# Geometry of adaptive control, part II: optimization and geodesics

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**Abstract**

Two incompatible topologies appear in the study of adaptive systems: the graph topology in control design, and the coefficient topology in system identification. Their incompatibility is manifest in the stabilization problem of adaptive control. We argue that this problem can be approached by changing the geometry of the sets of control systems under consideration: estimating $n_p$ parameters in an $n_p$-dimensional manifold whose points all correspond to stabilizable systems. One way to accomplish this is using the properties of the algebraic Riccati equation. Parameter estimation in such a manifold can be approached as an optimal control problem akin to the deterministic Kalman filter, leading to algorithms that can be used in conjunction with standard observers and controllers to construct stable adaptive systems.

## 1 Topologies in adaptive control

Two topologies appear in the study of adaptive systems. Relevant for feedback control is the graph topology, induced by both the gap and graph metrics; it is the coarsest topology on sets of linear systems for which feedback stability is a robust property [1, 13]. System identification, on the other hand, makes implicit use of the topology induced by the metric in which the distance between two systems is given by the Euclidean distance between the coefficients of their transfer functions. Even on sets of linear systems with dimension not exceeding a given $n$, on which both are defined, these topologies are not compatible: one is neither finer nor coarser than the other, that is, a set open in the graph topology may not be an open set in the coefficient topology, and vice-versa.

Adaptive controllers are characterized by a double feedback loop: the control and the adaptation loops. The incompatibility between the topologies underlying the design of the loops manifests itself in the form of the stabilization problem. In fact, the parameter values for which the design model, upon which certainty-equivalence control laws are designed, loses stabilizability, are exactly those for which the operations of addition and multiplication of transfer functions are discontinuous under the graph topology.

Myriad adaptive algorithms in the literature start by designing certainty-equivalence controllers, and jury-rig alternative feedback signals to be used when the certainty-equivalence control laws approach a singularity (see for instance [5] and its references, and [11], where an analysis similar to that of §5 is carried out). This is generally effected via logic-based hybrid control or time-varying feedback; but although switched controllers might be desirable for reasons of performance, there is no clear indication that they are necessary for stabilization. A second way to deal with this incompatibility is to develop alternative parametrizations for sets of linear systems, in order to exclude the singular points – those corresponding to systems that are not stabilizable. Among the few references to this idea in the literature are [2, 4, 10]. Unfortunately the resulting parameter sets often do not have convexity properties needed for the use of conventional estimation techniques.

The approach to system identification we advocate is to change the geometry of the tuned parameter set, and is based on the observation that the set of stabilizable systems can be identified with the set of matrix pairs for which the algebraic Riccati equation has a positive-definite solution. Rather than estimating $n_p$ parameters in $\mathbb{R}^{n_p}$ (or some subset thereof), we can tune them in an $n_p$-dimensional manifold comprising a hypersurface in a space that includes the terms of solution to the matrix Riccati equation, as well as the usual design model parameters.

An application of these ideas to the control of simple, one-dimensional SISO systems has been presented in [12]. Here they are generalized to higher-dimensional SISO processes. In §2 we set up the framework of the adaptive control problem that we wish to solve. Relevant geometric properties of the Riemannian manifold whose points correspond to stabilizable design models are studied in §3. Posing parameter estimation on this manifold as an optimal control problem akin to the Kalman filter permits the development of two algorithms, presented in §4. The analysis of an overall adaptive system constructed according to our recipes is given in §5. Some facts from optimal control gather informally in Appendix A.

# 2 Framework

We are concerned with designing an adaptive controller with basis on the SISO design model

$$\begin{aligned} \dot{x}_D(t) &= (A + dc)x_D(t) + bu_D(t) \\ y_D(t) &= cx_D(t). \end{aligned} \qquad (\Sigma_D)$$

Here $x_D \in \mathbb{R}^n$, $u_D, y_D \in \mathbb{R}$, the matrix pair $(c, A)$ is observable and fixed with $A$ stable, and $d, b \in \mathbb{R}^n$ are vectors of design model parameters. The transfer function of $\Sigma_D$ is

$$c(sI - A - dc)^{-1}b = \frac{c(sI - A)^{-1}b}{1 - c(sI - A)^{-1}d}, \qquad (2.1)$$

a fact that can be verified either by direct matrix manipulation or by rewriting $\Sigma_D$ as $\dot{x}_D = Ax_D + bu_D + dy_D$. Because of observability there is enough freedom to assign the poles of $\Sigma_D$ via suitable choice of $d$, and the zeroes via choice of $b$; moreover the pole-zero cancellations that correspond to eigenvalues of $A$ in (2.1) are stable, therefore $\Sigma_D$ is adequate for designing adaptive controllers for processes about which there is a considerable amount of uncertainty, provided we have reason to believe that they can be effectively controlled using a design based on an $n$-dimensional model. The existence of values of $b$, $d$ for which $(A + dc, b)$ is not stabilizable, and there are unstable cancellations in (2.1), is the origin of the stabilization problem we wish to avoid altogether. We shall defer making specific assumptions about the process itself until they are needed for analysis in §5; suffices to say informally that we are concerned with classes of processes that could be controlled using a design based upon $\Sigma_D$, if we just knew the parameters $b$ and $d$.

