Near Optimal LQR Performance for Uncertain First Order Systems *

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Abstract

In adaptive control, the objective is to provide stability and acceptable performance in the face of significant plant uncertainty. However, often there are large transients in the plant output and the control signal can become excessively large. Here we consider the first order case with the plant parameters restricted to a compact set; we show how to design a (linear time-varying) adaptive controller which provides near optimal LQR performance. This controller is periodic with each period split into two parts: during the Estimation Phase, an estimate of the optimal control signal is formed; during the Control Phase, a suitably scaled estimate of this signal is applied to the system. We demonstrate the technique with a simulation and discuss the benefits and limitations of the approach.

Keywords: Simultaneous stabilization, optimal control, time-varying control.

1 Introduction

In many situations we have an accurate nominal plant model, but at some point a structural change occurs, such as an actuator fault, which yields a new model. In this situation it may be possible to know in advance a list of common faults, and thereby a finite set of possible models. This leads to the simultaneous stabilization problem formulated by Saeks and Murray [12] and Vidyasagar and Viswanadham [13]: given a family of plants, find a single controller which stabilizes each of these plants.

After the formulation of this problem, a lot of research effort has been devoted to the development of existence conditions for simultaneous stabilization, e.g. see [13], [4], [12], [2]. In [13] and [8], necessary and sufficient conditions are provided for the simultaneous stabilization of two systems

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using an LTI controller. However, there are simple situations where no LTI controller exists: no LTI controller will simultaneously stabilize $\frac{1}{s-1}$ and $\frac{-1}{s-1}$. The stabilization of more than two LTI systems by LTI feedback laws remains a challenging issue.

Recent work on linear time-varying control has demonstrated that using a time-varying controller can overcome the limitations of LTI controllers. The use of time-varying feedback laws for simultaneous stabilization has been investigated: in [6] the idea is to first design a LTI deadbeat controller for each P_i, and then use these to construct a single linear time-varying controller which provides a dead-beat response for every P_i; the transient behavior would typically be quite poor. In [5] the simultaneous stabilization of a finite family of LTI systems is established with a generalized hold(GH). However, it is not clear how well it performs, since GH based controllers often have poor inter-sample behavior [3]. More recently, in [1] the simultaneous stabilization problem is solved using a multi-rate sampled-data controller, but it is still based on a generalized hold, so its inter-sample behavior is also expected to be poor. However, [10] has overcome this problem: for each possible model they choose a standard LQR-type cost function, and then show how to design a linear time-varying controller which not only simultaneously stabilizes each model but also provides a level of performance as close as we'd like to the optimal one. The approach adopted provides good performance for a finite set of plants.

In this paper our goal is to extend the LQR result of [10] to a set with a continuum of elements. We choose a first order plant $\frac{b}{s-a}$, and solve the problem of simultaneous stabilization for all possible models in a compact set. In fact, we show how to design a linear time-varying controller which not only simultaneously stabilizes every model but also provides near optimal LQR performance.

Here we propose an approach based on that of [10] and [9], which yields a linear periodic controller. We divide each period into two phases: the Estimation Phase and the Control Phase. Rather than estimating the plant parameters, instead in the Estimation Phase we estimate what the control signal would be if the plant parameters were known; in the Control Phase, we apply a suitably weighted estimate of the optimal control signal. We are able to do so in a linear fashion. Our approach requires fast actuators, and the nearer we wish to get to optimality, the smaller the sampling period needs to be. Although the controller gains are large, the control signal is modest. At this point, we observe that an early version of this work appeared in an MASc thesis of the first author [7], and that most proofs are omitted for space considerations.

We use standard notation throughout this paper. Let $\mathbf{Z}$ denote the set of integers, $\mathbf{R}$ denote the set of real numbers, $\mathbf{N}$ denote the set of positive integers.

We use the Holder 2-norm for vectors and the corresponding induced norm for matrices, and denote the norm of a vector or matrix by $\|\cdot\|$. We let $PC(\mathbf{R}^{n\times m})$ denote the set of piecewise continuous functions from $\mathbf{R}^+$ to $\mathbf{R}^{n\times m}$. We measure the size of a piecewise continuous signal $y \in PC(\mathbf{R}^{n\times m})$ by

$$\|y\|_2 := [\int_0^\infty \|y(\tau)\|^2 \, d\tau]^{1/2}.$$

We say that $f : \mathbf{R}^+ \to \mathbf{R}^{n\times m}$ is of order T^j, and write $f = \mathcal{O}(T^j)$, if there exist constants $c_1 > 0$ and $T_1 > 0$ so that

$$\|f(T)\| \leq c_1 T^j, \ T \in (0, T_1).$$

On occasion we have a function f which depends not only on $T \geq 0$ but also on a pair (a, b) restricted to a compact set $\Gamma \subset \mathbf{R}^2$; we say that $f = \mathcal{O}(T^j)$ if there exist constants $c_1 > 0$ and

