# Canonical Realizations of Linear Time-Varying Systems

**F.L. Neerhoff and P. van der Kloet**
**Department of Electrical Engineering**
**Delft University of Technology**
**Mekelweg 4**
**2628 CD Delft**
**The Netherlands**

**Abstract**

In this article, general scalar linear time-varying systems are addressed. In particular, canonical realizations with integrators, multipliers and adders are presented. Essentially, it is shown that the well-known configurations for constant systems can be generalized to the time-varying context by replacing the conventional eigenvalues by the earlier introduced dynamic eigenvalues. However, it is also shown that at least one configuration is not suitable for such a generalization.

## 1 Introduction

As is well-known, the solution of a (scalar) linear differential equation with constant coefficients can be simulated at the output of a linear time-invariant (LTI) signal processing filter with the known right-hand side of the differential equation as input. For such a filter, consisting of integrators, multipliers and adders, different but equivalent canonical realizations are known [1, 2].

In this article, canonical realizations of linear *time-varying* (LTV) systems are derived. To be more precize, possible generalizations of known LTI-configurations to the LTV-context are studied. It turns out that at least one LTI-configuration is not suitable for such a generalization. It is also shown that two other LTI-configurations, viz. a direct realization with the coefficients of the associated differential equation as multipliers, together with the so-called cascade realization can be generalized indeed. In the latter, the conventional algebraic eigenvalues have to be replaced by the earlier introduced *dynamic* eigenvalues [3, 4, 5, 6].

Related results can be found in [7] and [8, 9]. Essentially, these contributions are based on well-known mathematical methods for the factorization of the associated polynomial differential system operator (see also [10]).

In contrast, our approach uses the Riccati transformation as described in [11]. Basically, this transformation effectuates an appropriate order reduction and a subsequent decoupling of the associated LTV system equations. In the next section, first the original scalar differential equation is rewritten in a state-space system description. Then, an associated direct filter realization is easily obtained. As we explained earlier, a successive application of the above

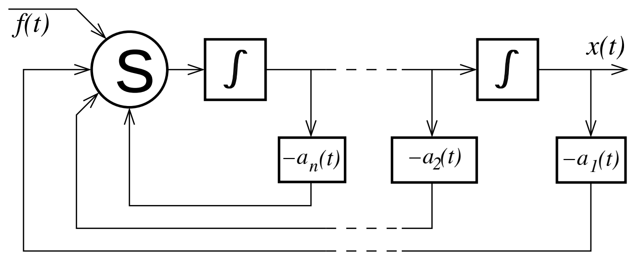


Figure 1: A canonical direct realization.

mentioned Riccati transformation *triangularizes* the accompanying LTV system matrix step-by-step [5, 6, 12, 13].

In Section 3, first an earlier obtained alternative method for the classical Cauchy-Floquet decomposition is shortly repeated [14]. Then, the cascade realization as described in [15] follows immediately. In this realization, the *dynamic* eigenvalues play the role of time-varying multipliers. As another result, it is shown that any triangularization step produces a next canonical realization with one extra dynamic eigenvalue as a multiplier of the LTV filter.

Finally, in Section 4 it is shown that an alternative direct realization for LTI systems does not have a LTV antipode.

## 2 A direct realization

Consider the inhomogeneous scalar linear differential equation for the unknown $x = x(t)$ with normalized time-varying coefficients $a_i = a_i(t)$ $(i = 1, \ldots, n)$

$$D^n x + a_n D^{n-1} x + \cdots + a_2 Dx + a_1 x = f \quad , \tag{2.1}$$

in which $D = d/dt$ and $f = f(t)$ is a known function, respectively. By introducing the new variables $\{x_1, x_2, \ldots, x_n\}$ as

$$x_1 = x, x_2 = \dot{x}_1, x_3 = \dot{x}_2, \ldots, x_n = \dot{x}_{n-1} \quad , \tag{2.2}$$

where the dot stands for a time-derivative, we obtain from (2.1)

$$\dot{x}_n = -a_n x_1 - \cdots - a_1 x_n + f \quad . \tag{2.3}$$

On account of (2.2) and (2.3), the canonical direct realization of Figure 1 is easily recognized. Next, introduce the $n$-dimensional column-vector $\boldsymbol{x}$ with $\boldsymbol{x}^T = [\, x_1, \ldots, x_n \,]$ ($T$ denotes the transpose) and unit vectors $\boldsymbol{e}_j^{(n)}$, $(1 \le j \le n)$, with $[\boldsymbol{e}_j^{(n)}]^T = [\,\delta_{1j}, \ldots, \delta_{nj}\,]$, where $\delta_{ij}$ denotes the Kronecker symbol. Moreover, the partitioned $n \times n$ companion system row matrix $\mathbf{A}$ is introduced as