Let $T(p)$ be the invertible change of state variable matrix such that $(cT^{-1}, TAT^{-1}, Tp) = (p, A^\top, c^\top)$ and construct the system

$$\begin{aligned} \dot{x} &= A_I x + b_I u + d_I y \\ \hat{x} &= \begin{bmatrix} T(d) & T(b) \end{bmatrix} x \\ e_I &= \begin{bmatrix} d^\top & b^\top \end{bmatrix} x - y, \end{aligned} \tag{$\Sigma_I$}$$

with

$$A_I = \begin{bmatrix} A^\top & 0 \\ 0 & A^\top \end{bmatrix}; \ d_I = \begin{bmatrix} c^\top \\ 0 \end{bmatrix}; \ \text{ and } b_I = \begin{bmatrix} 0 \\ c^\top \end{bmatrix}.$$

The transfer function of $(\Sigma_I)$ from $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} d^\top & b^\top \end{smallmatrix}\right] x$ is the same as the transfer function of $\Sigma_D$ from $\left[\begin{smallmatrix} y_D \\ u_D \end{smallmatrix}\right]$ to $y_D$; in fact if we were to carefully set $\Sigma_D$'s input $y$ to equal its output $\left[\begin{smallmatrix} d^\top & b^\top \end{smallmatrix}\right] x$ we would obtain precisely $\Sigma_D$'s transfer function. Because of the preceding we are justified in calling $\Sigma_I$ an adaptive observer or identifier appropriate for use in conjunction with design model $\Sigma_D$. In this application $u$ and $y$ will be set to equal the input and the output of a controlled process respectively, and $2n$-vector-valued signal $x$ may be called a regressor since it appears as a set of coefficients to the parameters in an affine error equation.

In view of the above, an appropriate regulator to go with $\Sigma_I$ is given by:

$$u_R = f_R(b, d)\hat{x}. \tag{$\Sigma_R$}$$

Together $\Sigma_I$ and $\Sigma_R$ form a parameterized controller, and $f_R(b, d)$ is chosen so that, were it connected to $\Sigma_D$ as explained above and were $u_D$ to be set equal to feedback control signal $u_R$, the resulting system would be internally stable.

To construct the remaining ingredients of a parameter adaptive control system, namely an expression for parameterized feedback $f_R(b, d)$ and a tuner, usual steps involve borrowing a linear control design and and estimation algorithm, using the latter to tune estimates $(d, b)$ on $\mathbb{R}^{2n}$, and finally combining both via some sort of certainty-equivalence. This brings problems in that, unless restrictive hypotheses are made, the parameter space ends up including points for which $\Sigma_D$ loses stabilizability. At such points the equations defining $f$ are sure to hit a singularity. Rather than tackle the stabilizability issue with modifications on standard tuners or feedback control designs, we question the assumption that the Euclidean space is the correct set on which to estimate parameters. The following section discusses what the geometry of an appropriate set might look like.

# 3 Geometry of the Riccati manifold

Perhaps the most transparent, general purpose control design paradigm, which may be applied to any stabilizable, detectable linear system of a known dimension, is linear-quadratic optimal control. In fact, if $b$ and $d$ are such that $\Sigma_D$ meets those conditions — and the values of parameters for which it does not are exactly those responsible for the loss of stabilizability problem in adaptive control — then there exists a symmetric, positive definite solution $P$ to the algebraic Riccati equation

$$(A + dc)^\top P + P(A + dc) - Pbr^{-1}b^\top P + Q = 0. \tag{3.2}$$

Here $r > 0$ and $Q > 0$ are arbitrary design parameters. Consider that (3.2) defines a $2n$-dimensional manifold as a subset of $\mathbb{R}^{n\times n\times n^2}$, uniquely identified by the requirement that $P > 0$. We shall denote this manifold, depicted in Figure 1, $\mathcal{R}icc$. Let $\theta$ be a parametrization of the manifold, in the sense that $\theta(d, b, P)$ is smooth and together with (3.2) and $P = P^\top$ forms a smooth bijection. The domain of the parametrization is $\{\theta \in \mathbb{R}^{2n} : (A + d(\theta)c, b(\theta))$ stabilizable$\}$; we shall refer to points of $\mathbb{R}^{2n}$ outside this domain as the singularity. We may take $\theta = \left[\begin{smallmatrix} d \\ b \end{smallmatrix}\right]$, which is in keeping with the idea of indirect adaptive control, and shall indeed do so in this paper, but not before remarking that other parametrizations may be of interest for alternative adaptive control designs. Notice also that (3.2) together with

$$f = -r^{-1}b^\top P \tag{3.3}$$

can be viewed as one among many possible choices of feedback controls $f_R$ in $\Sigma_R$.

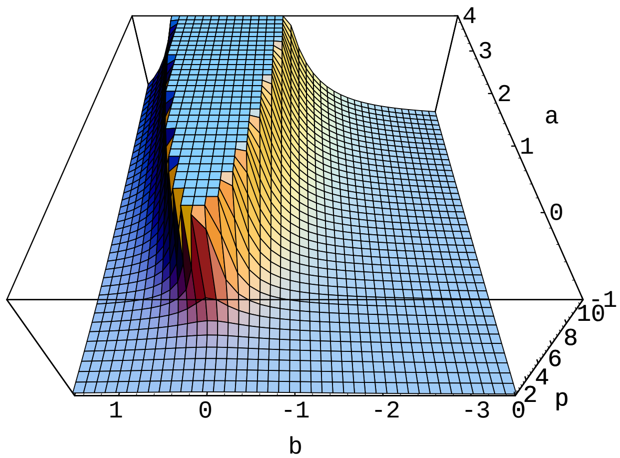


Figure 1: The 2-dimensional $\mathcal{R}icc$ manifold

The main goal of this section is to characterize $\mathcal{R}icc$ as a Riemannian manifold by means of its metric $G(\theta)$, which can be written as a matrix or alternatively as a second-rank tensor with elements $g_{ij}$. We shall consider a metric on $\mathcal{R}icc$ induced by the "natural" inner product on $(d, b, P)$-space, namely the Euclidean space $\mathbb{R}^{n\times n\times n^2}$, so that

$$\mathrm{d}\theta^\top G(\theta)\mathrm{d}\theta = \mathrm{d}d^\top \mathrm{d}d + \mathrm{d}b^\top \mathrm{d}b + \operatorname{trace} \mathrm{d}P^\top \mathrm{d}P.$$