$T_1 > 0$ so that

$$\|f(T)\| \le c_1 T^j,\ T \in (0, T_1),\ (a,b) \in \Gamma.$$

2 Problem Formulation

We consider the single-input single-output plant P:

$$\dot{y}(t) = ay(t) + bu(t),\ y(0) = y_0, \tag{2.1}$$

with $y(t) \in \mathbf{R}$ as the plant state and $u(t) \in \mathbf{R}$ as the plant input. The associated transfer function from u to y is given by

$$P(s) := \frac{b}{s-a}.$$

Here we don't know the plant parameters exactly; instead we assume that (a, b) is controllable (so that $b \neq 0$) and that they lie in a compact set Γ:

$$\Gamma \subset \{(a,b) \in \mathbf{R}^2 : b \neq 0\}.$$

The control objective can be stated as the following: Construct a linear time-varying controller which not only provides closed loop stability but also provides near optimal performance for every $(a, b) \in \Gamma$. Here we define performance in an LQR sense. Given $(a, b) \in \Gamma$, the LQR problem can be stated as follows: with $q > 0$ and $r > 0$ fixed weights, find, for every y_0, the control signal u which makes $y(t) \to 0$ as $t \to \infty$ and minimizes the cost

$$J(a, b, y_0) = \int_0^\infty [qy(t)^2 + ru(t)^2]\, dt. \tag{2.2}$$

Without loss of generality, **henceforth assume that** $q = 1$.

As is well known, the optimal controller is state-feedback:

$$u^0(t) = fy(t). \tag{2.3}$$

The gain f can be computed by first solving the Algebraic Riccati Equation(ARE)

$$2aP - \frac{1}{r}b^2P^2 + 1 = 0$$

for the unique positive root P:

$$P(a,b) = \frac{ar}{b^2} + \frac{1}{b^2}\sqrt{a^2r^2 + b^2r},$$

and then set the feedback gain to be

$$f(a,b) = -\frac{bP(a,b)}{r} = -\frac{a}{b} - \frac{1}{rb}\sqrt{a^2r^2 + b^2r}. \tag{2.4}$$

This gives rise to an optimal cost of the form

$$J^0(a, b, y_0) = P(a,b)y_0^2. \tag{2.5}$$

Here, we consider sampled-data controllers of the form

$$\begin{aligned} z(k+1) &= F(k)z(k) + G(k)y(kh),\ z(0) = z_0 \in \mathbf{R}^l, \\ u(kh+\tau) &= J(k)z(k),\ \tau \in [0,h), \end{aligned} \tag{2.6}$$

with the controller gains F, G, J periodic of period $p \in \mathbf{N}$ and $h > 0$; the period of the controller is $T := ph$, and we associate this system with the 5-tuple (F, G, J, T, p). In fact, controller (2.6) can be implemented with a sampler, a zero-order-hold and an l^{th} order periodically time-varying discrete-time system of period p.

In this paper, the notion of closed loop stability is the natural one:

Definition 2.1. *The sampled-data controller (2.6) stabilizes (2.1) if, for every $y_0 \in \mathbf{R}$ and $z_0 \in R^l$, we have $\lim_{t\to\infty} y(t) = 0$ and $\lim_{k\to\infty} z(k) = 0$.*

Assume that for every $(a,b) \in \Gamma$, the corresponding cost is independent of z_0. (This may seem odd at first glance but our controller has this property.) Suppose that the sampled-data controller (2.6) stabilizes every model in Γ and we label the cost function corresponding to $(a,b) \in \Gamma$ as $J(a,b,y_0)$. The main objective of this paper is to design a controller of the form (2.6), parameterized by $\delta > 0$, so that we have

$$J(a,b,y_0) - J^0(a,b,y_0) \leq \delta \parallel y_0 \parallel^2,\ (a,b) \in \Gamma.$$

At this point we provide a high level motivation of our approach. We divide the period $[kT,(k+1)T)$ into two phases: the Estimation Phase and the Control Phase. In the Estimation Phase, we estimate $f(a,b)y(kT)$ by applying a sequence of test signals, which are constructed on the fly. In the Control Phase we apply a suitably weighted estimate of $f(a,b)y(kT)$ (see Figure 1).

