Cautious hierarchical switching control of stochastic linear systems

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Abstract

Standard switching control methods are based on the certainty equivalence philosophy in that, at each switching time, the supervisor selects the candidate controller that is better tuned to the currently estimated process model. If the estimated model does not appropriately describe the process, this procedure may result in the selection of a controller that is not appropriate for the actual process.

In this paper, we propose a supervisory switching logic that takes into account the uncertainty on the process description when performing the controller selection. Specifically, a probability measure describing the likelihood of the different models is computed on-line based on the collected data and, at each switching time, the supervisor selects the candidate controller that, according to this probability measure, performs the best on the average. If the candidate controller set is hierarchically structured, the supervisor automatically selects the controller that appropriately compromises robustness and performance, given the actual level of uncertainty on the process description. The use of randomized algorithms makes the supervisor implementation computationally tractable.

1 Introduction

Suppose that a process with transfer function $G^\circ(z^{-1})$ has to be regulated by choosing a controller in some candidate controller set $\{K(\gamma, z^{-1}) : \gamma \in \Gamma\}$. In a standard optimal control setting, the control performance achieved by applying controller $K(\gamma, z^{-1})$ to the process $G^\circ(z^{-1})$ is typically measured by a (positive) cost criterion $J(G^\circ(z^{-1}); K(\gamma, z^{-1}))$: the lower the value of $J(G^\circ(z^{-1}); K(\gamma, z^{-1}))$, the more satisfactory the control performance. If the process is known, an optimal controller is computed by minimizing J over the candidate controller set. In this context, J can represent any cost, e.g., of the H_2 or H_∞ type.
Consider now the case of interest – i.e., when the process is not known – and suppose that a parametric class of admissible process models is introduced $\{G(\vartheta, z^{-1}) : \vartheta \in \Theta\}$. Then, the problem of selecting the best controller according to J can be addressed by introducing a state variable representing the unknown parameter vector, and solving the optimal control problem on the so-obtained augmented state-space representation of the process. The resulting controller incorporates a self-adjusting mechanism, in that it selects a control input that

realizes an appropriate compromise between the control and the identification objectives (dual action, see, e.g., [1]). However, such an optimal dual control approach is generally difficult, except in a few simple cases, where computing the solution to the optimization problem is actually feasible.

A computationally feasible – though sub-optimal – approach to the design of self-adjusting controllers is the so-called switching control design method originally introduced in [2] and further developed in, e.g., [3]-[7]. The switching control scheme is composed of an inner loop where a candidate controller is connected in closed-loop with the process, and an outer loop where a supervisor decides which controller to select and when to switch to a different one, based on the input-output data.

The switching time instants are chosen so as to avoid switching that is too fast with respect to the system's settling time, thus causing instability. As for the controller selection procedure, it is typically an "estimator-based" procedure ([3, 4]). Specifically, at any switching time instant, a performance signal – given by the integral norm of an estimation error – is computed for each admissible model parameter. The supervisor then selects the candidate controller that is optimal for the model that minimizes the performance signal (certainty equivalence approach). Implementation and analysis of the switching control scheme are typically simplified by considering a finite set of candidate controllers with the characteristic that, associated to each admissible process model, there is a candidate controller that ensures stability when placed in closed-loop with it. This set is called a *finite controller cover* ([8, 9]).

In a standard switching control scheme, the compromise between robustness and performance is made when one designs the controller cover and the map associating each process model with a particular controller. When the controller cover is composed by a few controllers, each one stabilizing a wide set of models, then robust stability is generally guaranteed in the transient phase, but in the long run the resulting performance is typically low. In contrast, when the controller cover is composed by a large number of controllers, each one tailored to a narrow set of models, a highly performing control system is potentially achieved, but poor performance will most likely occur until there is sufficient data to obtain an accurate estimate of the process model.

In this paper, we propose a *cautious switching logic* that still relies on the introduction of a parameterized class of admissible process models but, differently from the certainty equivalence-based logic, takes into account the uncertainty in the process description when performing the controller selection.

The controller choice is based on an on-line computed probability measure $\mathcal{P}_t$ describing the likelihood of the different process models. At any switching time instant t, the supervisor selects the controller that minimizes the average control cost $c_t(\gamma) := \mathrm{E}_{\mathcal{P}_t}[J(\vartheta, \gamma)]$, $\gamma \in \Gamma$, where $J(\vartheta, \gamma)$ is the short-hand-notation for $J(G(\vartheta, z^{-1}), K(\gamma, z^{-1}))$ and $\mathrm{E}_{\mathcal{P}_t}[J(\vartheta, \gamma)]$ is the expectation of $J(\vartheta, \gamma)$ with respect to the measure $\mathcal{P}_t$ for ϑ. Minimizing $c_t(\cdot)$ corresponds to optimizing the average control system behavior where different models are given different weights according to their likelihood at time t (cautious control, [10],[1, pag. 438]).

With cautious switching, we can overcome the difficulty in standard switching control that arises from being forced to establish an a-priori compromise between robust stability and performance by associating to each model a single candidate controller. To achieve this, we propose to integrate in the cautious switching scheme a hierarchically-structured set of candidate controllers composed of different finite controller covers: the lower level cover contains controllers with a high level of robustness, but low performance guarantees, and, as we go up in the hierarchical structure, we have controller covers with increasing performance, while progressively penalizing robustness. When the distribution $\mathcal{P}_t$ is spread out over the set Θ, it is expected that the cautious supervisor will select a controller that is robust with respect to stability, though low performing. As time goes by, more and more information is accumulated and the distribution $\mathcal{P}_t$ is expected to become more and more sharply peaked around the model that better describes the actual process. Consequently, the cautious supervisor will select controllers better tailored to the true process, ultimately resulting in an improvement of performance. Thus, in finite time the control scheme is robust, and it progressively becomes better performing.
The use of average control cost criteria was originally proposed in [11] in the context of robust control and then extended to the adaptive control context in [12]. In these references, randomized algorithms are used to make the minimization of the average control cost computationally tractable. Inspired by this, we suggest a stochastic algorithm for the implementation of the cautious switching logic.
The design of the proposed switching scheme is carried out for a discrete-time single input/single output process affected by white Gaussian noise. We analyze the performance of the resulting control algorithm and prove that the closed-loop is stable.

2 Problem formulation

We address the problem of regulating a stochastic linear process described by

$$\mathcal{A}(\vartheta^\circ, z^{-1})y_{t+1} = \mathcal{B}(\vartheta^\circ, z^{-1})u_t + w_{t+1},$$

where the polynomials $\mathcal{A}(\vartheta^\circ, z^{-1}) = 1 - \sum_{i=1}^{n_s} a_i^\circ z^{-i}$, $n_s \geq 1$, and $\mathcal{B}(\vartheta^\circ, z^{-1}) = \sum_{i=1}^{m_s} b_i^\circ z^{-(i-1)}$, $m_s \geq 1$, depend on the *unknown* parameter vector $\vartheta^\circ = [a_1^\circ, \dots, a_{n_s}^\circ,\ b_1^\circ, \dots, b_{m_s}^\circ]^T$ and the signal w is a noise process described by the following assumption.