To proceed with the computations, let us rewrite the equations that define the manifold using index notation:

$$(A_{ji} + d_j c_i)P_{jk} + P_{ij}(A_{jk} + d_j c_k) - P_{ij} b_j r^{-1} b_l P_{lk} + Q_{ik} = 0 \tag{3.4}$$

In this expression, we dropped the summation signs, a common act of laziness referred to as the Einstein summation convention: any repeated index is assumed to be summed over. So, the only free indices in (3.4) are $i$ and $k$. Any other indices are summed over, from 1 to $n$. To obtain partial derivatives of (3.4) we employ the rules

$$\frac{\partial x_i}{\partial x_\ell} = \delta_{i\ell}\,, \quad \frac{\partial x_{ij}}{\partial x_{\ell m}} = \delta_{i\ell}\,\delta_{jm}\,,$$

where $\delta_{ij}$ is the Kronecker delta, 1 when $i = j$, 0 otherwise. We also adopt the convention that the derivative of a scalar with respect to a column (row) vector is a column (respectively row) vector. An index preceded by comma denotes differentiation.

For concreteness we now derive an expression of the matrix of the metric $G(\theta)$. It is straightforward to obtain the following expression for the metric by differentiation:

$$g_{mn} = \frac{\partial d_i}{\partial \theta_m}\frac{\partial d_i}{\partial \theta_n} + \frac{\partial b_i}{\partial \theta_m}\frac{\partial b_i}{\partial \theta_n} + \frac{\partial P_{ij}}{\partial \theta_m}\frac{\partial P_{ij}}{\partial \theta_n}.$$

Now for the parametrization $\theta = \left[\begin{smallmatrix} d \\ b \end{smallmatrix}\right]$

$$g_{mn} = \delta_{mn} + \begin{bmatrix} \partial P_{ij}/\partial d_m \\ \partial P_{ij}/\partial b_m \end{bmatrix} \cdot \begin{bmatrix} \partial P_{ij}/\partial d_m & \partial P_{ij}/\partial b_m \end{bmatrix}. \tag{3.5}$$

Now compute the partials with respect to $d_m$ in (3.4)

$$\delta_{jm} c_i P_{jk} + (A_{ji} + d_j c_i)\frac{\partial P_{jk}}{\partial d_m} + \frac{\partial P_{ij}}{\partial d_m}(A_{jk} + d_j c_k) + P_{ij}\delta_{jm} c_k - \frac{\partial P_{ij}}{\partial d_m} b_j r^{-1} b_l P_{lk} - P_{ij} b_j r^{-1} b_l \frac{\partial P_{lk}}{\partial d_m} = 0$$

from which follows, using $f_i = -P_{il} b_l r^{-1}$ from definition (3.3),

$$(A_{ji} + d_j c_i + b_j f_i)\frac{\partial P_{jk}}{\partial d_m} + \frac{\partial P_{ij}}{\partial d_m}(A_{jk} + d_j c_k + b_j f_k) + c_i P_{mk} + P_{im} c_k = 0. \tag{3.6}$$

Performing analogous calculations with respect to $b_m$ gives

$$(A_{ji} + d_j c_i + b_j f_i)\frac{\partial P_{jk}}{\partial b_m} + \frac{\partial P_{ij}}{\partial b_m}(A_{jk} + d_j c_k + b_j f_k) + f_i P_{km} + P_{im} f_k = 0. \tag{3.7}$$

Equations (3.6) and (3.7) above can be made a tad more explicit using, for instance, Kronecker products. All that matters here is that they are nonsingular linear equations, because all eigenvalues of $A + dc + bf$ are negative, and thus have a unique solution, which in turn leaves expression (3.5) for the metric uniquely defined for each value of the parameter $\theta$.

Once a metric has been chosen it alone is enough to define $\mathcal{R}icc$ as a $2n$-dimensional Riemannian manifold, independently of the original embedding in $\mathbb{R}^{2n+n^2}$ which motivated the metric's definition. This intrinsic point of view is the one taken in the sequel. The salient feature of $\mathcal{R}icc$ is that points for which $\Sigma_D$ loses stabilizability are "at infinity," that is, $G(\theta)$ becomes unbounded so that any path on $\mathcal{R}icc$ that tends towards the singularities of a certainty-equivalence feedback has infinite length.