Hence if control signal is modest in size during the Estimation Phase, and if the Estimation Phase is short in duration, then

$$\begin{aligned} y[(k+1)T] &= e^{aT}y(kT) + \int_{kT}^{(k+1)T} e^{a[(k+1)T-\tau]}bu(\tau)\,d\tau \\ &\approx e^{aT}y(kT) + bTf(a,b)y(kT) \\ &\approx e^{[a+bf(a,b)]T}y(kT), \end{aligned}$$

which is what happens if the optimal controller were applied. Of course, the above is very hand wavy and it is not clear how to carry out the Estimation Phase. Hence, in the next section we will devise a linear time-varying candidate controller and then prove that it achieves our objective.

3 The Approach

Here our goal is not only to provide the closed loop stability for every model in Γ, but also to achieve near optimal LQR performance. Our optimal control law is as given in (2.3):

$$\begin{aligned} u^0(t) &= f(a,b)y(t) \\ &= [-\frac{a}{b} - \frac{1}{rb}\sqrt{a^2r^2 + b^2r}]y(t). \end{aligned} \tag{3.7}$$

We note that in the gain of the control signal, we don't know a and b, so some form of on-line estimation for $u^0(t)$ is required. Here we make precise exactly how to carry out the estimation discussed in the last chapter. The first step is to approximate our feedback gain by a polynomial.

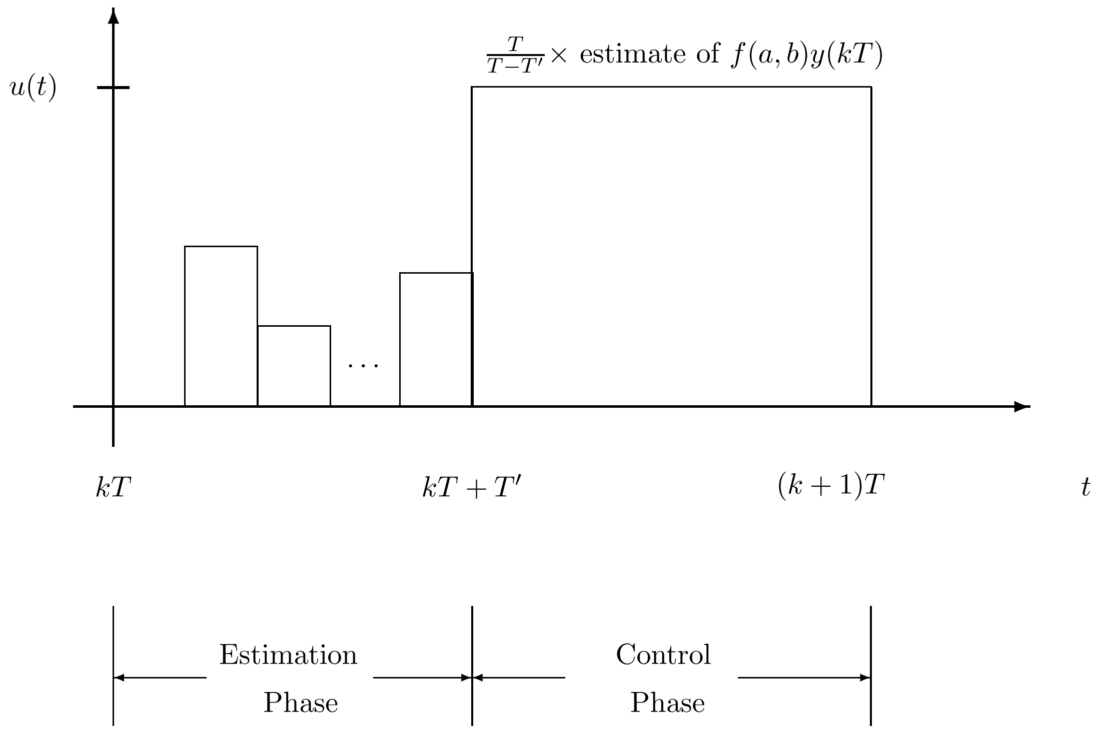

Figure 1: Estimation Phase and Control Phase

3.1 Approximation of $f(a,b)$ by a Polynomial

We see from (3.7) that the optimal feedback gain $f(a,b)$ is a highly nonlinear function of a and b. It turns out that it will be easy to estimate

$$a^j b^i y(t),$$

so it would be much more convenient if it were a polynomial in the a and b parameters. Fortunately, the set of uncertainty Γ is compact, so by the Stone-Weirstrass Approximation Theorem [11], for every $\varepsilon > 0$ there exists a polynomial $\hat{f}_\varepsilon(a,b)$ so that

$$\mid f(a,b) - \hat{f}_\varepsilon(a,b) \mid \leq \varepsilon, \;\; (a,b) \in \Gamma. \tag{3.8}$$

It turns out that most proofs of this result are non-constructive, while others are difficult to implement in practise. For special cases we can obtain a nice expressions, e.g. see Section 5.

