Data driven local coordinates

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Abstract

In this paper we study a rather novel parametrization for state-space systems: data driven local coordinates (DDLC) as introduced in [10]. We provide some insights into the geometry and topology of the DDLC construction and show a number of results for this parametrization which are also important for actual computations using DDLC.

1 Introduction

In this paper a novel parametrization for classes of linear systems is analyzed: *Data driven local coordinates* (DDLC) have been introduced by [10] and are claimed to be advantageous from a numerical point of view. Similar ideas can be found in [13] in an LFT-type parametrization setting. We will provide a theorem stating the main topological and geometrical results for this parametrization. This will be in the spirit of [5] and [11], summarizing and extending the investigation given in the above mentioned papers.

We will be concerned with linear, time invariant, discrete-time stochastic state-space systems of the following form

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + K\varepsilon_t \\ y_t &= Cx_t + Du_t + \varepsilon_t \end{aligned} \tag{1.1}$$

Here, x_t is the n-dimensional state vector which is not directly observed in general, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{s\times n}$, $D \in \mathbb{R}^{s\times m}$ and $K \in \mathbb{R}^{n\times s}$ are parameter matrices; y_t and u_t are the observed s-dimensional outputs and the m-dimensional exogenous inputs, respectively. In addition, (ε_t) is a s-dimensional white noise process, i.e. $\mathbb{E}\varepsilon_t = 0$ and $\mathbb{E}\varepsilon_s\varepsilon_t' = \delta_{s,t}\Sigma$, $\Sigma > 0$ for all $s, t \in \mathbb{Z}$.

The *impulse reponse* of the linear system (1.1) is given by the sequence of so called *Markov parameters*

$$(L_j, K_j)_{j\in\mathbb{N}} = ((D, I), C(B, K), CA(B, K), CA^2(B, K), \ldots)$$

The corresponding *transfer function* from (u_t, ε_t) to y_t is given by

$$(l(z), k(z)) = C(z^{-1}I - A)^{-1}(B, K) + (D, I) \tag{1.2}$$

where z denotes a complex variable. For $|z|$ sufficiently small, $(l(z), k(z))$ coincides with its power series expansion $\sum_{j=1}^{\infty} CA^{j-1}(B, K)z^j + (D, I)$. Usually, we will assume $k(z)$ to be stable and strictly minimum phase, but the results of this paper do not depend on this assumption.

1.1 Spaces of transfer functions

Let U_A be the set of all rational and causal $s \times (m+s)$ transfer functions of the form (1.2) for arbitrary state dimension n. U_A is endowed with the so called pointwise topology which corresponds to the relative topology in the product space $(\mathbb{R}^{s\times(m+s)})^{\mathbb{N}}$ for the coefficients $((L_j, K_j)|j \in \mathbb{N})$.

U_A is infinite dimensional and may be broken into finite dimensional bits U_α, $\alpha \in I$, say. Usually these bits are described by a subset of an Euclidean space (to be more precise, by the set of free parameters). We will consider the case where the sets U_α are subsets of the class $\mathbb{M}(n)$ of rational and causal $s \times (m+s)$ transfer functions of fixed McMillan degree n. Clearly, $\mathbb{M}(n) \subset U_A = \cup_{i\in\mathbb{N}}\mathbb{M}(i)$.

It is well known that $\mathbb{M}(n)$ is a real analytic manifold of dimension $2ns + m(n+s)$; see e.g. [7]. $\bar{\mathbb{M}}(n)$ denotes the closure of $\mathbb{M}(n)$ in U_A and satisfies: $\bar{\mathbb{M}}(n) = \cup_{i\leq n}\mathbb{M}(i)$.

Note that $\mathbb{M}(n)$ consists of $n+1$ pathwise connected components in the SISO case ($s = 1$, $m = 0$) and is pathwise connected otherwise; see [2] and [6]. Finally, note that the same holds true if we restrict ourselves to stable systems (see [8]) or to stable and strictly minimum phase systems (see [4]).