# 4 Estimation algorithms

## 4.1 An optimal control problem

In order to develop algorithms for parameter estimation on $\mathcal{R}icc$, we consider the following optimal control problem with initial cost: minimize

$$\int_{\sigma}^{\tau} \Big(\dot{\theta}^{\top} G(\theta)\dot{\theta} + (x^{\top}\theta - y)^{\top} Q (x^{\top}\theta - y)\Big)\,\mathrm{d}t + \big(\theta(\sigma) - \theta_0\big)^{\top} S\big(\theta(\sigma) - \theta_0\big). \tag{4.8}$$

Here $(\theta(t), \dot{\theta}(t))$ describes a curve on $\mathcal{R}icc$ parametrized by $t \in [\sigma, \tau]$, $G(\theta)$ is the matrix expression of the metric studied in §3, $y(t) \in \mathbb{R}^{n_y}$ and $x(t) \in \mathbb{R}^{2n \times n_y}$ are respectively a vector of data and the regressor as explained in §2, and $\theta_0$ is some $2n$-vector. Matrices $S$ and $Q$ are positive-definite design parameters of dimensions $2n \times 2n$ and $n_y \times n_y$ respectively. We have considered the measurement $y$ to be vector-valued since there is no extra difficulty in doing so; to consider the single-output case simply set $n_y = 1$ so that $y$ and $Q$ are scalars.

If only the first parcel inside the integral were present, and initial and final conditions were imposed on $\theta$, the problem would become one of minimizing a measure of the length of the parametrized curve — and in fact its solution would be the geodesic on $\mathcal{R}icc$ connecting $\theta(\sigma)$ and $\theta(\tau)$ (such a curve exists because $\mathcal{R}icc$ is geodesically complete). The parcel weighting the identification error is more familiar in the estimation literature, and the parcel outside the integral, which weights the deviation of initial condition from some a priori guess, serves to regularize the problem.

Both to guide the search for a solution and to further motivate the formulation of the optimization problem, it is useful to pose it as an equivalent filtering problem: minimize

$$J(\sigma, \tau, \theta) = \int_{\sigma}^{\tau} \big(w^{\top} G(\theta) w + e_I^{\top} Q e_I\big)\,\mathrm{d}t + \big(\theta(\sigma) - \theta_0\big)^{\top} S\big(\theta(\sigma) - \theta_0\big) \tag{4.9}$$

subject to

$$\dot{\theta}(t) = w(t) \tag{4.10}$$

$$y(t) = x^{\top}(t)\theta(t) - e_I(t). \tag{4.11}$$

This problem is a particular case of the deterministic Kalman filter with the complication that the weighting of the input depends on the state $\theta$. An interpretation is that we search for $\theta$ which best explains a linear relationship between data $y$ and $x$, in the sense that $w$, $e_I$, and $\theta(\sigma)$ are minimized according to functional (4.9). An amount of parameter drift that would be small (in terms of its effect on feedback design stability) for points far from the singularity rapidly becomes unacceptable if it leads $\theta$ towards values which correspond to non-stabilizable systems, thus we are less inclined to take into account data that points in such a direction. This provides a motivation, without recourse to geometry, for the choice of a $\theta$-dependent input weighting matrix.

In a kind of least-squares estimator often employed in the adaptive control literature, the matrix $G$ is altogether absent, $\theta$ being considered fixed. Such estimators are known to converge even in the absence of persistent excitation, however their ability to track parameter changes such as those caused by model changes or process faults is poor, a fact that is often dealt with by a number of ad hoc modifications. Explicitly introducing a weighting on parameter drift, which is discussed for instance in [3], might be preferable in its own right, besides opening the possibility of introducing geometric considerations.

## 4.2 Estimation algorithm 1

We now develop two recursive solutions to the equivalent problems (4.8) and (4.9): in the first we consider some fixed value for $\theta(\sigma)$ and in the second we further optimize with respect to all possible trajectories of $\theta$. First let us convert (4.9) into a more standard problem by augmenting $\theta$ with the definition $\bar{\theta} = \left[\begin{smallmatrix}\theta\\1\end{smallmatrix}\right]$ so that

$$J = \int_\sigma^\tau \left(w^\top G(\theta) w + \bar{\theta}^\top \bar{Q} \bar{\theta}\right) \mathrm{d}t + \bar{\theta}^\top(\sigma)\bar{S}(\sigma)\bar{\theta}(\sigma) \tag{4.12}$$

subject to

$$\dot{\bar{\theta}}(t) = \bar{B} w(t),$$

where

$$\bar{B} = \begin{bmatrix} I_{2n\times 2n} \\ 0_{2n\times 1} \end{bmatrix},\ \bar{Q} = \begin{bmatrix} xQx^\top & -xQy \\ -y^\top Q x^\top & y^\top Q y \end{bmatrix},\ \text{ and } \bar{S}(\sigma) = \begin{bmatrix} S & -S\theta_0 \\ -\theta_0^\top S & \theta_0^\top S \theta_0. \end{bmatrix}$$

Notice that the problem assumes a time-varying nature because $\bar{Q}$ depends on the data. Following Appendix A, form the Hamiltonian

$$\mathcal{H}(t, \bar{\theta}, w, \lambda) = w^\top G(\theta) w + \bar{\theta}^\top \bar{Q}\bar{\theta} - \bar{\lambda}\bar{B} w,$$

whose minimum value is attained for

$$w = \tfrac{1}{2} G^{-1}(\theta) \bar{B}^\top \bar{\lambda}^\top. \tag{4.13}$$

The differential equation for the row vector of Lagrange multipliers will involve derivatives of the matrix $G$, so we revert to index notation:

$$\begin{aligned}\dot{\bar{\lambda}}_l = \frac{\partial \mathcal{H}}{\partial \bar{\theta}_l} &= \frac{\partial}{\partial \bar{\theta}_l}\left(w_i G_{ij} w_j + \bar{\theta}_i \bar{Q}_{ij}\bar{\theta}_j - \bar{\lambda}_i \bar{B}_{ij} w_j\right) \\ &= w_i \frac{\partial G_{ij}}{\partial \bar{\theta}_l} w_j + \bar{Q}_{lj}\bar{\theta}_j + \bar{\theta}_i \bar{Q}_{il}.\end{aligned}$$