At this point, with $\varepsilon > 0$ we assume that we have approximated f by a polynomial of the form

$$\hat{f}_\varepsilon(a,b) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \ell_{j,i} a^j b^i, \;\; (a,b) \in \Gamma, \tag{3.9}$$

satisfying

$$|\hat{f}_\varepsilon(a,b) - f(a,b)| < \varepsilon;$$

here the $\ell'_{j,i}s$ are constants. We assume that

$$a + b\hat{f}_\varepsilon(a,b) < 0, \;\; (a,b) \in \Gamma;$$

this will clearly be the case for small ε. Indeed, by compactness of Γ and continuity of $\hat{f}_\varepsilon(a,b)$, we can choose $\bar{\varepsilon} > 0$ and $\bar{\lambda} < 0$ so that for all $\varepsilon \in (0, \bar{\varepsilon})$, we have

$$a_{cl}^{\varepsilon}(a,b) := a + b\hat{f}_\varepsilon(a,b) \le \bar{\lambda}, \ (a,b) \in \Gamma.$$

At this point, we would like to verify that $u = \hat{f}_\varepsilon(a,b)y$ is near optimal as $\varepsilon \to 0$. To this end, let u^0 denote the optimal control signal and y^0 denote the corresponding response; recall that we have already labelled $J^0(a,b,y_0)$ the optimal cost. We label the control signal corresponding to the feedback $\hat{f}_\varepsilon(a,b)$ by $\bar{u}$, with the corresponding output labelled $\bar{y}$ and the corresponding cost labelled $\bar{J}(a,b,y_0)$, so that

$$\bar{J}(a,b,y_0) := \int_0^\infty \{[\bar{y}(t)]^2 + r[\bar{u}(t)]^2\}dt. \tag{3.10}$$

Proposition 3.1. *There exixts a constant c so that for all $\varepsilon \in (0, \bar{\varepsilon})$, we have*

$$\bar{J}(a,b,y_0) - J^0(a,b,y_0) \le c\varepsilon y_0^2.$$

So at this point we fix $\varepsilon \in (0, \bar{\varepsilon})$ and write $\hat{f}_\varepsilon(a,b)$ as in (3.9). We see from Proposition 3 that the control law

$$u(t) = \hat{f}_\varepsilon(a,b)y(t) \tag{3.11}$$

will be a good approximation to the optimal one if ε is small enough. The problem is that we don't know a and b, so we will design a linear time-varying controller which will approximately implement this control law.

3.2 Motivating a Sampled-Data Controller

The goal here is to design a sampled-data control law of the form (2.6). We use h small and with m and n given in the above choice of $\hat{f}_\varepsilon(a,b)$, we choose $p > mn$. First we provide a conceptual description of the controller and a high-level description of why it should work. We do so in open loop and then explain how to convert it to a controller of the form (2.6). To motivate the approach, first observe that the sampled-data control law

$$u(t) = \begin{cases} 0 & t \in [kT, kT + mnh) \\ \frac{p}{p-mn}\hat{f}_\varepsilon(a,b)y(kT) & t \in [kT + mnh, (k+1)T) \end{cases}$$

should be a good approximation to (3.11) if h and $T = ph$ are both small. Here we will implement something similar to this: every period $[kT, (k+1)T)$ is divided into two phases:

- Estimation Phase: on the interval $[kT, kT + mnh)$ we estimate

  $$\hat{f}_\varepsilon(a,b)y(kT).$$

  While we do not set $u(t)$ equal to zero here, we ensure that the effect of the probing used in the estimation yields only a small effect; we do so through the use of a scaling factor ρ and by making the Estimation Phase much shorter than the Control Phase.

- Control Phase: on the interval $[kT + mnh, (k+1)T)$ we apply $\frac{p}{p-mn}$ times the above estimate.

We proceed by looking at a special case of $n = m = 2$. Let us look at the first period $[0, T)$. We start with the Estimation Phase. The first step is to form an approximation of

$$\sum_{i=0}^{1}\sum_{j=0}^{1} \ell_{j,i} \underbrace{a^j b^i y(0)}_{=:\,\phi_{j,i}(0)}.$$

(Recall the form of $\hat{f}_\varepsilon(a, b)$ given in (3.9)). Obviously, we can obtain an exact estimate of $\phi_{0,0}$:

$$\begin{aligned} \hat{\phi}_{0,0}(0) &:= y(0) \\ &= \phi_{0,0}(0). \end{aligned}$$