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_{n-1}^{+} & \boldsymbol{e}_{n-1}^{(n-1)} \\ -\boldsymbol{a}^T(t) & -a_1(t) \end{bmatrix} \quad , \tag{2.4}$$

where $\mathbf{I}_k^+$ denotes the square shift matrix of size $k$, given by

$$\mathbf{I}_k^+ = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{bmatrix} \quad , \tag{2.5}$$

while the varying system parameters $\{a_2, a_3, \dots, a_n\}$ are collected in the time-dependent row vector $\boldsymbol{a}^T = [\, a_n, \dots, a_2 \,]$. Then, the state-space description of (2.1) follows as

$$\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x} + \boldsymbol{e}_n^{(n)} f(t) \quad , \tag{2.6}$$

with read-out equation

$$x = x_1 \quad . \tag{2.7}$$

As a result, equations (2.6) and (2.7) are realized by the LTV-filter in Figure 1.

# 3 The cascade realization

As we showed earlier, there exists a lower triangular Riccati transformation matrix $\mathbf{R}$

$$\boldsymbol{x}(t) = \mathbf{R}(t)\boldsymbol{y}(t) \tag{3.1}$$

by which system (2.6) is transformed into

$$\dot{\boldsymbol{y}}(t) = \begin{bmatrix} \lambda_1(t) & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 \\ 0 & \dots & \dots & \lambda_n(t) \end{bmatrix} \boldsymbol{y}(t) + \boldsymbol{e}_n^{(n)} f(t) \tag{3.2}$$

with $\boldsymbol{y}^T = [\, y_1, \dots, y_n \,]$, and (2.7) into

$$x = y_1 \quad . \tag{3.3}$$

Now, it is immediately observed that the original differential equation (2.1) is given by the Cauchy-Floquet decomposition [14]

$$(D - \lambda_n(t))(D - \lambda_{n-1}(t)) \dots (D - \lambda_1(t))x = f(t) \quad . \tag{3.4}$$

Secondly, the result (3.4) constitutes the canonical cascade signal processing filter, see Figure 2. Note, that these mulptipliers may be complex valued functions of time. As we explained earlier [5, 6, 12], each triangularization step needs a *paticular* solution of a vector Riccati

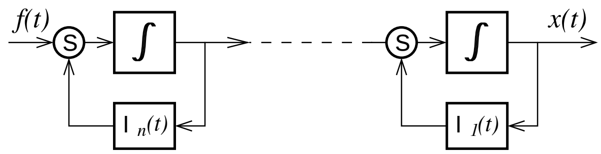


Figure 2: The canonical cascade realization.

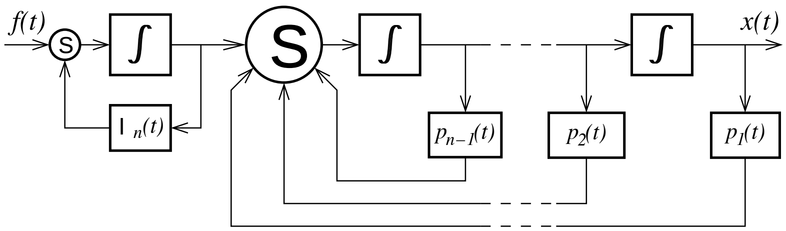


Figure 3: A canonical direct realization after one Riccati transformation step.

differential equation. If $p_1, p_2, \dots, p_{n-1}$ denote the components of the solution vector of the first Riccati-equation, it can be shown that the topology depicted in Figure 3 is equivalent to the LTV filter realization of Figure 1. The matrix in (3.2) indeed confirms that the functions $\lambda_i(t)$ are a kind of eigenvalues. To show this regorously, consider the homogeneous equation (3.2). This equation is investigated for modal solutions of the form

$$\boldsymbol{y}_j(t) = \boldsymbol{u}_j(t) \exp\left[\int_0^t \lambda_j(\tau) d\tau\right] \tag{3.5}$$

with

$$\boldsymbol{u}_j(t) = [\, u_{1,j}(t), \dots, u_{j-1,j}(t), 1, 0, \dots, 0\,]^T \quad . \tag{3.6}$$

