subseteq \mathcal{S}_N \circ \phi(x) \tag{3.6}$$

In [11] it is shown that this notion, local in nature, is equivalent to a more intuitive and global relation between Σ_M and Σ_N.

Proposition 3.1 ([11]) *Let Σ_M and Σ_N be two control systems defined on manifolds M and N, respectively and let $\phi : M \longrightarrow N$ be a map. Control system Σ_N is ϕ-related to Σ_M iff for every trajectory $c(t)$ of Σ_M, $\phi(c(t))$ is a trajectory of Σ_N.*

Informally speaking, a category is a collection of *objects* and *morphisms* between objects, that relate the structure of the objects. Choosing manifolds for objects leads to the the natural choice of smooth maps for morphisms and defines **Man**, the category of smooth manifolds. In this section we introduce the category of control systems which we regard as the natural framework to study quotients of control systems. We defer the reader to [5] for further details on the elementary notions of category theory used throughout the paper and to [14] for the proofs of the results given in this and the forthcoming sections.

The category of control systems, denoted by **Con**, has as objects control systems as described in Definition 2.1. The morphisms in this category extend the concept of ϕ-related control systems described by Definition 3.1.

Definition 3.2 (Morphisms of Control Systems) *Let Σ_M and Σ_N be two control systems defined on manifolds M and N, respectively. A morphism f from Σ_M to Σ_N is a pair of maps $f = (\phi, \varphi)$, $\phi : M \longrightarrow N$ and $\varphi : U_M \longrightarrow U_N$ making the following diagrams commutative:*

$$\begin{array}{ccc} U_M & \xrightarrow{\varphi} & U_N \\ \downarrow{\scriptstyle \pi_{U_M}} & & \downarrow{\scriptstyle \pi_{U_N}} \\ M & \xrightarrow[\phi]{} & N \end{array} \qquad \begin{array}{ccc} U_M & \xrightarrow{\varphi} & U_N \\ \downarrow{\scriptstyle F_M} & & \downarrow{\scriptstyle F_N} \\ TM & \xrightarrow[T\phi]{} & TN \end{array} \tag{3.7}$$

It will be important for later use to also define isomorphisms:

Definition 3.3 (Isomorphisms of Control Systems) *Let Σ_M and Σ_N be two control systems defined on manifolds M and N, respectively. System Σ_M is isomorphic to system Σ_N iff there exist morphisms f_1 from Σ_M to Σ_N and f_2 from Σ_N to Σ_M such that $f_1 \circ f_2 = (id_N, id_{U_N})$ and $f_2 \circ f_1 = (id_M, id_{U_M})$.*

In this setting, feedback transformations can be seen as special isomorphisms. Consider an isomorphism (ϕ, φ) with $\varphi : U_M \longrightarrow U_M$ such that $\phi = id_M$. In trivializing coordinates (x, v) adapted to the fibers, the isomorphism has a coordinate expression for φ of the form $\varphi = (x, \beta(x, v))$. The fiber term $\beta(x, v)$ representing the new control inputs is interpreted as a feedback transformation since it depends on the state at the current location as well as the former inputs v. We shall therefore refer to feedback transformations as isomorphisms over the identity since we have $\phi = id_M$.

The relation between the notions of ϕ-related control systems (3.1) and **Con** morphisms (3.2) is stated in the next proposition.

Proposition 3.2 *Let Σ_M and Σ_N be two control systems defined on M and N, respectively. Control system Σ_N is ϕ-related to Σ_M iff $f = (\phi, \varphi)$ is a* **Con** *morphism from Σ_M to Σ_N for some fiber preserving lift φ of ϕ.*

We now see that if there is a morphism f from Σ_M to Σ_N, then this morphism carries trajectories of Σ_M to trajectories of Σ_N in virtue of Proposition 3.1.

4 Quotients of Control Systems

Given a control system Σ_M and an equivalence relation on the manifold M we can regard the quotient control system as an abstraction since some modeling details propagate from Σ_M to the quotient while other modeling details disappear in the factorization process. This fact motivates the study of quotient control systems as they represent lower complexity (dimension) objects that can be used to verify properties of the original control system. Quotients are also important from a design perspective since a control law for the quotient object can be regarded as a specification for the desired behavior of the original control system. In this spirit we will address the following questions:

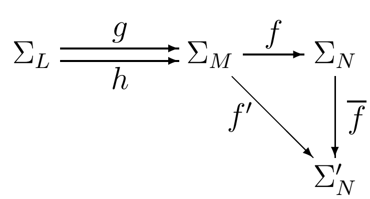
Existence: Given a control system Σ_M defined on a manifold M and an equivalence relation $\sim_M$ on M when does there exist a control system on $M/\sim_M$, the quotient manifold, and a fiber preserving lift p_U of the projection $p_M : M \longrightarrow M/\sim_M$ such that (p_M, p_U) is a **Con** morphism?