Suppose that we initially set

$$u(t) = 0, \ t \in [0, mh) = [0, 2h).$$

Since a is constrained to a compact set, it follows that

$$\begin{aligned} y(kh) &= e^{akh}y(0) \\ &= [1 + akh + \frac{a^2k^2h^2}{2!} + \mathcal{O}(h^3)]y(0), \ k = 1, 2. \end{aligned}$$

So we can define

$$\hat{\phi}_{1,0}(0) := \frac{y(h) - y(0)}{h} = ay(0) + \mathcal{O}(h)y(0) = \phi_{1,0}(0) + \mathcal{O}(h)y(0),$$

To form an estimate of $\phi_{j,1}(0), j = 0, 1$, we will carry out some experiments. With $\rho > 0$ a constant, set

$$u(t) = \rho\hat{\phi}_{0,0}(0), \ t \in [mh, 2mh) = [2h, 4h).$$

Then

$$\begin{aligned} y[(2 + k)h] &= e^{a(2+k)h}y(0) + \int_0^{kh} e^{a\tau}b\rho\hat{\phi}_{0,0}(0)\, d\tau \\ &= [1 + (2 + k)ah + (2 + k)^2a^2h^2/2! + +\mathcal{O}(h^3)]y(0) + \\ &\quad \rho b[kah + k^2a^2h^2/2 + \mathcal{O}(h^3)]\hat{\phi}_{0,0}(0), \ \ k = 0, 1, 2. \end{aligned}$$

So we can define

$$\begin{aligned} \hat{\phi}_{0,1}(0) &:= \frac{1}{\rho h}[y(3h) - y(2h) - y(h) + y(0)] \\ &= b\hat{\phi}_{0,0} + \mathcal{O}(h)y(0), \end{aligned}$$

and

$$\begin{aligned} \hat{\phi}_{1,1}(0) &:= \frac{1}{\rho h^2}[y(4h) - 2y(3h) + 2y(2h) - y(h)] \\ &= ab\phi_{1,1}(0) + \mathcal{O}(h)y(0). \end{aligned}$$

At the end of the Estimation Phase, we are at $t = mnh = 4h$, and we have estimates of $\phi_{j,i}(0)$, $i, j = 0, 1$. To ensure $\hat{\phi}_{j,i}(0)$ is a good estimate of $\phi_{j,i}(0)$, we require $|h| << 1$.

Now we turn to the Control Phase. There is a lot of flexibility in how long one can make this phase: the higher the percentage that we carry out control, the closer that the magnitude will be to the ideal one; however, if we make the percentage too large then we will need a very small h to ensure that T is small enough, which will exacerbate noise tolerance. In any event, at this point we fix

$$p > nm = 4$$

with a corresponding controller period of $T = ph$, and we set

$$u(t) = \frac{p}{p-nm} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \ell_{j,i} \hat{\phi}_{j,i}(0) = \frac{p}{p-4} \sum_{i=0}^{1} \sum_{j=0}^{1} \ell_{j,i} \hat{\phi}_{j,i}(0), \quad t \in [ph, T) = [4h, T).$$

It follows that

$$\begin{aligned} y(ph) &= y(T) \\ &= e^{aT} y_0 + \int_0^T e^{a(T-\tau)} b u(\tau) d\tau \\ &= [1 + aT + b\hat{f}_\varepsilon(a,b)T] y(0) + \mathcal{O}(T) y(0) + \frac{1}{p} \mathcal{O}(1) y(0). \end{aligned}$$

The above analysis was carried out for one step for a special case. We now move to the general case with a more careful analysis.

3.3 The Proposed Controller

Now we return to the general case with $n, m \in \mathbf{N}$. The approach is based, to some extent, on that of [9]. As above, we first explain how to do the estimation on the interval $[0, T)$ in open loop and then we provide a closed loop description. First, we present some notation and a key estimation result.

Define two $(m+1) \times (m+1)$ matrices:

$$S_m = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^m \\ & & \vdots & & \\ 1 & m & m^2 & \cdots & m^m \end{bmatrix}, \quad H_m(h) = \begin{bmatrix} 1 & & & & \\ & h & & & \\ & & h^2/(2!) & & \\ & & & \ddots & \\ & & & & h^m/(m!) \end{bmatrix}. \qquad (3.12)$$

Since we are using a sequence of samples of y to do our estimation, we shall define

$$\mathcal{Y}(t) := \begin{bmatrix} y(t) \\ y(t+h) \\ \vdots \\ y(t+mh) \end{bmatrix}.$$

Lemma 3.1. *There exists constants $\gamma > 0$ and $\bar{h} > 0$ so that for all $t_0 \in \mathbf{R}^+$, $y_0 \in \mathbf{R}$, and $\bar{u} \in \mathbf{R}$, we have that the solution of (2.1) with*

$$u(t) = \bar{u}, \quad t \in [t_0, t_0 + mh]$$

has the following property:

$$\|H_m(h)^{-1}S_m^{-1}\mathcal{Y}(t_0) - \begin{bmatrix} 1 \\ a \\ \vdots \\ a^m \end{bmatrix} y(t_0) - \begin{bmatrix} 0 \\ b \\ \vdots \\ a^{m-1}b \end{bmatrix} \bar{u}\| \leq \gamma h(\|\mathcal{Y}(t_0)\| + \|\bar{u}\|).$$

Proof: This is a straight-forward application of Taylor series.