1.2 Spaces of state-space realizations

The set of all state-space systems (A, B, C, D, K) for fixed m and s, but *variable* n is denoted by S_A. If n is fixed too, we denote the corresponding set of state-space systems (A, B, C, D, K) by $S(n) \subset S_A$. In the following, we always identify (A, B, C, D, K) with

$$\begin{pmatrix} vec(A) \\ vec(\tilde{B}) \\ vec(C) \\ vec(D) \end{pmatrix}$$

where $\tilde{B} = (B, K)$ and $vec(.)$ stacks the first, second, etc. column of the matrix argument on top of each other. Note that for $m = 0$, i.e. in the case where no exogenous inputs are present and the state-space system is given by (A, K, C, I), this embedding degenerates to $(vec(A)', vec(K)', vec(C)')'$.

$S(n)$ is endowed with the Euclidean norm for (A, B, C, D, K).

Finally, we introduce the set $S_m(n) = \{(A, B, C, D, K) \in S(n)|(A, B, C, D, K) \text{ is minimal } \}$.

1.3 The mapping π

By π the mapping attaching transfer functions to state-space matrices is denoted:

$$\begin{array}{cccc} \pi : & S_n & \rightarrow & U_A \\ & (A,B,C,D,K) & \mapsto & C(z^{-1}I-A)^{-1}(B,K)+(D,I) \qquad n \in \mathbb{N} \end{array}$$

Note that π is actually a family of mappings because the domain of definition is different for different n. However, we will neglect this fact in the sequel and speak of π as one mapping for some (arbitrary) $n \in \mathbb{N}$. It is evident that π is continuous.

It is well known that for every $(l,k) \in \mathbb{M}(n)$, the (l,k)-equivalence class of minimal systems, i.e. the inverse image $\pi^{-1}(l,k)$ in $S_m(n)$, is of the form

$$\left\{(TAT^{-1}, T\tilde{B}, CT^{-1}, D), \quad T \in GL(n)\right\} \tag{1.3}$$

where $GL(n)$ denotes the set of non singular $n \times n$ matrices and (A,B,C,D,K) is any minimal realization of (l,k), i.e. $\pi(A,B,C,D,K) = (l,k)$. This set constitutes a n^2 dimensional real analytic manifold consisting of two disconnected components; see [11]. Using the relation

$$vec(XYZ) = Z' \otimes X vec(Y) \text{ where } X \otimes Y = \begin{pmatrix} X_{11}Y & \dots & X_{1q}Y \\ \vdots & & \vdots \\ X_{p1}Y & \dots & X_{pq}Y \end{pmatrix} \tag{1.4}$$

with $X \in \mathbb{R}^{p \times q}$ and $Y \in \mathbb{R}^{r \times s}$, the vectorized form of (1.3) becomes

$$\left\{ \underbrace{\begin{pmatrix} T^{-1'} \otimes T & 0 & 0 & 0 \\ 0 & I_{m+s} \otimes T & 0 & 0 \\ 0 & 0 & T^{-1'} \otimes I_s & 0 \\ 0 & 0 & 0 & I_m \otimes I_s \end{pmatrix}}_{T_{vec} \in \mathbb{R}^{n^2+2ns+m(n+s) \times n^2+2ns+m(n+s)}} \cdot \begin{pmatrix} vec(A) \\ vec(\tilde{B}) \\ vec(C) \\ vec(D) \end{pmatrix}, \quad T \in GL(n) \right\} \tag{1.5}$$

Note that T_{vec} has full rank for all $T \in GL(n)$ as $rk(A \otimes B) = rk(A) \cdot rk(B)$.

1.4 A few basics in real algebraic geometry

An *algebraic subset of* $\mathbb{R}^d$ is the set of zeros of some polynomial set $\mathcal{B} \subset \mathbb{R}[x_1, \dots, x_d]$, where $\mathbb{R}[x_1, \dots, x_d]$ denotes the polynomial ring with coefficients in $\mathbb{R}$. *Semi-algebraic subsets of* $\mathbb{R}^d$ are subsets of $\mathbb{R}^d$ of the form $\cup_{i=1}^{k} \cap_{j=1}^{r_k} \{x \in \mathbb{R}^d | f_{i,j} *_{i,j} 0\}$ where $f_{i,j} \in \mathbb{R}[x_1, \dots, x_d]$ and $*_{i,j}$ is either $<$ or $=$, for all i, j. Clearly, every algebraic subset is also semi-algebraic.