Uniqueness: Is the lift p_U of p_M, when it exists, unique?

Structure of the quotient control bundle: What is the structure of the quotient control system control bundle?

We remark that the characterization of the quotient control system system map $F : U \longrightarrow T(M/\sim_M)$ was already solved for the case of control affine systems in [12] where a constructive algorithm for its computation was proposed. To clarify our discussion we formalize the notion of quotient control systems:

Definition 4.1 (Quotient Control System) *Let Σ_L, Σ_M, Σ_N be control systems defined on manifolds L, M and N, respectively and g, h two morphisms from Σ_L to Σ_M. The pair (f, Σ_N) is a quotient control system of Σ_M if $f \circ g = f \circ h$ and for any other pair (f', Σ'_N) such that $f' \circ g = f' \circ h$ there exists one and only one morphism $\overline{f}$ from Σ_N to Σ'_N such that the following diagram commutes:*



(4.8)

Intuitively, we can read diagram (4.8) as follows. Assume that the set $\sim = \{(u, v) \in U_M \times U_M \;:\; (u, v) = (g(l), h(l))$ for some $l \in U_L\}$ is a regular equivalence relation [1]. Then, the condition $f \circ g = f \circ h$ simply means that f respects the equivalence relation, that is, $u \sim v \quad \Rightarrow \quad f(u) = f(v)$. Furthermore it asks that for any other map f' respecting relation $\sim$, there exists a unique map $\overline{f}$ such that $f' = \overline{f} \circ f$. This is a usual characterization of quotient manifolds [1] that we here use as a definition. The same idea must, therefore, hold for control systems and this means that control system Σ_N must also satisfy a unique factorization property in order to be a quotient control system.

The first two questions of the previous list are answered in the next theorem which asserts that quotients exist under very moderate conditions:

Theorem 4.1 *Let Σ_M be a control system on a manifold M and $\phi : M \longrightarrow N$ a surjective submersion. If the distribution $(T\mathcal{S}_M + Ker(TT\phi))/Ker(TT\phi)$ has constant rank, then there exists a control system Σ_N on N and a* unique *fiber preserving lift $\varphi : U_M \longrightarrow U_N$ of ϕ such that the pair $((\phi, \varphi), \Sigma_N)$ is a quotient control system of Σ_M.*

This result provides the first characterization of quotient objects in **Con**. It shows that given a regular equivalence relation on the base (state) space of a control system and a mild regularity condition[2], there always exists a quotient control system on the quotient manifold[3]. Furthermore it also shows that the regular equivalence relation on M or the map ϕ uniquely determines a fiber preserving lift φ which describes how pairs state/input of the control system on M relate to the pairs state/input of the quotient control system. Having answered the first two questions from the previous list, we concentrate on the characterization of the quotient control bundle on the remaining papaer.

5 Projectable Control Sections

We now extend the notion of projectable vector fields from [7] and of projectable families of vector fields from [8] to control sections. The notion of projectable control sections is weaker then projectable vector field or families of vector fields but nonetheless stronger than **Con** morphisms. The motivation for introducing this notion comes from the fact that projectability of control sections will be a fundamental ingredient in characterizing the structure of the quotient control bundle.

Definition 5.1 *Let M be a manifold, $\mathcal{S}_M$ a control section on M and $\phi : M \longrightarrow N$ a surjective submersion. We say that $\mathcal{S}_M$ is projectable with respect to ϕ iff $\mathcal{S}_M$ induces a control section $\mathcal{S}_N$ on N such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{P}(TM) & \xrightarrow{T\phi} & \mathcal{P}(TN) \\ \uparrow{\scriptstyle \mathcal{S}_M} & & \uparrow{\scriptstyle \mathcal{S}_N} \\ M & \xrightarrow[\phi]{} & N \end{array} \tag{5.9}$$

We see that if $\mathcal{S}_M$ is in fact a vector field we recover the notion of projectable vector fields. Sufficient and necessary conditions for projectability of control sections are given in the next result.

[2]The constant rank condition on $(Ker(TT\phi) + T\mathcal{S}_M)/Ker(TT\phi)$ is only required to ensure that $\mathcal{S}_N$ is a manifold. If one does not require a control section to be a manifold, then this condition can be weakened.