To see how this result can be applied, we set

$$u(t) = 0, \quad t \in [kT, kT + mh).$$

It follows that we can define

$$\begin{bmatrix} \hat{\phi}_{0,0}(kT) \\ \vdots \\ \hat{\phi}_{m-1,0}(kT) \end{bmatrix} := \begin{bmatrix} I_m & 0 \end{bmatrix} H_m(h)^{-1}S_m^{-1}\mathcal{Y}(kT) \approx \begin{bmatrix} 1 \\ a \\ \vdots \\ a^{m-1} \end{bmatrix} y(kT).$$

Now set

$$u(t) = \rho\hat{\phi}_{0,0}(kT), \quad t \in [kT + mh, kT + 2mh);$$

then we see that

$$\begin{aligned} H_m(h)^{-1}S_m^{-1}\mathcal{Y}(kT + mh) &\approx \begin{bmatrix} 1 \\ a \\ \vdots \\ a^m \end{bmatrix} y(kT) + \begin{bmatrix} 0 \\ b \\ \vdots \\ a^{m-1}b \end{bmatrix} \rho\hat{\phi}_{0,0}(kT) \\ &\approx \begin{bmatrix} 1 \\ a \\ \vdots \\ a^m \end{bmatrix} y(kT + mh) + \begin{bmatrix} 0 \\ b \\ \vdots \\ a^{m-1}b \end{bmatrix} \rho y(kT), \end{aligned}$$

which means that we should define

$$\begin{bmatrix} \hat{\phi}_{0,1}(kT) \\ \vdots \\ \hat{\phi}_{m-1,1}(kT) \end{bmatrix} := \frac{1}{\rho} \begin{bmatrix} 0 & I_m \end{bmatrix} H_m(h)^{-1}S_m^{-1}[\mathcal{Y}(kT + mh) - \mathcal{Y}(kT)].$$

Given this observation, we can clearly repeat this and yield a good estimate of

$$\hat{f}_\varepsilon(a, b)y(kT) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \ell_{j,i}a^j b^i y(kT)$$

by the end of the interval $[kT, kT + nmh)$; during the Control Phase we apply a suitably scaled version of this estimate. We now define the proposed controller, defined in open loop over one period.

THE PROPOSED CONTROLLER (OPEN LOOP)

Estimation Phase: $[kT, kT + nmh)$

$$\begin{bmatrix} \hat{\phi}_{0,i} \\ \vdots \\ \hat{\phi}_{m-1,i} \end{bmatrix} (kT) = \begin{cases} \begin{bmatrix} I_m & 0 \end{bmatrix} H_m(h)^{-1} S_m^{-1} \mathcal{Y}(kT) & i = 0, \\ \frac{1}{\rho} \begin{bmatrix} 0 & I_m \end{bmatrix} H_m(h)^{-1} S_m^{-1} [\mathcal{Y}(kT + imh) - \mathcal{Y}(kT)] & i = 1, ..., n-1, \end{cases} \tag{3.13}$$

$$u(t) = \begin{cases} 0 & t \in [kT, kT + mh), \\ \rho \hat{\phi}_{0,i-1}(kT) & t \in [kT + imh, kT + (i+1)mh], \ i = 1, ..., n-1. \end{cases} \tag{3.14}$$

Control Phase: $[kT + nmh, (k+1)T)$

$$u(t) = \frac{p}{p - nm} \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} \ell_{j,i} \hat{\phi}_{j,i}(kT), \quad t \in [kT + mnh, (k+1)T). \tag{3.15}$$

It turns out that the probing and control aspects of the Estimation Phase can be implemented in an iterative fashion so that we end up with a causal description. Indeed, we can prove the following:

Lemma 3.2. *The control law described by (3.13)-(3.15) has a representation of the form (2.6) given by (F, G, J, T, p) which is deadbeat; in fact, $F(0) = 0$.*

At this point we will will examine the closed loop behaviour provided over a single period. We let $\hat{u}$ denote the control signal provided by (3.13)-(3.15), and let $\hat{y}$ and $\hat{J}(a, b, y_0)$ denote the corresponding output and cost, respectively.