The last two parcels are equal and read, in matrix notation,

$$\begin{bmatrix} xQx^\top & -xQy \\ -y^\top Q x^\top & y^\top Q y \end{bmatrix} \cdot \begin{bmatrix} \theta \\ 1 \end{bmatrix} = \begin{bmatrix} xQx^\top\theta - xQy \\ -y^\top Q x^\top \theta + y^\top Q y \end{bmatrix},$$

and because $G$ does not depend on the last element of $\bar\theta$ the expression of $\bar\lambda$ reduces to

$$\begin{aligned}\dot\lambda_l &= w_i G_{ij,l} w_j + 2(xQe_I)_l,\\ \dot{\bar\lambda}_{2n+1} &= -y^\top Q e_I.\end{aligned} \tag{4.14}$$

Here $\lambda = \bar\lambda \bar B$ and $G_{ij,l}$ denotes $\partial G_{ij}/\partial\theta_l$. Together (4.10), (4.13), and (4.14) express an algorithm, summarized here for ease of reference, for estimation of a parameter we shall follow fashion in calling $\hat\theta$.

**Algorithm 1** Choose some initial condition $\hat\theta(\sigma)$ (say, $\theta_0$) and set parameter estimate $\hat\theta$ according to

$$\begin{aligned}\dot{\hat\theta}(t) &= w\\ \dot\lambda_l(t) &= w_i G_{ij,l}(\hat\theta) w_j + 2(xQe_I)_l\\ w(t) &= \tfrac{1}{2} G^{-1}(\hat\theta)\lambda^\top,\end{aligned} \tag{4.15}$$

with $\lambda(\sigma) = 2(\theta(\sigma) - \theta_o)^\top S$.

Existence and uniqueness of solutions to (4.15) on the interval $[\sigma, \tau]$ is not a question provided that $G$ remains nonsingular, its derivative bounded, and that $(x, y)$ are bounded. Nonsingularity follows from the definition of $G$ and boundedness of $G_{ij,l}$ follows from differentiability, which implies boundedness in any compact subset, together with the fact that any trajectory for which $G$ becomes unbounded must be on infinite length and thus incompatible with a finite $J$.

An alternative characterization of Algorithm 1 can be obtained rewriting (4.13) as

$$G_{kj} w_j + w_i G_{ik} = \lambda_k,$$

and then taking derivatives with respect to $t$ and identifying $w$ with $\dot\theta$:

$$G_{kj,l}\dot\theta_l\dot\theta_j + G_{ik,l}\dot\theta_l\dot\theta_i + G_{kj}\ddot\theta_j + G_{ik}\ddot\theta_i = \dot\theta_i G_{ij,k}\dot\theta_j + 2(xQe_I)_k.$$

Using symmetry of $G$ results

$$2G_{km}\ddot\theta_m - G_{ij,k}\dot\theta_i\dot\theta_j + G_{kj,i}\dot\theta_i\dot\theta_j + G_{ik,j}\dot\theta_i\dot\theta_j = 2(xQe_I)_k,$$

or finally

$$\ddot\theta_m + \Gamma^m_{ij}\dot\theta_i\dot\theta_j = \big(G_{km}\big)^{-1}(xQe_I)_k. \tag{4.16}$$

Here $\Gamma^m_{ij} = \frac{1}{2}G^{-1}_{km}\big(G_{kj,i} + G_{ik,j} - G_{ij,k}\big)$ are known as the Christoffel symbols and the right-hand side is the expression of the gradient of $e_I^2/2$ on $\mathcal{R}icc$. When $e_I$ is zero, the second-order differential equation (4.16) is nothing but the expression of a geodesic on the manifold of metric $G$. The relationship of form (4.16) of Algorithm 1 with the second-order tuner discussed in [7] might be worth exploring.

## 4.3 Estimation algorithm 2

While Algorithm 1 defines a recursive procedure for constructing parameter estimates in its own right, one thing that is still lacking is to optimize with respect to all trajectories. This is more in keeping with the spirit of Kalman filtering and appears to have some advantages. To accomplish the minimization, notice that once we have decided to use optimizing law (4.15) the final condition $\theta(\tau)$ biunivocally identifies the initial value $\theta(\sigma)$; therefore in order to optimize with respect to initial conditions one can alternatively minimize the optimal accumulated cost $V(\tau, \bar{\theta})$ with respect to $\bar{\theta}$.