Assumption 2.1. *$\{w_t\}$ is a sequence of independent and identically distributed Gaussian random variables with zero mean and variance $\sigma^2 > 0$.*

We suppose that some a-priori knowledge is available on ϑ°. Specifically, we assume that

Assumption 2.2. *ϑ° is an interior point of the compact set $\Theta \subset \mathbb{R}^{n_s+m_s}$.*

Model class: We consider the class of models

$$\mathcal{A}(\vartheta, z^{-1})y_{t+1} = \mathcal{B}(\vartheta, z^{-1})u_t + w_{t+1}, \qquad \vartheta \in \Theta.$$

Each model can be expressed in the regression-like form

$$y_{t+1} = \varphi_t^T \vartheta + w_{t+1}, \tag{2.1}$$

where $\varphi_t = [\, y_t,\ \ldots,\ y_{t-n_s+1},\ u_t,\ \ldots,\ u_{t-m_s+1}\,]^T$ is the regression vector.

Candidate controller set: We consider a finite set of candidate controllers, where each controller is described by

$$\mathcal{C}(\gamma, z^{-1})u_t = \mathcal{E}(\gamma, z^{-1})y_t,$$

with the polynomials $\mathcal{C}(\gamma, z^{-1}) = 1 - \sum_{i=1}^{m_c} \chi_i z^{-i}$ and $\mathcal{E}(\gamma, z^{-1}) = \sum_{i=0}^{n_c} \eta_i z^{-i}$ depending on $\gamma = [\eta_0\ \eta_1 \ldots \eta_{n_c}\ \chi_1\ \chi_2 \ldots \chi_{m_c}]^T \in \Gamma \subseteq \mathbb{R}^{n_c+m_c+1}$.
We assume that the candidate controller set is sufficiently rich to appropriately control any admissible model. To make this precise, let us denote by $\mathcal{S}(\vartheta, \gamma)$ the closed-loop system for which the model with parameter ϑ is controlled by the controller with parameter γ. Given $\lambda \in [0, 1)$, we say that $\mathcal{S}(\gamma, \vartheta)$ is λ-stable if all the eigenvalues of the characteristic polynomial of $\mathcal{S}(\gamma, \vartheta)$ have absolute value smaller than or equal to λ. We assume that the control performance of $\mathcal{S}(\gamma, \vartheta)$ is measured by

$$J(\vartheta, \gamma) := \begin{cases} \dfrac{\alpha J'(\vartheta, \gamma)}{1 + \alpha J'(\vartheta, \gamma)}, & \text{if } \mathcal{S}(\vartheta, \gamma) \text{ is } \lambda\text{-stable} \\ 1, & \text{otherwise,} \end{cases} \tag{2.2}$$

where $J'(\vartheta, \gamma)$ is some positive cost criterion (e.g., an H_2 or H_∞ cost), and α is a positive constant. The criterion J thus combines both stability and performance. Note that J' is normalized so that J takes values in $[0, 1]$. This is done for technical reasons related to the implementation of the cautious switching logic.
We can then formalize the required richness of the candidate controller set as follows.

Assumption 2.3. $\bar{J} := \sup_{\vartheta \in \Theta} \inf_{\gamma \in \Gamma} J(\vartheta, \gamma) < 1.$

This means that for any admissible model, there is a candidate controller which ensures a closed-loop performance of at least $\bar{J}$.

Supervisory switching logic: The task of supervisor is to generate a switching signal that determines, at each time instant, which is the candidate controller to place in closed-loop with the process, based on the data collected. In a switching control system, the switching rate is slowed down so as to avoid switching that is too fast with respect to the system's settling time, destabilizing it. We adopt the so-called *dwell-time switching logic* where a dwell time is forced between consecutive switching instants ([4, 13, 14, 15]). Similarly to [15], we use a varying dwell time, which is adaptively selected based on the data collected from the system. We postpone to Subsection 2.2 the description of the method used for dwell-time selection. In the next subsection, we describe the candidate controller selection procedure, which is the distinguishing feature of the proposed approach.

2.1 Cautious controller selection

At each switching time t, the supervisor selects the next candidate controller by minimizing the average cost $c_t(\gamma) = \mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]$ over the controller set Γ. The controller selection procedure involves two tasks: i) computing the probability measure $\mathcal{P}_t$, and ii) minimizing the average cost $c_t(\cdot)$. We address these two issues next.

i) Computing $\mathcal{P}_{\mathbf{t}}$

In order to derive a method for computing $\mathcal{P}_t$, it is mathematically convenient to assume that ϑ° is stochastic and randomly chosen according to a distribution $\mathcal{P}$. Under this assumption, we can in fact think of $\mathcal{P}_t$ as the a-posteriori distribution of ϑ° conditioned to the observations. This line of reasoning dates back to [16], [17], and has become common practice in adaptive control, see, e.g., [18], [19], [20]. In these references, ϑ° is supposed to be a Gaussian random variable independent of the noise process $\{w_t\}$ and therefore the a-posterior distribution of ϑ°, given all the observations up to time t, is still Gaussian with mean and variance that can be computed using the Kalman filter equations ([21]).
Here, we assume that ϑ° is a random variable, independent of the noise process $\{w_t\}$, taking values in Θ and distributed according to a Gaussian distribution truncated to Θ. Specifically, we set $\mathcal{P} \sim \mathcal{N}_\Theta(M,V)$, where $\mathcal{N}_\Theta(M,V)$ denotes the rescaled Gaussian distribution with mean M and variance V, whose support is restricted to the set Θ. We assume that $V > 0$. This allows us to embed in the stochastic framework for computing $\mathcal{P}_t$ the a-priori knowledge on ϑ° provided by Assumption 2.2. Moreover, the results proven in [21] for the standard Gaussian case can be extended to our framework. Indeed, denote by M_t and V_t the mean and variance of the posterior distribution of ϑ° in the case when ϑ° is a Gaussian random variable with mean M and variance V. Since the posterior distribution of ϑ° satisfies

$$\mathrm{P}(\vartheta^\circ|y_j, j=0,1,\dots,t) = \mathrm{P}(y_j, j=0,1,\dots,t|\vartheta^\circ)\frac{\mathcal{P}(\vartheta^\circ)}{\mathrm{P}(y_j, j=0,1,\dots,t)}, \qquad \vartheta^\circ \in \mathbb{R}^{n_s+m_s},$$

where $\mathrm{P}(y_j, j=0,1,\dots,t|\vartheta^\circ)$ depends only on the considered value for ϑ° and $\mathcal{P} \sim \mathcal{N}_\Theta(M,V)$, one concludes that $\mathcal{P}_t(\vartheta^\circ) \sim \mathcal{N}_\Theta(M_t,V_t)$.

Therefore, $\mathcal{P}_t$ can be recursively updated as follows.

Algorithm 2.1.