Lemma 3.3. *There exist constants $c > 0$ and $\bar{T} > 0$ so that for all $T \in (0, \bar{T})$ and $p > nm$, the closed loop behaviour satisfies*

$$\begin{aligned} |\hat{y}(t) - e^{a_{cl}^{\varepsilon}(t-kT)} \hat{y}(kT)| &\leq c(T + \frac{nm}{p}) T |\hat{y}(kT)|, \quad t \in [kT, (k+1)T) \\ |\hat{u}(t)| &\leq c |\hat{y}(kT)|, \quad t \in [kT, (k+1)T) \\ |\hat{u}(t) - \frac{p}{p - mn} \hat{f}_{\varepsilon}(a, b) \hat{y}(kT)| &\leq c(T + \frac{nm}{p}) |\hat{y}(kT)|, \quad t \in [kT + \frac{nm}{p} T, (k+1)T). \end{aligned}$$

Lemmas 3.1 and 3.2 describe what happens over a single period. The following explains what happens on $[0, \infty)$.

Proposition 3.2. *There exist constants $c > 0$ and $\bar{T} > 0$ so that for all $T \in (0, \bar{T})$ and $p > nm$, the closed loop behaviour satisfies*

$$|\hat{J}(a, b, y_0) - \bar{J}(a, b, y_0)| \leq c(T + \frac{nm}{p}) \|y_0\|^2.$$

4 The Main Result

Now we can leverage the results of Propositions 3.1 and 3.2 to prove the main result.

Theorem 4.1. *For every $\delta > 0$ there exists a controller of the form (2.6) so that*

$$J(a, b, y_0) - J^0(a, b, y_0) \leq \delta \|y_0\|^2, \ \ (a, b) \in \Gamma.$$

Proof:
Fix $\delta > 0$. By Proposition 3.1 there exists an $\varepsilon > 0$ so that

$$\bar{J}(a, b, y_0) - J^0(a, b, y_0) \leq \frac{\delta}{2} \|y_0\|^2, \ \ (a, b) \in \Gamma.$$

By Proposition 3.2 follows that there exist constants $c > 0$ and $\bar{T} > 0$ so that for all $p > nm$ and $T \in (0, T)$, we have

$$|\hat{J}(a, b, y_0) - \bar{J}(a, b, y_0)| \leq c(\frac{nm}{p} + T) \|y_0\|^2, \ \ (a, b) \in \Gamma.$$

Now choose $p > nm$ large and $T \in (0, \bar{T})$ small so that

$$c(\frac{nm}{p} + T) < \frac{\delta}{2},$$

so that

$$|\hat{J}(a, b, y_0) - \bar{J}(a, b, y_0)| \leq \frac{\delta}{2} \|y_0\|^2, \ \ (a, b) \in \Gamma.$$

Hence,

$$|\hat{J}(a, b, y_0) - J^0(a, b, y_0)| \leq \delta \|y_0\|^2, \ \ (a, b) \in \Gamma,$$

as required.

□

5 An Example

Here we discuss a prototype example to illustrate the proposed design methodology. We have $r = 1$ and set

$$\Gamma = \{(a, b) : a \in [-1, 1], \ b \in \{-1, 1\}\}.$$

Obviously,

$$\frac{1}{b} = b, \ \ b \in \{-1, 1\},$$

so the optimal control gain

$$f(a, b) = -ab - b\sqrt{1 + a^2}. \tag{5.16}$$

We notice that no linear time-invariant controller can stabilize all models in the compact set, since it contains $\frac{1}{s-1}$ and $\frac{-1}{s-1}$. The objective is to find a linear time-varying controller that provides a quadratic performance within 25% of the optimal.

We approximate (5.16) by the polynomial

$$\hat{f}_\varepsilon(a,b) = -ab - b(1 + \frac{1}{2}a^2).$$

In Figure 2 we plot the closed loop response for a particular choice of controller parameters; the solid lines correspond to the proposed controller while the dotted lines correspond to the optimal control law. We see that the controller stabilizes both systems; the performance is near optimal for the first plant but somewhat sub-optimal for the second one. To obtain a maximum of 25% error, we need to decrease h further:

$$T = 0.002, \quad p = 20, \quad h = 0.001, \quad \rho = 0.1.$$

The obvious drawback of the controller is that we require fast actuators for the controller to work properly.

6 Summary and Conclusions

In this paper we have considered the problem of designing a linear periodic controller for a class of first order models whose parameters lie in a compact set. This includes cases in which there is no LTI controller which simultaneously stabilizes every admissible model. Here we show how to construct a linear time-varying controller to not only simultaneously stabilize every model in the compact set, but also provide near optimal LQR performance.