The value $\hat{\vartheta}$ which minimizes the accumulated cost at time $\tau$ must satisfy

$$\frac{\partial V}{\partial \theta}(\tau, \hat{\vartheta}) = 0. \tag{4.17}$$

Along the trajectory which leads to $\hat{\vartheta}$ at time $\tau$, $w(\tau) = \frac{1}{2} G^{-1}(\hat{\vartheta}) \frac{\partial V}{\partial \theta}(\tau, \hat{\vartheta})^\top = 0$, an observation that greatly simplifies the calculations that follow if they are correct — a big if considering that the deadline for submitting this paper has passed and the second author has been known since his college days for leaving pieces hanging when playing chess. Although (4.17) does not provide an explicit formula for $\hat{\vartheta}$, it serves to obtain a recursive expression as follows. Considering $\hat{\vartheta}$ as a function of $\tau$ and differentiating gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\partial V}{\partial \theta}(\tau, \hat{\vartheta}) = \frac{\partial^2 V}{\partial \tau\, \partial \theta}(\tau, \hat{\vartheta}) + \dot{\hat{\vartheta}}^\top \frac{\partial^2 V}{\partial \theta^2}(\tau, \hat{\vartheta}) = 0. \tag{4.18}$$

To solve this equation for $\hat{\vartheta}$, first compute

$$\frac{\partial^2 V}{\partial \tau\, \partial \theta}(t, \hat{\vartheta}) = \frac{\partial^2 V}{\partial \theta\, \partial \tau}(t, \hat{\vartheta}) = \frac{\partial}{\partial \theta}(w^\top G(\theta) w + e_I Q e_I)\bigg|_{\theta = \hat{\vartheta}} = 2xQe_I(\tau, \hat{\vartheta}).$$

Second define $\Psi(\tau, \hat{\vartheta}) = \frac{\partial^2 V}{\partial \theta^2}(\tau, \hat{\vartheta})$ and use (A.26) from Appendix A to write

$$\dot{\Psi} = \frac{\partial^2 \mathcal{H}}{\partial \theta^2} - \left(\frac{\partial w}{\partial \theta}\right)^\top \frac{\partial^2 \mathcal{H}}{\partial w^2} \left(\frac{\partial w}{\partial \theta}\right) = w^\top \partial^2_{\theta\theta} G(\theta) w + 2xQx^\top - 2\partial_\theta w^\top G(\theta) \partial_\theta w. \tag{4.19}$$

Here indices were dropped for readability, $w^\top \partial^2_{\theta\theta} G(\theta) w$ and $\partial_\theta w$ being understood as $2n \times 2n$ matrices. From $2G_{ij} w_j = \lambda_i$ follows

$$2G_{ik,s} w_k + 2G_{ij} \frac{\partial w_i}{\partial \theta_s} = \frac{\partial \lambda_i}{\partial \theta_s} = \Psi_{is}.$$

Substituting into (4.19) gives

$$\dot{\Psi}_{rs} = 2xQx^\top - \tfrac{1}{2} \Psi G^{-1} \Psi.$$

Inversion of matrix $\Psi$ can be avoided defining $\Pi = 2\Psi^{-1}$ so that $\dot{\Pi} = -2\Psi^{-1}\dot{\Psi}\Psi^{-1} = \frac{1}{2}\Pi\dot{\Psi}\Pi$, and solving (4.18) for $\vartheta$ results in

**Algorithm 2** Set parameter estimate $\hat{\vartheta}$ according to

$$\dot{\hat{\vartheta}}(t) = -\Pi x Q e_I(t, \hat{\vartheta}),$$
$$\dot{\Pi}(t) = G^{-1}(\hat{\vartheta}) - \Pi x Q x^\top \Pi,$$

with initial conditions $\hat{\vartheta}(\sigma) = \theta_0$ and $\Pi(\sigma) = S$.

# 5 Analysis

The time is now ripe to start making some hypotheses and proving theorems. So far we have essentially discussed signal processing algorithms with little regard for the origin of the signals, however stability and other steady-state properties so beloved by control theorists are eminently noncausal in that they cannot be ascertained with any finite amount of data. Making statements about stability requires assumptions that bind the future behavior of a process after a finite amount of measurements. The hypotheses needed for the first theorem amount to little more than existence of some time interval where all signals are bounded.

**Theorem 5.1.** *Assume that $y, x$ exist and are bounded on the interval $[\sigma, \tau]$, and moreover that there exist an instant $s \in [\sigma, \tau]$ and a constant $C_1$ such that*

$$\int_\sigma^s \left(|x(t)|^2 + |y(t)|^2\right) \mathrm{d}t \le C_1. \tag{5.20}$$

*Further suppose that there exist $\theta_*$ and $C_2(\sigma, \tau)$ such that*

$$\int_\sigma^\tau \left(x^\top(t)\theta_* - y(t)\right)^\top Q \left(x^\top(t)\theta_* - y(t)\right) \le C_2(\sigma, \tau). \tag{5.21}$$

*Then if $\hat{\theta}(t)$ is chosen according to Algorithm 1 or to Algorithm 2 there exists a constant $C_3$ such that*

$$\int_\sigma^\tau \left(\dot{\hat{\theta}}^\top G(\hat{\theta})\dot{\hat{\theta}} + e_I^\top Q e_I\right) \mathrm{d}t \le C_3 + C_2(\sigma, \tau). \tag{5.22}$$

*The same holds for $\hat{\vartheta}$ if chosen according to Algorithm 2.*

**Proof:** Pick a time $s \in [\sigma, \tau]$ and consider the following trajectory in $\theta$-space: for $t \in [\sigma, s)$, $\theta(t)$ follows a geodesic on $\mathcal{R}icc$ with $\theta(\sigma) = \theta_0$, and for $t \in [s, \tau]$, $\theta(t) = \theta_*$. Along this trajectory $\theta$ is bounded, so one may choose a constant $C_4$ such that $|\theta(t)| + 1 \le C_4$. The distance between $\theta_0$ and $\theta_*$ is given by

$$\ell(\theta_0, \theta_*) = \int_\sigma^s \sqrt{\dot{\theta}^\top(t) G\big(\theta(t)\big)\dot{\theta}(t)}\, \mathrm{d}t.$$