1. *compute* M_t *and* V_t *through the Kalman filter equations*

$$\begin{aligned}
K_{t-1} &= V_{t-1}\varphi_{t-1}/(\varphi_{t-1}^T V_{t-1}\varphi_{t-1} + \sigma^2) \\
M_t &= M_{t-1} + K_{t-1}(y_t - \varphi_{t-1}^T M_{t-1}) \\
V_t &= V_{t-1} - V_{t-1}\varphi_{t-1}\varphi_{t-1}^T V_{t-1}/(\varphi_{t-1}^T V_{t-1}\varphi_{t-1} + \sigma^2),
\end{aligned}$$

   initialized with $M_0 = M$ *and* $V_0 = V$.

2. *set* $\mathcal{P}_t \sim \mathcal{N}_\Theta(M_t, V_t)$.

ii) Minimizing $\mathbf{c_t}(\cdot)$

An exact minimization of $c_t(\cdot)$ is computationally hard because $c_t(\gamma) = \mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]$ is the integral over Θ of the cost $J(\vartheta,\gamma)$ with respect to measure $\mathcal{P}_t$ for ϑ. For many control objectives, the integrand function $J(\vartheta,\gamma)$ cannot be computed in a closed-form and even the evaluation of $J(\vartheta,\gamma)$ for a given pair (ϑ,γ) may be time consuming. The approach adopted here to overcome this difficulty follows to a large extent the ideas in [11, 12] and is based on the use of randomized methods. The resulting minimizers are not rigorously optimal (i.e., they do not minimize $\mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]$ with probability 1). Nevertheless, it is possible to see that minimization is achieved in a weaker sense.

Algorithm 2.2. *Given* $\epsilon \in (0,1)$ *and* $\delta \in (0,1)$, *do the following:*

1. *extract at random* $M(\epsilon,\delta) \geq \frac{2}{\epsilon^2}\ln\frac{2|\Gamma|}{\delta}$ *independent model parameters* $\vartheta_{1,t}, \vartheta_{2,t}, \ldots, \vartheta_{M(\epsilon,\delta),t}$ *according to the probability distribution* $\mathcal{P}_t$;

3. *compute* $\hat{\mathrm{E}}_{\mathcal{P}_t,M(\epsilon,\delta)}[J(\vartheta,\gamma)] := \dfrac{1}{M(\epsilon,\delta)}\displaystyle\sum_{i=1}^{M(\epsilon,\delta)} J(\vartheta_{i,t},\gamma)$;

4. *choose* $\gamma_t := \arg\min_{\gamma\in\gamma} \hat{\mathrm{E}}_{\mathcal{P}_t,M(\epsilon,\delta)}[J(\vartheta,\gamma)]$.

We next prove that the controller parameter γ_t obtained through the stochastic Algorithm 2.2 is an approximate minimizer of $\mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]$ over Γ. In the following proposition, the phrase "with probability not less than $1-\delta$" makes reference to the probability involved in the random extraction of γ_t, once the past up to time t has been fixed.

Proposition 2.1. *The controller parameter* γ_t *computed via Algorithm 2.2 is an approximate minimizer of* $\mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]$ *to accuracy* ϵ *with confidence* $1-\delta$, *i.e.,* $\mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma_t)] \leq \inf_{\gamma\in\Gamma}\mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)] + \epsilon$, *with probability not less than* $1-\delta$.

Proof. γ_t is the minimizer of the sampling estimate $\hat{\mathrm{E}}_{\mathcal{P}_t,M(\epsilon,\delta)}[J(\vartheta,\gamma)]$, which is based on a random selection of parameters $\vartheta_i \in \Theta$ and, as such, it is a random variable over the space $\Theta^{M(\epsilon,\delta)} := \Theta \times \Theta \times \ldots \times \Theta$, $M(\epsilon,\delta)$ times. Consider the multi-samples $\theta \in \Theta^{M(\epsilon,\delta)}$ such that $\hat{\mathrm{E}}_{\mathcal{P}_t,M(\epsilon,\delta)}[J(\vartheta,\gamma)]$ is an uniformly good approximation to $\mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]$ over the set Γ to accuracy $\epsilon/2$, namely $\mathcal{Q} := \big\{\theta \in \Theta^{M(\epsilon,\delta)} : \sup_{\gamma\in\Gamma} |\hat{\mathrm{E}}_{\mathcal{P}_t,M(\epsilon,\delta)}[J(\vartheta,\gamma)] - \mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]| \leq \frac{\epsilon}{2}\big\}$. Then, $\forall\theta \in \mathcal{Q}$, letting $\gamma_t := \arg\min_{\gamma\in\Gamma} \hat{\mathrm{E}}_{\mathcal{P}_t,M(\epsilon,\delta)}[J(\vartheta,\gamma)]$, and $\gamma_t^\circ := \arg\min_{\gamma\in\Gamma} \mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]$, we have that

$$\begin{aligned} \mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma_t)] &\leq \hat{\mathrm{E}}_{\mathcal{P}_t,M(\epsilon,\delta)}[J(\vartheta,\gamma_t)] + \frac{\epsilon}{2} \leq \hat{\mathrm{E}}_{\mathcal{P}_t,M(\epsilon,\delta)}[J(\vartheta,\gamma_t^\circ)] + \frac{\epsilon}{2} \\ &\leq \mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma_t^\circ)] + \frac{\epsilon}{2} + \frac{\epsilon}{2} = \inf_{\gamma\in\Gamma} \mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)] + \epsilon. \end{aligned} \tag{2.3}$$

We next show that the probability that the multisample $\theta_t := \{\vartheta_{1,t},\ldots,\vartheta_{M(\epsilon,\delta),t}\}$ belongs to $\mathcal{Q}$ is greater than or equal to $1-\delta$. In view of (2.3), this concludes the proof.
By the definition of $\mathcal{Q}$ and the fact that the parameters $\vartheta_{1,t},\ldots,\vartheta_{M(\epsilon,\delta),t}$ are independently extracted according to $\mathcal{P}_t$, we have that

$$\begin{aligned} \Pr\{\theta_t \in \mathcal{Q}\} &= 1 - \mathcal{P}_t^{M(\epsilon,\delta)}\big\{\theta_t \in \Theta^{M(\epsilon,\delta)} : \sup_{\gamma\in\Gamma} |\hat{\mathrm{E}}_{\mathcal{P}_t,M(\epsilon,\delta)}[J(\vartheta,\gamma)] - \mathrm{E}_{\mathcal{P}_t}[J(\vartheta,\gamma)]| > \frac{\epsilon}{2}\big\} \\ &\geq 1 - 2|\Gamma|e^{-M(\epsilon,\delta)\epsilon^2/2}, \end{aligned}$$

where the last inequality follows from Hoeffding's inequality ([22]). Given that $M(\epsilon,\delta) \geq \frac{2}{\epsilon^2}\ln\frac{2|\Gamma|}{\delta}$, it is straightforward to verify that $\Pr\{\theta_t \in \mathcal{Q}\} \geq 1-\delta$. $\square$