The controller operates by periodically estimating what the control signal would be if the plant parameters were known, and then applying this estimate; this is carried out in a linear fashion. The desirable features of the approach are that it provides near optimal performance and the control signal is modest in size. The undesirable features are that it requires fast actuators to ensure that the estimation is accurate, it may require large gains, which may yield poor noise tolerance, and it does not tolerate time-varying parameters. Further work is required to remove the undesirable features as well as to extend the approach to the high order situation.

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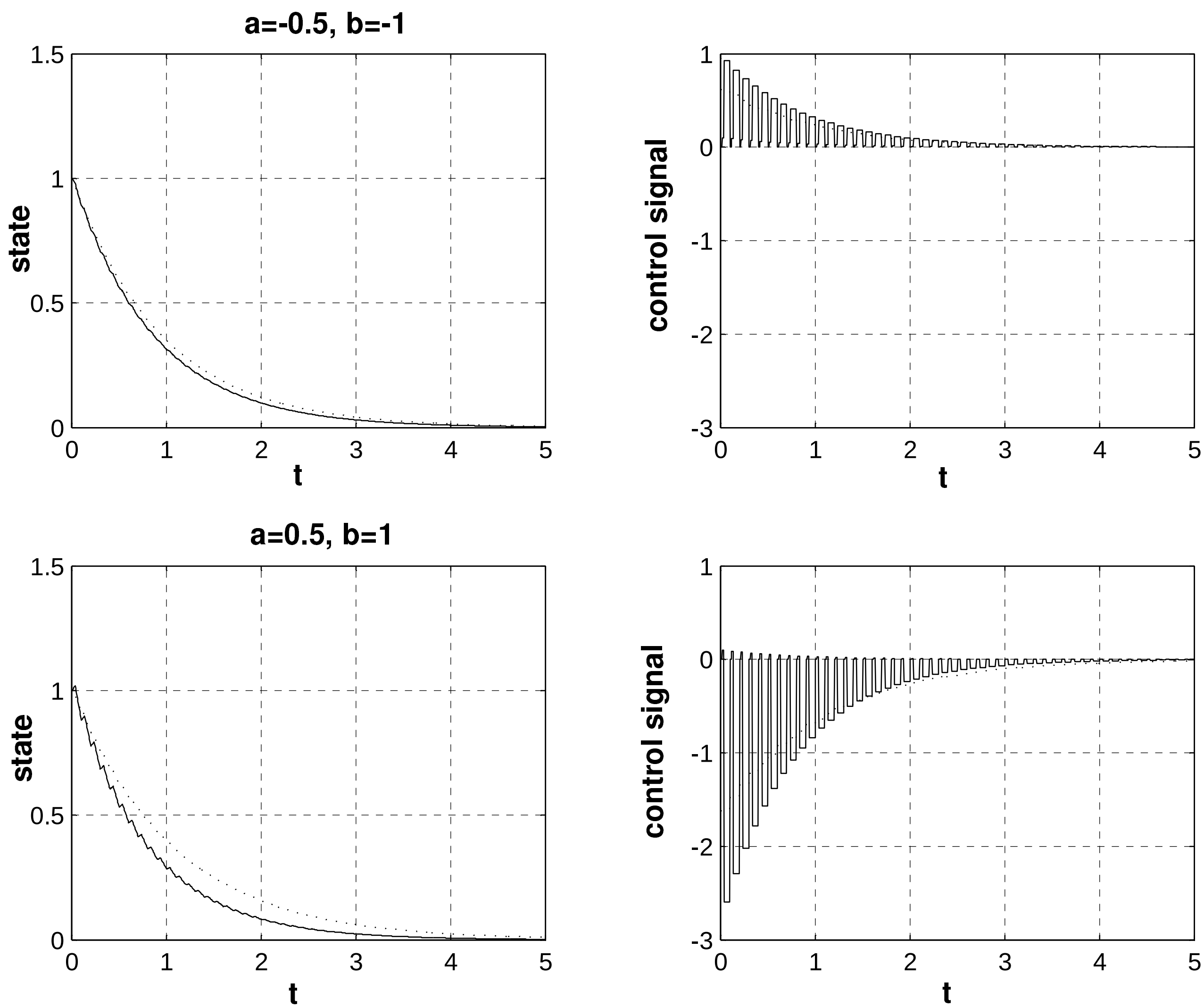

Figure 2: Responses of $(-0.5, -1)$ and $(0.5, 1)$ with $h = 0.005s$, $p = 20$ and $\rho = 0.1$.

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