Further specify the trajectory by choosing $\dot{\theta}^\top G(\theta)\dot{\theta}$ constant in the interval $[\sigma, s]$, so that

$$\ell(\theta_0, \theta_*) = (\sigma - s)\sqrt{\dot{\theta}^\top(t) G\big(\theta(t)\big)\dot{\theta}(t)}.$$

Thus

$$\int_\sigma^s \left(\dot\theta^\top G(\theta)\dot\theta + e_I^\top Q e_I\right) \mathrm{d}t$$

$$\leq \int_\sigma^s \left(\frac{\ell(\theta_0,\theta_*)}{s-\sigma}\sqrt{\dot\theta^\top G(\theta)\dot\theta} + qc_4(|x|+|y|)^2\right) \mathrm{d}t \leq \frac{\ell^2(\theta_0,\theta_*)}{s-\sigma} + 2qC_1C_4 = C_2,$$

where $q = \max_{ij}|Q_{ij}|$. Because $\theta = \theta_*$ after time $s$

$$\int_\sigma^s \left(\dot\theta^\top G(\theta)\dot\theta + e_I^\top Q e_I\right) \mathrm{d}t + \int_s^\tau e_I^\top Q e_I \,\mathrm{d}t \leq \frac{\ell^2(\theta_0,\theta_*)}{s-\sigma} + qC_1 + C_2(\sigma,\tau).$$

Now recall that the trajectory defined by Algorithm 1 is optimal among all trajectories starting from $\theta_0$ on the interval $[\sigma,\tau]$, so the cost $J(\sigma,\tau,\hat\theta)$ is also bounded by the expression on the right-hand side of the inequality above. The statement of the theorem now follows immediately. ∎

We now state a result that concerns the stabilization capabilities of the type of controller described in this paper.

**Theorem 5.2.** *Suppose that a controller composed of parameterized identifier $\Sigma_I$ and feedback regulator $\Sigma_R$, with the parameters given by Algorithm 1 or 2, is applied to a process $\Sigma_P$ whose input $u$ and output $y$ are such that*

$$y(t) = \bar y(t) + v(t),$$

*where $\bar y$ is the output of an $n$-dimensional, linear time-invariant, detectable and stabilizable siso system whose input is $u$. Further suppose that there exists a constant $\gamma$ and a function $\nu(\cdot)$ such that*

$$||v||_t \leq \gamma||u||_t + \nu(t)$$

*on any interval $[0,t)$ on which all signals exist and are finite. Then there exist values $\gamma_*$ and $\nu_*$ such that, if $\gamma \leq \gamma_*$ and $\nu(t) \leq \nu_*$, all signals in the overall adaptive system are bounded on $[0,\infty)$.*

**Sketch of proof:** First argue that the conditions of Theorem 5.1 are met, thus $||e_I|| \leq C_3 + C_2(\gamma||u|| + \nu)$. Then invoke the certainty-equivalence stabilization theorem [6] to state that the overall system $\Sigma$ composed of $\Sigma_P$, $\Sigma_I$, and $\Sigma_R$ is detectable through $e_I$. Therefore there exists an output injection that converts $\Sigma$ into a parameterized system which is stable for each fixed value of the parameters, and whose input $e_I$ is "small." Because the parameters vary "slowly," stability for each fixed value of them is enough to guarantee stability of the time-varying system, which in turn implies that all signals remain bounded, at least in the case $\gamma = 0$. To argue stability when $\gamma > 0$, that is, the case when higher-order, unmodeled dynamics may be present, stronger versions of the certainty-equivalence stabilization theorem and of the nondestabilizing property of slowly time-varying linear systems are needed; such results can be found in [8, 9].

# A Some facts from optimal control theory

Here some facts from optimal control gather informally. Sufficient smoothness is assumed so that all relevant derivatives exist and are smooth. All variables in this section are local in scope, that is, their definition here has no impact on their use elsewhere in the paper.

Consider the initial-cost problem of minimizing the functional

$$J(\sigma, \tau, x, u) = \int_\sigma^\tau q(t, x, u)\,\mathrm{d}t + p(x(\sigma))$$

subject to the differential equation

$$\dot{x}(t) = f(t, x, u).$$

The optimal accumulated cost (or Bellman value function) $V(t, x)$ must satisfy

$$V(s, x) = \int_\sigma^s q(t, x, u_*)\,\mathrm{d}t + p(x(\sigma))$$

for the optimal control $u_*(t)$, so that

$$\dot{V}(t, x) = q(t, x, u_*).$$

Since $V$ is a function of $t$ and $x$ only, along the optimal trajectory

$$\partial_t V + (\partial_x V) f(t, x, u_*) = q(t, x, u_*) \tag{A.23}$$

the Hamilton-Jacobi-Bellman equation for the initial cost problem. Define the appropriate Hamiltonian

$$\mathcal{H}(t, x, u, \lambda) = q(t, x, u) - \lambda f(t, x, u).$$

The optimal control is that which minimizes $\mathcal{H}$ with $\lambda = \partial_x V(t, x)$. Now take partial derivatives with respect to $x$ in (A.23):

$$\frac{\partial^2 V}{\partial x_j \partial t} + \frac{\partial^2 V}{\partial x_j \partial x_i} f_i + \frac{\partial V}{\partial x_i}\frac{\partial f_i}{\partial x_j} + \frac{\partial V}{\partial x_i}\frac{\partial f_i}{\partial u_k}\frac{\partial u_k}{\partial x_j} = \frac{\partial q}{\partial x_j} + \frac{\partial q}{\partial u_k}\frac{\partial u_k}{\partial x_j}$$