Substitution of (3.5) and (3.6) in the homogeneous form (3.2) yields that (3.5) is indeed a solution, only if $\boldsymbol{u}_j$ satisfies

$$\dot{\boldsymbol{u}}_j(t) = \left\{ \begin{bmatrix} \lambda_1(t) & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \dots & \dots & \lambda_n(t) \end{bmatrix} - \lambda_j(t)\mathbf{I}_n \right\} \boldsymbol{u}_j(t) \quad , \tag{3.7}$$

where $\mathbf{I}_n$ is the $n$-dimensional unity matrix. For a linear time-invariant system, $\boldsymbol{u}_i(t)$ and $\lambda_i(t)$ are constants and, as a consequence, the left hand side of (3.7) becomes zero. Hence, the classical eigenvalue problem results. As we argued earlier, this justifies to call $\lambda_i(t)$ a *dynamic* eigenvalue and $\boldsymbol{u}_i(t)$ a *dynamic* eigenvector [3, 4].
Now, it is clear that

$$\mathbf{U}(t) = [\, \boldsymbol{u}_1(t), \dots, \boldsymbol{u}_n(t)\,] \tag{3.8}$$

is a transformation matrix that transforms with

$$\boldsymbol{y}(t) = \mathbf{U}(t)\boldsymbol{z}(t) \tag{3.9}$$

the homogeneous equation (3.3) into

$$\dot{\boldsymbol{z}}(t) = \begin{bmatrix} \lambda_1(t) & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & \dots & \lambda_n(t) \end{bmatrix} \boldsymbol{z}(t) \quad . \tag{3.10}$$

Hence, the fundamental matrix $\boldsymbol{\Phi}$ of (2.1) is given by

$$\boldsymbol{\Phi}(t) = \mathbf{R}(t)\mathbf{U}(t) \begin{bmatrix} e^{\gamma_1(t)} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & \dots & e^{\gamma_n(t)} \end{bmatrix} \quad . \tag{3.11}$$

# 4 An alternative configuration

Finally, consider the filter topology of Figure 4.

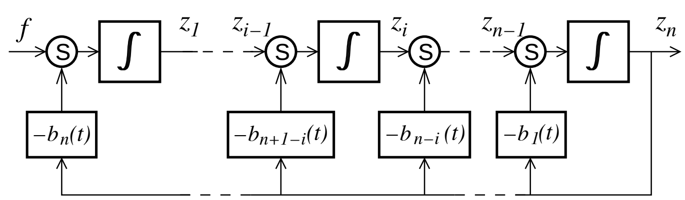


Figure 4: An alternative configuration.

We find the set of associated equations as

$$\dot{z}_1 = -b_n z_n + f \quad , \tag{4.1}$$

$$\dot{z}_i = z_{i-1} - b_{n+1-i} z_n \quad (i = 2, \dots, n) \quad . \tag{4.2}$$

Equation (4.2) yields after $(i-1)$ differentations

$$D^i z_i = D^{i-1} z_{i-1} - D^{i-1}(b_{n+1-i} z_n) \quad (i = 2, \dots, n) \quad . \tag{4.3}$$

Adding all equations in (4.3), we obtain

$$D^n z_n + \sum_{l=1}^{n} D^{n-l}(b_l z_n) = f \quad . \tag{4.4}$$

Since the Leibniz-rule of differentiation gives

$$D^{n-l}(b_l z_n) = \sum_{k=0}^{n-l} \binom{n-l}{k} D^{n-l-k} b_l D^k z_n \quad , \tag{4.5}$$

equation (4.4) can be rewritten as

$$D^n z_n + \sum_{k=0}^{n-1} \left[ \sum_{l=1}^{n-k} \binom{n-l}{k} D^{n-l-k} b_l \right] D^k z_n = f \quad . \tag{4.6}$$

We now conclude that if

$$a_{n-l} = \sum_{l=1}^{n-k} \binom{n-l}{k} D^{n-l-k} b_l \tag{4.7}$$

and

$$z_n = x \quad , \tag{4.8}$$

then (4.6) is equalent to (2.1). Finally, it is observed that for LTI systems equation (4.7) reduces to

$$a_{n-k} = b_{n-k} \tag{4.9}$$

Then, and only then, the realizations in Figure 1 and Figure 3 are equivalent.

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