The following result can be found in [1]; see theorem 2.3.6:

Theorem 1.1. *Every semi-algebraic subset of* $\mathbb{R}^d$ *is the disjoint union of a finite number of semi-algebraic sets, each of them being semi-algebraically homeomorphic to an open hypercube* $(0,1)^l \subset \mathbb{R}^l$ *for some* $l \in \mathbb{N}$ *(with* $(0,1)^0$ *being a point).*

The dimension $dim(A)$ of a semi-algebraic set A can be defined algebraically. From the theorem above we know that each semi-algebraic subset A of $\mathbb{R}^d$ can be written as $A = \cup_{i=1}^{p} A_i$ where each A_i is semi-algebraically homeomorphic to an open hypercube $(0,1)^{l_i}$. It can be shown that the dimension $dim(A)$ coincides with $\max(l_1, \ldots, l_p)$; see corollary 2.8.9 in [1]. Moreover, if the semi-algebraic set is a real analytic submanifold of $\mathbb{R}^d$ of dimension r (with the usual definition of dimension for real analytic manifolds, i.e. for each point on the real analytic manifold there exists an open neighborhood being diffeomorphic to an open subset of $\mathbb{R}^r$), then the two dimension concepts agree: $dim(A) = r$; see proposition 2.8.14 in [1].

1.5 Organization of the paper

The paper is organized as follows: In section 2, DDLC is briefly introduced. Section 3 then gives the main theorem together with remarks on their practical relevance. The proof then comes in section 4. Finally, in section 5 a short summary is given to conclude this contribution.

2 Data driven local coordinates

A possible approach to parametrize transfer functions in $\mathbb{M}(n)$ is the following: Take $S_m(n)$ as a parameter space, i.e. consider all matrix entries of minimal state-space representations (A, B, C, D, K) to be free parameters. This "full state space parametrization" has certain drawbacks, however: As has been mentioned above, for any given transfer function $(l, k) \in \mathbb{M}(n)$, the corresponding (l, k)-equivalence class in $S_m(n)$ is a real analytic manifold of dimension n^2. This in turn means that there are n^2 essentially unnecessary coordinates when using the full state-space parametrization.

The idea now is to avoid this drawback by only considering the $2ns+m(n+s)$ dimensional ortho-complement to the tangent space to a certain (l, k)-equivalence class in $S_m(n)$ at a given (A, B, C, D, K) as a parameter space. Here, (A, B, C, D, K) is obtained by some initial estimate, and this is the reason for calling the parametrization *data driven local coordinates*. Clearly, the parameter space will then be of dimension $2ns + m(n + s)$ rather than $n^2 + 2ns + m(n + s)$ and thus has no unnecessary coordinates.

The construction of the tangent space to the n^2 dimensional equivalence class (1.3) is obtained by differentiation. The tangent space is given by (see section 5.6 in [9])

$$\{(\dot{T}AT^{-1} - TAT^{-1}\dot{T}T^{-1}, \dot{T}\tilde{B}, -CT^{-1}\dot{T}T^{-1}, 0), \quad \dot{T} \in \mathbb{R}^{n\times n}\} \tag{2.6}$$

Using (1.4) together with $(A \otimes B)(C \otimes D) = (AC \otimes BD)$, this can be vectorized to yield

$$\left\{ \begin{pmatrix} T^{-1'} A' \otimes I_n - T^{-1'} \otimes TAT^{-1} \\ \tilde{B}' \otimes I_n \\ -T^{-1'} \otimes CT^{-1} \\ 0_{sm \times n^2} \end{pmatrix} \cdot vec(\dot{T}), \quad \dot{T} \in \mathbb{R}^{n \times n} \right\} = \tag{2.7}$$

$$\left\{ \begin{pmatrix} T^{-1'} \otimes T & 0 & 0 & 0 \\ 0 & I_{m+s} \otimes T & 0 & 0 \\ 0 & 0 & T^{-1'} \otimes I_s & 0 \\ 0 & 0 & 0 & I_m \otimes I_s \end{pmatrix} \underbrace{\begin{pmatrix} A' \otimes I_n - I_n \otimes A \\ \tilde{B}' \otimes I_n \\ -I_n \otimes C \\ 0_{sm \times n^2} \end{pmatrix}}_{Q} (I_n \otimes T^{-1}) \cdot vec(\dot{T}), \quad \dot{T} \in \mathbb{R}^{n \times n} \right\}$$