2.2 Dwell-time selection

We adaptively select the dwell-time interval between consecutive switching times based on the model parameters extracted in step 1 of Algorithm 2.2. Specifically, we introduce the *dwell-time function* $\tau_D : 2^\Theta \times \gamma \to \mathbb{N}$, where 2^Θ is the power set of Θ, and compute the switching time sequence $\{t_i\}$ by the recursive equation

$$t_{i+1} = t_i + \tau_D(\theta_{t_i}, \gamma_{t_i}), \qquad i = 0,1,\ldots \tag{2.4}$$

initialized with $t_0 = 0$, where $\theta_{t_i} := \{\vartheta_{1,t_i}, \vartheta_{2,t_i}, \ldots, \vartheta_{M(\epsilon,\delta),t_i}\}$ is the set of $M(\epsilon,\delta)$ model parameters extracted at point 1 of Algorithm 2.2 at time $t = t_i$.
To define the dwell-time function, we first introduce the following notation: Consider the closed-loop system

$$\begin{cases} \mathcal{A}(\vartheta, z^{-1})\, y_{t+1} = \mathcal{B}(\vartheta, z^{-1})\, u_t + w_{t+1}, \\ \mathcal{C}(\gamma, z^{-1}) u_t = \mathcal{E}(\gamma, z^{-1}) y_t. \end{cases} \tag{2.5}$$

By letting $x_t := [y_t \ldots y_{t-(n-1)}\, u_{t-1} \ldots u_{t-(m-1)}]^T$ where $n := \max\{n_s, n_c+1\}$ and $m := \max\{m_s, m_c+1\}$, system (2.5) can be represented by

$$\begin{cases} x_{t+1} = A(\vartheta)\, x_t + B(\vartheta) u_t + C w_{t+1}, \\ u_t = L(\gamma) x_t, \end{cases}$$

where

$$A(\vartheta) = \left[\begin{array}{cccc|cccc} a_1 & \dots & a_{n-1} & a_n & b_2 & \dots & b_{m-1} & b_m \\ 1 & 0 & \dots & & 0 & \dots & & 0 \\ & \ddots & \ddots & & & \ddots & & 0 \\ & & 1 & 0 & & & & 0 \\ \hline 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & 1 & 0 & & \\ & \ddots & \ddots & & & \ddots & \ddots & \\ & & 0 & 0 & & & 1 & 0 \end{array}\right], \; B(\vartheta) = \left[\begin{array}{c} b_1 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \\ 0 \\ \vdots \\ 0 \end{array}\right], \; C = \left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{array}\right],$$

$$L(\gamma) = \left[\begin{array}{cccc|cccc} \eta_0 & \cdots & \eta_{n-2} & \eta_{n-1} & \chi_1 & \cdots & \chi_{m-2} & \chi_{m-1} \end{array}\right],$$

with $a_i = 0$ if $i > n_s$, $\eta_i = 0$ if $i > n_c$, $b_i = 0$ if $i > m_s$, $\chi_i = 0$ if $i > m_c$, thus leading to the state-space representation $x_{t+1} = F(\vartheta, \gamma)\, x_t + C w_{t+1}$, where $F(\vartheta, \gamma) = A(\vartheta) + B(\vartheta)L(\gamma)$. The condition that $\mathcal{S}(\vartheta, k)$ is λ-stable implies that matrix $F(\vartheta, \gamma)$ has all eigenvalues with absolute value smaller than or equal to λ.
Fix a *contraction constant* $\mu \in [0, 1)$. Then, τ_D is defined by

$$\tau_D(\bar{\Theta}, \gamma) := \min\{\tau \geq 1 : \sup_{\vartheta \in \bar{\Theta} : F(\vartheta,\gamma)\ \lambda\text{-stable}} \|F(\vartheta, \gamma)^\tau\| \leq \mu\}, \qquad \bar{\Theta} \subseteq \Theta, \gamma \in \Gamma, \tag{2.6}$$

with the understanding that, if the set $\{\vartheta \in \bar{\Theta} : F(\vartheta, \gamma)\ \lambda\text{-stable}\}$ is empty, then $\tau_D(\bar{\Theta}, \gamma) = 1$. Thus, by (2.4), the controller with parameter γ_{t_i} is kept in the loop until the possible overshoot of any system $\mathcal{S}(\vartheta, \gamma_{t_i})$, $\vartheta \in \theta_{t_i}$, stabilized by it has decayed of a factor μ.

3 Stability Analysis

In this section, we analyze the cautious switching control scheme:

$$\begin{cases} y_{t+1} = [1 - \mathcal{A}(\vartheta^\circ, z^{-1})]\, y_{t+1} + \mathcal{B}(\vartheta^\circ, z^{-1})\, u_t + w_{t+1} \\ u_t = \mathcal{E}(\sigma_t, z^{-1})\, y_t + [1 - \mathcal{C}(\sigma_t, z^{-1})]\, u_t, \end{cases} \tag{3.7}$$

where σ_t is the switching signal given by

$$\sigma_t := \begin{cases} \gamma_{t_i}, & \text{if } t = t_i \\ \sigma_{t-1}, & \text{otherwise.} \end{cases}$$

In particular, we shall prove that the closed-loop system is L^2-stable in the following sense:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} [u_t^2 + y_t^2] < \infty, \text{ a.s. (almost surely)}, \tag{3.8}$$

for all ϑ° satisfying Assumption 2.2.

Theorem 3.1. *The cautious switching control scheme is L^2-stable.*

Note that, differently from Proposition 2.1, the statement (3.8) is not of average type, instead it is guaranteed to hold true for all interior points of Θ. As a matter of fact, the interpretation of ϑ° as a stochastic random variable is just instrumental to computing $\mathcal{P}_t$. In this section, ϑ° is viewed simply as a deterministic parameter satisfying Assumption 2.2.
The following proposition is needed to prove Theorem 3.1.

Proposition 3.1. *The model parameters $\vartheta_{1,t}, \vartheta_{2,t}, \ldots, \vartheta_{M(\epsilon,\delta),t}$ independently extracted in step 1 of Algorithm 2.2 according to $\mathcal{P}_t \sim \mathcal{N}_\Theta(M_t, V_t)$ computed in Algorithm 2.1 satisfy*

$$(\vartheta_{j,t} - \vartheta^\circ)^T \sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T(\vartheta_{j,t} - \vartheta^\circ) = o\Big(\sum_{s=1}^{t} \|\varphi_{s-1}\|^2\Big), \qquad j = 1, \ldots, M(\epsilon, \delta), \ \textit{a.s.}$$