[3]This fact can be put in a more general context by introducing a forgetful functor from **Con** to **Man** that associates with each control system Σ_M defined over M the manifold M and to each morphism from Σ_M to Σ_N the map ϕ. In this context the previous result assumes the form of a universal arrow for this functor.

Proposition 5.1 (Projectable Control Sections) *Let $\mathcal{S}_M$ be a control section, $\phi : M \longrightarrow N$ a surjective submersion and $0^e = T\pi_{U_M}^{-1}(0)$. Given any control parameterization (U_M, F_M) of $\mathcal{S}_M$ and any $\overline{F_M} \in F_M^e$, $\mathcal{S}_M$ is projectable with respect to ϕ iff:*

$$[\overline{F_M}, Ker(T\phi^e)] \subseteq Ker(T\phi^e) + [\overline{F_M}, 0^e] \tag{5.10}$$

If a control section is projectable then locally we can always chose $\overline{F_M} = F_M^l$ and therefore recover the conditions for local controlled invariance from [3]:

Theorem 5.1 ([3]) *Let Σ_M be a control system over a manifold M and $\phi : M \longrightarrow N$ a surjective submersion. The distribution $Ker(T\phi)$ is locally controlled invariant for F_M iff $\mathcal{S}_M$ is projectable with respect to ϕ.*

From the study of symmetries of nonlinear control systems [4, 9] it was already known that the existence of symmetries or partial symmetries implies controlled invariance. This shows that control systems that are projectable comprise quotients induced by symmetries and controlled invariance. However there are also quotients for which projectability does not hold as we describe in the next section.

6 The Structure of Quotient Control Systems

We start by characterizing the fiber preserving lift φ of ϕ. Recall that if $f = (\phi, \varphi)$ is a morphism from Σ_M to Σ_N we have the following commutative diagram:

$$\begin{array}{ccc} U_M & \xrightarrow{\varphi} & U_N \\ {\scriptstyle F_M}\downarrow & & \downarrow{\scriptstyle F_N} \\ TM & \xrightarrow[T\phi]{} & TN \end{array} \tag{6.11}$$

Since φ is a surjective submersion we know that U_N is diffeomorphic to $U_M/\sim$, where $\sim$ is the regular equivalence relation induced by φ. This means that to understand the structure of U_N it is enough to determine the regular and involutive distribution on U_M given by $Ker(T\varphi)$. However the map φ is completely unknown, so we will resort to the elements that are available, namely F_M and ϕ to determine $Ker(T\varphi)$. Differentiating[4] diagram (6.11) we get:

$$\begin{array}{ccc} TU_M & \xrightarrow{T\varphi} & TU_N \\ {\scriptstyle TF_M}\downarrow & & \downarrow{\scriptstyle TF_N} \\ TTM & \xrightarrow[TT\phi]{} & TTN \end{array} \tag{6.12}$$

[4]The operator sending manifolds to their tangent manifolds and maps to their tangent maps is an endofunctor on **Man**, also called the tangent functor.

from which we conclude:

$$Ker(TT\phi \circ TF_M) = Ker(TF_N \circ T\varphi) = Ker(T\varphi) \tag{6.13}$$

where the last equality holds since F_N is an immersion by definition of control parameterization. We can now attempt to understand what is factored away and what is propagated from U_M to U_N since $Ker(T\varphi)$ is expressible in terms of F_M and ϕ. The first step is to clarify the relation between $Ker(T\varphi)$ and $Ker(T\phi)$. Since φ is a fiber preserving lift of ϕ the following diagram commutes:

$$\begin{array}{ccc} TU_M & \xrightarrow{T\varphi} & TU_N \\ {\scriptstyle T\pi_{U_M}}\downarrow & & \downarrow{\scriptstyle T\pi_{U_N}} \\ TM & \xrightarrow[T\phi]{} & TN \end{array} \tag{6.14}$$

which implies that:

$$T\pi_{U_M}(Ker(T\varphi)) \subseteq Ker(T\phi) \tag{6.15}$$

However this only tell us that the reduction on M due to ϕ cannot be "smaller" than the reduction on the base space of U_M due to φ. This leads to the interesting phenomena which occurs when, for *e.g.* :

$$T\pi_{U_M}(Ker(T\varphi)) = \{0\} \subseteq Ker(T\phi) \tag{6.16}$$

The above expression implies that the base space of U_M is not reduced by φ. However, U_N is a fiber bundle with base space N and therefore the points reduced by ϕ must necessarily lift to the fibers of U_N. This will not happen if we can ensure the existence of a distribution $\mathcal{D} \subseteq Ker(T\varphi)$ such that $T\pi_{U_M}(\mathcal{D}) = Ker(T\phi)$. The existence of such a distribution turns out to be related with projectability as asserted in the next proposition:

Proposition 6.1 *Let $\Sigma_M = (U_M, F_M)$ be a control system over a manifold M, $\phi : M \longrightarrow N$ a surjective submersion and $\varphi : U_M \longrightarrow U_N$ a fiber preserving lift of ϕ. There exists a regular distribution $\mathcal{D}$ on U_M satisfying $\mathcal{D} \subseteq Ker(T\varphi)$ and $T\pi_{U_M}(\mathcal{D}) = Ker(T\phi)$ iff $\mathcal{S}_M$ is projectable with respect to ϕ.*

Proposition 6.1 shows that projectability characterizes the structure of the quotient control system in the sense that states lift to the fibers when the control section is *not* projectable. However we can be a little more detailed in our analysis and try to determine if the fibers of U_M are reduced or if the fibers of U_M are in fact diffeomorphic to the fibers of U_N and reduction takes place only on the base space. The answer is given in the next proposition:

Proposition 6.2 *Let $\Sigma_M = (U_M, F_M)$ be a control system over a manifold M, $\phi : M \longrightarrow N$ a surjective submersion, $\varphi : U_M \longrightarrow U_N$ a fiber preserving lift of ϕ and $\overline{F_M}$ any vector field in F_M^e. A regular and involutive distribution $\mathcal{E}$ on U_M such that $T\pi_{U_M}(\mathcal{E}) = \{0\}$ satisfies $\mathcal{E} \subseteq Ker(T\varphi)$ iff:*

$$[\overline{F_M}, \mathcal{E}] \subseteq Ker(T\phi^e) \tag{6.17}$$

Collecting the results given by Propositions 6.1 and 6.2 we can now characterize both φ and U_N.

Theorem 6.1 (Structure of Quotients) *Consider a control system $\Sigma_M = (U_M, F_M)$ over a manifold M, $(f, \Sigma_N) = ((\phi, \varphi), (U_N, F_N))$ a quotient of Σ_M, and any vector field $\overline{F_M}$ in F_M^e. Let $\mathcal{E}$ be the involutive distribution defined by $\mathcal{E} = \{X \in 0^e \ : \ [\overline{F_M}, X] \in Ker(T\phi^e)\}$, which we assume to be regular, and denote by $R_\mathcal{E}$ the regular equivalence relation induced by $\mathcal{E}$. Under these assumptions:*

Reduction from states to states and from inputs to inputs - *Fiber bundle U_N has base space diffeomorphic to N, and standard fiber $\mathcal{F}_N$ diffeomorphic to $\mathcal{F}_M/R_\mathcal{E}$ iff $\mathcal{S}_M$ is projectable with respect to ϕ.*

Reduction from states to inputs and from inputs to inputs - *Fiber bundle U_N has base space diffeomorphic to N, and standard fiber $\mathcal{F}_N$ diffeomorphic to $(\mathcal{F}_M/R_\mathcal{E}) \times \mathcal{K}$ iff:*

1. $[\overline{F_M}, Ker(T\phi^e)] \cap \left(Ker(T\phi^e) + [\overline{F_M}, 0^e]\right) = \{0\}$;
2. $[\overline{F_M}, Ker(T\phi^e)] \neq \{0\}$.

where $\mathcal{K}$ is any leaf of the foliation on M induced by the distribution $Ker(T\phi)$.

We see that the notion of projectability is fundamentally related to the structure of quotient control systems. If the controlled section $\mathcal{S}_M$ is projectable then the control inputs of the quotient control system are the same or a quotient of the original control inputs. Projectability can therefore be seen as a structural property of a control system in the sense that it admits special decompositions [10]. However, for general systems that are not projectable, it is still possible to construct quotients by lifting the neglected state information to the fibers. The states of the original system that are factored out by ϕ are regarded as control inputs in the quotient control system. This shows that from a hierarchical synthesis point of view, control systems that are not projectable are much more appealing since one can design control laws for the abstracted system, that when pulled-down to the original one are regarded as specifications for the dynamics on the neglected states.

7 Conclusions

In this paper quotients of fully nonlinear control systems were investigated. We showed that under mild conditions quotients always exist and we characterized the structure of the quotient control bundle. This was achieved by introducing the category of control systems which was the natural framework to discuss quotients of control systems. One of the important ingredients of the characterization of quotients was the notion of projectable control section, which being equivalent to controlled invariance allowed to understand the difference between general quotients and those induced by symmetries, partial symmetries or controlled invariance.

Other directions being currently investigated include similar results for mechanical control systems [15] as well as hybrid control systems [16].

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