Inverting the order of derivatives and identifying $\lambda_i = \partial V / \partial x_i$ results

$$\frac{\partial}{\partial t}\lambda_j + \left(\frac{\partial}{\partial x_i}\lambda_j\right) f_i + \lambda_i \frac{\partial f_i}{\partial x_j} = \frac{\partial q}{\partial x_j} + \left(\frac{\partial q}{\partial u_k} - \lambda_i \frac{\partial f_i}{\partial u_k}\right)\frac{\partial u_k}{\partial x_j}$$

or

$$\dot{\lambda} + \lambda\, \partial_x f(t, x, u_*) = \partial_x q(t, x, u_*). \tag{A.24}$$

To obtain the equation above we used the fact that the optimal $u_*$ must satisfy

$$\frac{\partial \mathcal{H}}{\partial u} = \frac{\partial q}{\partial u} - \lambda \frac{\partial f}{\partial u} = 0, \tag{A.25}$$

which in fact together with (A.24) is an expression of Pontryagin's maximum principle for the problem under consideration. One way of formally proving that (A.23) is sufficient, and (A.24) is necessary, for a control to be optimal would be to reverse time, transforming the problem into a final-cost problem, and applying the usual optimality principles.

We shall also have an occasion to use a recursion on $\Psi(t,x) = \partial_x^2 V$ that can be obtained by further computing derivatives with respect to $x$ in (A.24):

$$\frac{\partial^2 \lambda_j}{\partial x_k \partial t} + \frac{\partial^2 \lambda_j}{\partial x_k \partial x_i} f_i + \frac{\partial \lambda_j}{\partial x_i}\left(\frac{\partial f_i}{\partial x_k} + \frac{\partial f_i}{\partial u_l}\frac{\partial u_l}{\partial x_k}\right) + \frac{\partial \lambda_i}{\partial x_k}\frac{\partial f_i}{\partial x_j} + \lambda_i\left(\frac{\partial^2 f_i}{\partial x_k \partial x_j} + \frac{\partial^2 f_i}{\partial x_j \partial u_l}\frac{\partial u_l}{\partial x_k}\right) = \frac{\partial^2 q}{\partial x_k \partial x_j} + \frac{\partial^2 q}{\partial x_j \partial u_l}\frac{\partial u_l}{\partial x_k}$$

So

$$\frac{\partial \Psi_{jk}}{\partial t} + \frac{\partial \Psi_{jk}}{\partial x_i} f_i + \Psi_{ji}\frac{\partial f_i}{\partial x_k} + \frac{\partial f_i}{\partial x_j}\Psi_{ik} = \frac{\partial^2 q}{\partial x_k \partial x_j} - \lambda_i \frac{\partial^2 f_i}{\partial x_k \partial x_j} + \left(\frac{\partial^2 q}{\partial x_j \partial u_l} - \lambda_i \frac{\partial^2 f_i}{\partial x_j \partial u_l} - \frac{\partial \lambda_j}{\partial x_i}\frac{\partial f_i}{\partial u_l}\right)\frac{\partial u_l}{\partial x_k}$$

But taking derivatives with respect to $x_j$ in (A.25) gives

$$\frac{\partial^2 q}{\partial x_j \partial u_l} + \frac{\partial^2 q}{\partial u_l \partial u_m}\frac{\partial u_m}{\partial x_j} - \lambda_i\left(\frac{\partial^2 f_i}{\partial x_j \partial u_l} + \frac{\partial^2 f_i}{\partial u_l \partial u_m}\frac{\partial u_m}{\partial x_j}\right) - \frac{\partial \lambda_i}{\partial x_j}\frac{\partial f_i}{\partial u_l} = 0,$$

thus

$$\frac{\partial^2 q}{\partial x_j \partial u_l} - \lambda_i \frac{\partial^2 f_i}{\partial x_j \partial u_l} - \frac{\partial \lambda_i}{\partial x_j}\frac{\partial f_i}{\partial u_l} = \left(\lambda_i \frac{\partial^2 f_i}{\partial u_l \partial u_m} - \frac{\partial^2 q}{\partial u_l \partial u_m}\right)\frac{\partial u_m}{\partial x_j} = -\frac{\partial^2 \mathcal{H}}{\partial u_l \partial u_m}\frac{\partial u_m}{\partial x_j}.$$

Hence we can write

$$\dot{\Psi}_{jk} + \Psi_{ji}\big(\partial_x f\big)_{ik} + \big(\partial_x f\big)_{ij}\Psi_{ik} = \big(\partial_x^2 \mathcal{H}\big)_{jk} - \frac{\partial u_m}{\partial x_j}\frac{\partial^2 \mathcal{H}}{\partial u_m \partial u_l}\frac{\partial u_l}{\partial x_k}, \tag{A.26}$$

which is the expression we wished to obtain. For instance, in the well-known linear-quadratic case, where $f = Ax + Bu$ and $q = u^\top R u + x^\top Q x$, from (A.25) results $u = R^{-1}B^\top \lambda^\top/2$ and (A.26) reduces to

$$\dot{\Psi} = -A^\top \Psi - \Psi A + 2Q - \tfrac{1}{2}\tfrac{\partial \lambda}{\partial x} B R^{-1} B^\top \tfrac{\partial \lambda^\top}{\partial x},$$

which reduces to the usual Kalman filtering Riccati differential equation when we substitute $\frac{\partial \lambda}{\partial x} = \Psi = 2P$.

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