Proof. Fix a real constant $\beta > 0$ and define

$$v_t := \log^{1+\beta}\big(\sum_{s=1}^{t} \|\varphi_{s-1}\|^2\big). \tag{3.9}$$

Observe that from equation $y_t = \varphi_{t-1}\vartheta^\circ + w_t$ it follows that $w_k^2 \le 2\max\{\|\vartheta^\circ\|^2, 1\}[\, y_k^2 + \|\varphi_{k-1}\|^2\,]$. Taking into account the fact that the autoregressive part of model (2.1) is not trivial ($n > 0$), this in turn implies that $w_k^2 \le 2\max\{\|\vartheta^\circ\|^2, 1\}[\, \|\varphi_k\|^2 + \|\varphi_{k-1}\|^2\,]$, hence, $\sum_{k=1}^{t} \|\varphi_{k-1}\|^2 \ge h \sum_{k=1}^{t-1} w_k^2$, where h is a suitable constant. By Assumption 2.1, we then have

$$t = O(\sum_{k=1}^{t} \|\varphi_{k-1}\|^2), \qquad \text{a.s.} \tag{3.10}$$

Define the set $S_t := \Big\{\vartheta \in \Theta : (\vartheta - M_t)^T V_t^{-1}(\vartheta - M_t) > v_t\Big\} \subseteq \Theta$. We next prove that,

$$\lim_{t\to\infty} \mathcal{P}_t(S_t) = 0, \qquad \text{a.s.} \tag{3.11}$$

at an appropriate rate. This will be key to prove the proposition.
Fix a time instant $t \ge 0$. Since $\mathcal{P}_t \sim N_\Theta(M_t, V_t)$, $\mathcal{P}_t(S_t)$ can be upper bounded as follows

$$\mathcal{P}_t(S_t) \le \frac{\displaystyle\int_{\vartheta\in\mathbb{R}^{n_s+m_s}\setminus E_t} p_g(\vartheta; M_t, V_t)d\vartheta}{\displaystyle\int_{\vartheta\in\Theta} p_g(\vartheta; M_t, V_t)d\vartheta}, \tag{3.12}$$

where $p_g(\cdot; M_t, V_t)$ is the density function associated with the Gaussian distribution with mean M_t and variance V_t, and E_t denotes the ellipsoid in $\mathbb{R}^{n_s+m_s}$ defined by $E_t := \{\vartheta \in \mathbb{R}^{n_s+m_s} : (\vartheta - M_t)^T V_t^{-1}(\vartheta - M_t) \le v_t\}$. We next bound the numerator and denominator of

the right-hand-side in (3.12).
By changing the integration variables, the numerator can be expressed as follows:

$$\int_{\vartheta\in\mathbb{R}^{n_s+m_s}\setminus E_t} p_g(\vartheta; M_t, V_t)d\vartheta = \frac{1}{(2\pi)^{(n_s+m_s)/2}}\int_{\|v\|>v_t} \exp\Big(-\frac{1}{2}\|v\|^2\Big)dv$$
$$= \frac{c}{(2\pi)^{(n_s+m_s)/2}}\int_{r>v_t} r^{n_s+m_s-1}\exp\Big(-\frac{1}{2}r^2\Big)dr, \qquad (3.13)$$

where $cr^{n_s+m_s-1}$, with $c := (n_s+m_s)\frac{\pi^{(n_s+m_s)/2}}{((n_s+m_s)/2)!}$, is the surface of the (n_s+m_s)-sphere of radius r. Observe that by (3.9) and (3.10), there exists a.s. $t' \geq 0$ such that

$$\frac{c}{(2\pi)^{(n_s+m_s)/2}}r^{n_s+m_s-1}\exp\Big(-\frac{1}{4}r^2\Big) \leq \frac{c}{(2\pi)^{(n_s+m_s)/2}}v_t^{n_s+m_s-1}\exp\Big(-\frac{1}{4}v_t^2\Big) \leq 1, \quad r > v_t,\ t \geq t'.$$

Then, using this bound in (3.13), we have that $\int_{\vartheta\in\mathbb{R}^{n_s+m_s}\setminus E_t} p_g(\vartheta; M_t, V_t)d\vartheta \leq \int_{r>v_t}\exp(-\frac{1}{4}r^2)dr$, $t \geq t'$. Since $\int_{r>v_t}\exp(-\frac{1}{4}r^2)dr \leq \frac{1}{\sqrt{v_t}}\exp(-\frac{1}{4}v_t^2)$, we finally obtain

$$\int_{\vartheta\in\mathbb{R}^{n_s+m_s}\setminus E_t} p_g(\vartheta; M_t, V_t)d\vartheta \leq \frac{1}{\sqrt{v_t}}\exp\Big(-\frac{1}{4}v_t^2\Big), \qquad t \geq t'. \qquad (3.14)$$

As for the denominator in the right-hand-side of (3.12), it can be bounded as follows:

$$\int_{\vartheta\in\Theta} p_g(\vartheta; M_t, V_t)d\vartheta \geq \int_{\vartheta\in\Theta} p_g(\vartheta; \vartheta^\circ, V_t)\exp\Big(-\frac{1}{2}\big(\vartheta-\vartheta^\circ\big)^T V_t^{-1}\big(\vartheta-\vartheta^\circ\big)\Big)d\vartheta$$
$$\exp\Big(-\big(\vartheta^\circ - M_t\big)^T V_t^{-1}\big(\vartheta^\circ - M_t\big)\Big), \qquad (3.15)$$

since $(\vartheta - M_t)^T V_t^{-1}(\vartheta - M_t) \leq 2(\vartheta-\vartheta^\circ)^T V_t^{-1}(\vartheta-\vartheta^\circ) + 2(\vartheta^\circ - M_t)^T V_t^{-1}(\vartheta^\circ - M_t)$.
By [23, Theorem 4.1] and [24], $\big(\vartheta^\circ - M_t\big)^T V_t^{-1}\big(\vartheta^\circ - M_t\big) = O(\log(\sum_{s=1}^t \|\varphi_{s-1}\|^2))$, a.s. This jointly with (3.9) and (3.10) implies that there exists a.s. a time instant $\tilde{t} \geq 0$ such that

$$\big(\vartheta^\circ - M_t\big)^T V_t^{-1}\big(\vartheta^\circ - M_t\big) \leq v_t, \qquad t \geq \tilde{t}. \qquad (3.16)$$

We next bound the term $\int_{\vartheta\in\Theta} p_g(\vartheta; \vartheta^\circ, V_t)\exp(-\frac{1}{2}\big(\vartheta-\vartheta^\circ\big)^T V_t^{-1}\big(\vartheta-\vartheta^\circ\big))d\vartheta$ in (3.15).
By some algebraic manipulations, it can be easily seen that V_t given by the Kalman filter equations satisfies the following equation

$$V_t^{-1} = \frac{1}{\sigma^2}\sum_{s=1}^{t}\varphi_{s-1}\varphi_{s-1}^T + V^{-1}, \qquad (3.17)$$

and, hence, $V_{t+1}^{-1} \geq V_t^{-1}$, $\forall t$. From Assumption 2.2 and this property of V_t, it follows that there exists a $\Delta > 0$ such that $D_t := \{\vartheta \in \mathbb{R}^{n_s+m_s} : (\vartheta-\vartheta^\circ)^T V_t^{-1}(\vartheta-\vartheta^\circ) \leq \Delta\} \subseteq \Theta$, $t \geq 0$. Indeed, since ϑ° is an interior point of the compact set Θ, there exists $\Delta > 0$ such that $D_0 \subseteq \Theta$. Then, $\mathcal{D}_t \subseteq \Theta$, $\forall t \geq 0$, follows from the monotonicity property $\mathcal{D}_{t+1} \subseteq \mathcal{D}_t$,

$\forall t \geq 0$, which is a consequence of the fact that $V_{t+1}^{-1} \geq V_t^{-1}$, $\forall t$. We then have that

$$\begin{aligned}\int_{\vartheta\in\Theta} \exp\Big(-\frac{1}{2}(\vartheta-\vartheta^\circ)^T V_t^{-1}(\vartheta-\vartheta^\circ)\Big)p_g(\vartheta;\vartheta^\circ,V_t)d\vartheta \\ &\geq \int_{\vartheta\in D_t} \exp\Big(-\frac{1}{2}(\vartheta-\vartheta^\circ)^T V_t^{-1}(\vartheta-\vartheta^\circ)\Big)p_g(\vartheta;\vartheta^\circ,V_t)d\vartheta \\ &\geq \exp(-\Delta)\int_{\vartheta\in D_t} p_g(\vartheta;\vartheta^\circ,V_t)d\vartheta = \exp(-\Delta)\int_{\|v\|\leq\Delta} p_g(v;0,I)dv.\end{aligned}$$

By introducing the map $\epsilon : [0,\infty) \to [0,1)$ defined by $\Delta \mapsto \int_{\|v\|\leq\Delta} p_g(v;0,I)dv$, we finally obtain that there exists a.s. $\tilde{t} \geq 0$ such that

$$\int_{\vartheta\in\Theta} \exp\Big(-\frac{1}{2}(\vartheta-\vartheta^\circ)^T V_t^{-1}(\vartheta-\vartheta^\circ)\Big)p_g(\vartheta;\vartheta^\circ,V_t)d\vartheta \geq \epsilon(\Delta)\exp(-\Delta), \qquad t \geq \tilde{t}.$$

Using this inequality and (3.16) in (3.15), we conclude that

$$\int_{\vartheta\in\Theta} p_g(\vartheta;M_t,V_t)d\vartheta \geq \epsilon(\Delta)\exp(-\Delta)\exp(-v_t), \qquad t \geq \tilde{t}. \tag{3.18}$$

Consider now equation (3.12). Using the bounds (3.14) and (3.18) in (3.12), we get

$$\mathcal{P}_t(S_t) \leq \frac{1}{\sqrt{v_t}}\exp\big(-\frac{1}{4}v_t^2 + v_t\big)\epsilon(\Delta)^{-1}\exp(\Delta), \qquad t \geq \tau := \max\{t', \tilde{t}\}. \tag{3.19}$$

Since by (3.10) $v_t \to \infty$, we conclude that (3.11) is satisfied.
We next show that with probability 1 there exists $\bar{t} \geq 0$ such that, for any $t \geq \bar{t}$, the $M(\epsilon,\delta)$ model parameters $\vartheta_{1,t},\ldots,\vartheta_{M(\epsilon,\delta),t}$ independently extracted at step 1 of Algorithm 2.2 according to $\mathcal{P}_t$ belong to set $\Theta \setminus S_t$. This concludes the proof because it entails that $(\vartheta_{j,t}-M_t)^T\sum_{s=1}^t \varphi_{s-1}\varphi_{s-1}^T(\vartheta_{j,t}-M_t) = o(\sum_{s=1}^t \|\varphi_{s-1}\|^2)$, $j=1,\ldots,M(\epsilon,\delta)$, where we used equation (3.17) and the fact that by (3.10) $v_t = o(\sum_{s=1}^t \|\varphi_{s-1}\|^2)$.
Consider the event when at least one of the parameters extracted at time t belongs to S_t, i.e., $A_t := \{\vartheta_{1,t} \in S_t \text{ or } \ldots \text{ or } \vartheta_{M(\epsilon,\delta)} \in S_t\}$. Since they are independently extracted from the same distribution $\mathcal{P}_t$, then, $\Pr\{A_t\} \leq M(\epsilon,\delta)\Pr\{\vartheta_{1,t} \in S_t\} = M(\epsilon,\delta)\mathcal{P}_t(S_t)$, and by (3.19), $\sum_{t=0}^\infty \Pr\{A_t\} \leq M(\epsilon,\delta)\sum_{t=0}^{\tau-1}\mathcal{P}_t(S_t) + M(\epsilon,\delta)\epsilon(\Delta)^{-1}\exp(\Delta)\sum_{t=\tau}^\infty \frac{1}{\sqrt{v_t}}\exp\big(-\frac{1}{4}v_t^2+v_t\big)$. We next show that

$$\frac{1}{\sqrt{v_t}}\exp\big(-\frac{1}{4}v_t^2+v_t\big) = o\big(\frac{1}{t^2}\big), \qquad \text{a.s.}, \tag{3.20}$$

which implies that $\sum_{t=0}^\infty \Pr\{A_t\} < \infty$, with probability 1. By Borel-Cantelly Lemma ([23]), we then have (i.o.=infinitely often) $\Pr\{A_t \ \ i.o.\} = 0$, thus, with probability 1 there exists $\bar{t} > 0$ such that all model parameters $\vartheta_{1,t},\ldots,\vartheta_{M(\epsilon,\delta),t}$ belong to the set $\Theta \setminus S_t$, $\forall t \geq \bar{t}$.
The following chain of equalities can be easily derived by (3.9) and (3.10)

$$\begin{aligned}t^2\frac{1}{\sqrt{v_t}}\exp\big(-\frac{1}{4}v_t^2+v_t\big) &= o\Big(t^2\exp\big(-\frac{1}{5}v_t^2\big)\Big) = o\Big(t^2\exp\big(-\frac{1}{5}\log^{2(1+\beta)}(t)\big)\Big) \\ &= o\Big(t^2\big(\exp(-\log t)\big)^{\frac{1}{5}\log^{(1+2\beta)}(t)}\Big) = o\Big(t^{-\frac{1}{5}\log^{(1+2\beta)}(t)+2}\Big) = o(1),\end{aligned}$$

which proves that equation (3.20) is satisfied. □

Before proving Theorem 3.1, we outline the structure of its proof.
We start by showing that for each switching time t_i there exists a parameter value $\bar{\vartheta}_{t_i}$ belonging to the set $\theta_{t_i} := \{\vartheta_{1,t}, \ldots, \vartheta_{M(\epsilon,\delta),t}\}$ such that $\mathcal{S}(\bar{\vartheta}_{t_i}, \gamma_{t_i})$ is λ-stable. We then define the sequence of parameters $\{\vartheta_t\}_{t\geq 0}$ by

$$\vartheta_t = \begin{cases} \bar{\vartheta}_{t_i}, & \text{if } t = t_i, \\ \vartheta_{t-1}, & \text{otherwise,} \end{cases} \tag{3.21}$$

and represent the closed-loop system (3.7) as a variational system with respect to the time varying closed-loop system $\mathcal{S}(\vartheta_t, \sigma_t)$ as follows

$$\begin{cases} y_{t+1} = [1 - \mathcal{A}(\vartheta_t, z^{-1})]\, y_{t+1} + \mathcal{B}(\vartheta_t, z^{-1})\, u_t + e_t + w_{t+1} \\ u_t = \mathcal{E}(\sigma_t, z^{-1})\, y_t + [1 - \mathcal{C}(\sigma_t, z^{-1})]\, u_t, \end{cases} \tag{3.22}$$

where $e_t := \varphi_t^T(\vartheta^\circ - \vartheta_t)$ is regarded as an exogenous input.
This representation has two nice properties:
1) the closed-loop system $\mathcal{S}(\vartheta_t, \sigma_t)$ is time invariant over each switching time interval $[t_i, t_{i+1})$, $i \geq 0$, and has a λ-stable dynamic matrix $F(\bar{\vartheta}_{t_i}, \gamma_{t_i})$;
2) by Proposition 3.1, $\vartheta_t - \vartheta^\circ$ appearing in e_t satisfies $(\vartheta_{t_i} - \vartheta^\circ)^T \sum_{s=1}^{t_i} \varphi_{s-1}\varphi_{s-1}^T(\vartheta_{t_i} - \vartheta^\circ) = o(\sum_{s=1}^{t_i} \|\varphi_{s-1}\|^2)$.

On the basis of 1), we are then able to show that i) the dwell-time interval sequence $\{t_{i+1} - t_i\}_{i\geq 0}$ is bounded, and ii) the time-varying system (3.22) where e_t is viewed as an exogenous input is exponentially stable, uniformly in time. Then, by 2) jointly with i), we conclude that the perturbation term feeding system (3.22) is L^2-bounded ([15]). These properties will finally lead to equation (3.8).

Proof of Theorem 3.1. Consider any switching time instant t_i and suppose by contradiction that there is no parameter $\vartheta \in \theta_{t_i}$ such that $\mathcal{S}(\vartheta, \gamma_{t_i})$ is λ-stable. Then, by (2.2), $\hat{\mathrm{E}}_{\mathcal{P}_{t_i}, M(\epsilon,\delta)}[J(\vartheta, \gamma_{t_i})] = 1$. Pick up any $\bar{\vartheta} \in \theta_{t_i}$. Since $\bar{\vartheta} \in \Theta$, by Assumption 2.3 there exists $\bar{\gamma} \in \Gamma$ such that $J(\bar{\vartheta}, \bar{\gamma}) \leq \bar{J} < 1$. Then, $\hat{\mathrm{E}}_{\mathcal{P}_{t_i}, M(\epsilon,\delta)}[J(\vartheta, \bar{\gamma})] \leq \frac{M(\epsilon,\delta)-1}{M(\epsilon,\delta)} + \bar{J} < 1 = \hat{\mathrm{E}}_{\mathcal{P}_{t_i}, M(\epsilon,\delta)}[J(\vartheta, \gamma_{t_i})]$, which contradicts the controller selection policy in Algorithms 2.2. This proves that, for every switching time t_i, there exists a parameter value $\bar{\vartheta}_{t_i} \in \theta_{t_i}$ such that $\mathcal{S}(\bar{\vartheta}_{t_i}, \gamma_{t_i})$ is λ-stable. Since the set $\{\vartheta \in \theta_{t_i} : F(\vartheta, \gamma_{t_i}) \text{ is } \lambda\text{-stable}\}$ is not empty, because of (2.4) and (2.6), we conclude that

$$t_{i+1} - t_i = \min\{\tau \geq 1 : \sup_{\vartheta \in \theta_{t_i} : F(\vartheta, \gamma_{t_i}) \text{ is } \lambda\text{-stable}} \|F(\vartheta, \gamma_{t_i})^\tau\| \leq \mu\}. \tag{3.23}$$

To each $\gamma \in \Gamma$, we can associate the set of parameters $\vartheta \in \Theta$ such that $\mathcal{S}(\vartheta, \gamma)$ is λ-stable, i.e., $\Theta_\gamma := \{\vartheta \in \Theta : |\lambda_{\max}(F(\vartheta, \gamma))| \leq \lambda\}$, where $\lambda_{\max}(F(\vartheta, \gamma))$ is the maximum eigenvalue of $F(\vartheta, \gamma)$. Since $\lambda_{\max}(F(\cdot, \gamma))$, $\gamma \in \Gamma$, is a continuous function of ϑ and Θ is compact, it then follows that Θ_γ is a compact set.

Consider now a parameter $\gamma \in \Gamma$ and fix $\nu \in (\lambda, 1)$. The matrix $\frac{1}{\nu} F(\vartheta, \gamma)$ is exponentially stable $\forall\, \vartheta \in \Theta_\gamma$, hence, the solution $P_\gamma(\vartheta)$ to the Lyapunov equation $\frac{1}{\nu} F(\vartheta, \gamma)^T P \frac{1}{\nu} F(\vartheta, \gamma) - P = -I$ is positive definite. This implies that $x^T \frac{1}{\nu} F(\vartheta, \gamma)^T P_\gamma(\vartheta) \frac{1}{\nu} F(\vartheta, \gamma) x \leq x^T P_\gamma(\vartheta) x$, $\forall x \in \mathbb{R}^{n+m-1}$. By applying τ times this equation, we get $x^T (\frac{1}{\nu^\tau} F(\vartheta, \gamma)^\tau)^T P_\gamma(\vartheta) \frac{1}{\nu^\tau} F(\vartheta, \gamma)^\tau x \leq x^T P_\gamma(\vartheta) x$, $\forall x \in \mathbb{R}^{n+m-1}$, which leads to

$$\|F(\vartheta, \gamma)^\tau x\| \leq c_\gamma \nu^\tau \|x\|, \qquad \forall x \in \mathbb{R}^{n+m-1}, \vartheta \in \Theta_\gamma, \tag{3.24}$$

where $c_\gamma := \max_{\vartheta \in \Theta_\gamma} \sqrt{\lambda_{max}(P_\gamma(\vartheta))/\lambda_{min}(P_\gamma(\vartheta))}$. Note that $c_\gamma < \infty$ because $P_\gamma(\vartheta)$ is continuous on the compact set Θ_γ ([25]).

Define $T_\gamma := \inf\{\tau \in \mathbb{N} : c_\gamma \nu^\tau \leq \mu\}$. Then, $\|F(\vartheta, \gamma)^{T_\gamma}\| = \sup_{\|x\| \neq 0} \frac{\|F(\vartheta, \gamma)^{T_\gamma} x\|}{\|x\|} \leq \mu$, $\forall\, \vartheta \in \Theta_\gamma$, $\forall \gamma \in \Gamma$. Therefore, $\{t_{i+1} - t_i\}_{i=0}$ is uniformly bounded by $\bar{T} := \max_{\gamma \in \gamma} T_\gamma$, since by (3.23) $t_{i+1} - t_i \leq \min\{\tau \geq 1 : \sup_{\vartheta \in \Theta_{\gamma_{t_i}}} \|F(\vartheta, \gamma_{t_i})^\tau\| \leq \mu\} \leq \bar{T}$.

Consider the state space representation $x_{t+1} = F(\vartheta_t, \sigma_t) x_t + C(e_t + w_{t+1})$ of the time-varying system (3.22) with e_t viewed as an exogenous input and ϑ_t given by (3.21). In each interval $[t_i, t_{i+1})$ this system is time invariant and by (3.24) its dynamic matrix $F(\vartheta_{t_i}, \gamma_{t_i})$ satisfies $\|F(\vartheta_{t_i}, \gamma_{t_i})^\tau\| = \sup_{\|x\| \neq 0} \frac{\|F(\vartheta_{t_i}, \gamma_{t_i})^\tau x\|}{\|x\|} \leq c \nu^\tau$ with $c := \max_{\gamma \in \gamma} c_\gamma$. Also, by (3.23) $\|F(\vartheta_{t_i}, \gamma_{t_i})^{t_{i+1} - t_i}\| \leq \mu$.

If t', t do not belong to the same dwell-time interval, say $0 \leq t' < t_i < \cdots < t_{j+1} < t$, then,

$$\begin{aligned}
\|x_t\| &= \|F(\vartheta_{t_{j+1}}, \gamma_{t_{j+1}})^{t - t_{j+1}} F(\vartheta_{t_j}, \gamma_{t_j})^{t_{j+1} - t_j} \ldots F(\vartheta_{t_i}, \gamma_{t_i})^{t_{i+1} - t_i} F(\vartheta_{t_{i-1}}, \gamma_{t_{i-1}})^{t_i - t'} x_{t'}\| \\
&\leq \|F(\vartheta_{t_{j+1}}, \gamma_{t_{j+1}})^{t - t_{j+1}}\| \|F(\vartheta_{t_j}, \gamma_{t_j})^{t_{j+1} - t_j}\| \ldots \|F(\vartheta_{t_i}, \gamma_{t_i})^{t_{i+1} - t_i}\| \|F(\vartheta_{t_{i-1}}, \gamma_{t_{i-1}})^{t_i - t'}\| \|x_{t'}\| \\
&\leq c \nu^{t - t_{j+1}} \mu^{j+1-i} c \nu^{t_i - t'} \|x_{t'}\| \leq c^2 \bar{\nu}^{t - t'} \|x_{t'}\|,
\end{aligned}$$

where $\bar{\nu} := \max\{\nu, \mu^{1/\bar{T}}\}$. If t', t belong to the same dwell-time interval, then, $\|x_t\| \leq c \nu^{t - t'} \|x_{t'}\| \leq c^2 \bar{\nu}^{t - t'} \|x_{t'}\|$, which proves the uniform exponential stability of $\mathcal{S}(\vartheta_t, \sigma_t)$.

By the boundedness of $\{t_{i+1} - t_i\}_{i=0}$ and the property that $(\vartheta_{t_i} - \vartheta^\circ)^T \sum_{s=1}^{t_i} \varphi_{s-1} \varphi_{s-1}^T (\vartheta_{t_i} - \vartheta^\circ) = o(\sum_{s=1}^{t_i} \|\varphi_{s-1}\|^2)$, a.s. (which follows from (3.21) and Proposition 3.1), we have that

$$\sum_{t=0,\ t \notin \mathcal{Q}_N}^{N-1} e_t^2 = o(\sum_{t=0}^{N-1} \|\varphi_t\|^2 + N), \qquad \text{a.s.}, \tag{3.25}$$

where $\mathcal{Q}_N$ is a set of instant points which depends on N, whose cardinality is uniformly bounded: $|\mathcal{Q}_N| \leq k, \forall N$ ([15, Proposition 3.3]).

To prove the theorem, it is convenient to adopt the following representation for (3.22). Fix $N > 0$. For all $t \in [0, N)$ system (3.22) can be represented as

$$x_{t+1} = \begin{cases} F^\circ(\vartheta_t)\, x_t + C w_{t+1}, & t \in \mathcal{Q}_N, \\ F(\vartheta_t)\, x_t + C[e_t + w_{t+1}], & t \notin \mathcal{Q}_N, \end{cases} \tag{3.26}$$

where $F^\circ(\vartheta, \gamma) = A(\vartheta^\circ) + B(\vartheta^\circ) L(\gamma)$, with $A(\vartheta^\circ)$, $B(\vartheta^\circ)$, $L(\gamma)$ defined in Subsection 2.2. Since $F^\circ(\vartheta)$ is a continuous function of ϑ in the compact set Θ, then, $\|F^\circ(\hat{\vartheta}_t)\|$ is uniformly bounded. From this fact and the uniform exponential stability of $x_{t+1} = F(\vartheta_t, \sigma_t) x_t$, it is straightforward to show that the x_t generated by (3.26) can be bounded as follows: $\|x_t\| \leq k_1\{\sum_{i=1}^{t} \nu^{t-i}|w_i| + \sum_{i=0, i \notin \mathcal{Q}_N}^{t-1} \nu^{t-i}|e_i|\}$, $t \leq N$, where k_1 and $\nu \in (0,1)$ are suitable constants, from which we get $\frac{1}{N}\sum_{t=1}^{N} \|x_t\|^2 \leq k_2\{\frac{1}{N}\sum_{t=1}^{N} w_t^2 + \frac{1}{N}\sum_{t=0, t \notin \mathcal{Q}_N}^{N-1} e_t^2\}$, where k_2 is a suitable constant, independent of N. By Assumption 2.1 and equation (3.25), $\frac{1}{N}\sum_{t=1}^{N} \|x_t\|^2 = O(1) + o(\frac{1}{N}\sum_{t=0}^{N-1} \|\varphi_t\|^2)$ a.s. Since $\frac{1}{N}\sum_{t=0}^{N-1} \|\varphi_t\|^2 \leq \frac{1}{N}\sum_{t=0}^{N} \|x_t\|^2$, this implies that $\frac{1}{N}\sum_{t=0}^{N-1} \|\varphi_t\|^2$ remains bounded, thus concluding the proof. □

4 Conclusions

In this paper, we proposed to combine cautious randomized control and switching control so as to overcome existing difficulties of both methods, while preserving their positive features. We consider the class of systems described by a linear input-output model affected by Gaussian noise. For these systems, we show that it is possible to conceive a randomized cautious switching control scheme that is robust in finite time and asymptotically stable. Based on this stability result, it is possible to obtain also self-tuning properties. For example, if a dither noise is added to the control input ([23, 12]), then the switching signal converges to the controller parameter that is optimal for the actual system.

Still, important issues remain open. This includes taking into consideration the presence of unmodeled dynamics when updating $\mathcal{P}_t$, and studying an easy-to-implement procedure for the candidate controllers design. These problems represent a stimulus for future research.

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