At $T = I$, i.e. at the given minimal realization (A, B, C, D, K), (2.6) reduces to

$$\left\{ (\dot{T}A - A\dot{T}, \dot{T}\tilde{B}, -C\dot{T}, 0), \quad \dot{T} \in \mathbb{R}^{n \times n} \right\} \tag{2.8}$$

and (2.7) becomes

$$\left\{ Q \cdot vec(\dot{T}), \quad \dot{T} \in \mathbb{R}^{n \times n} \right\} \tag{2.9}$$

The matrix $Q \in \mathbb{R}^{n^2+2ns+m(n+s)\times n^2}$ has full column rank n^2 for any minimal (A, B, C, D, K). In fact, it is rank deficient if and only if (A, B, C, D, K) becomes both unobservable *and* uncontrollable (see [10]). The columns of Q span the tangent space to the equivalence class at (A, B, C, D, K) and by $Q^\perp$ we denote a matrix the columns of which span the orthogonal complement to the tangent space given above. $Q^\perp$ can be obtained e.g. from a singular value decomposition of Q, and the parametrization is then obtained as follows:

Definition 2.1 (Data driven local coordinates (DDLC)). *Let a* minimal (A, B, C, D, K) *be given. The DDLC are given by the mapping*

$$\begin{aligned} \varphi_D : T_D &\rightarrow S_m(n) \\ \tau_D &\mapsto \begin{pmatrix} vec(A(\tau_D)) \\ vec(\tilde{B}(\tau_D)) \\ vec(C(\tau_D)) \\ vec(D(\tau_D)) \end{pmatrix} = \begin{pmatrix} vec(A) \\ vec(\tilde{B}) \\ vec(C) \\ vec(D) \end{pmatrix} + Q^\perp \tau_D \end{aligned} \tag{2.10}$$

Here, $T_D \subset \mathbb{R}^{2ns+m(n+s)}$ denotes the parameter space for DDLC, i.e. the set of all $\tau_D \in \mathbb{R}^{2ns+m(n+s)}$ such that $\varphi_D(\tau_D)$ is minimal. Let $V_D = \pi(\varphi_D(T_D))$.

Remark 2.1. *For any fixed minimal (A, B, C, D, K), the mapping φ_D from the parameter vectors τ_D to the state-space matrices is affine (and therefore continuous and analytic) as can be seen from (2.10). Clearly, $\varphi_D(\bar{T}_D) = \overline{\varphi_D(T_D)}$, and in the sequel* we will use the symbol $\pi(T_D) = \pi(\varphi_D(T_D))$ with slight abuse of notation.

3 Geometry and topology of DDLC

Before we state the main theorem of this section, we want to discuss a special case. This will be done in order to motivate and (hopefully) clarify the results of the theorem below.

3.1 An illustrative example

Consider the case where $n = s = 1$ and $m = 0$. Here, no exogenous inputs are present and thus $l(z)$ vanishes. Clearly, $S(1) = \mathbb{R}^3$ and $S_m(1) = \{(a,b,c)' \in \mathbb{R}^3 | (a,b,c) \text{ is minimal } \}$. Note that for a given $(a,b,c) \in S_m(1)$ corresponding to $k(z) \in \mathbb{M}(1)$, the scalar a is unique and the corresponding k-equivalence class in $S_m(1)$ is a hyperbola (with two branches) which is determined by a fixed a and $bc = const$; see, e.g. the thick line in figure 1. Non minimal systems (corresponding to the trivial transfer function $k(z) = 0$) are represented by the union of the planes $b = 0$ and $c = 0$, respectively. Finally, note that $\mathbb{M}(1)$ consists of two pathwise connected components, each of which corresponds to hyperbolae in $S_m(1)$ where $sign(bc) = +1$ and $sign(bc) = -1$, respectively.

Commencing from $(a,b,c) \in S_m(1)$ we have $Q = (0, b, -c)'$ and thus the columns of $Q^\perp$ may be chosen as $(1,0,0)'$ and $(0,c,b)'$. If we decide to choose a particular orthonormal basis, the DDLC parametrization is given by

$$\begin{pmatrix} a(\tau_D^1, \tau_D^2) \\ b(\tau_D^1, \tau_D^2) \\ c(\tau_D^1, \tau_D^2) \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \frac{c}{\sqrt{b^2+c^2}} \\ 0 & \frac{b}{\sqrt{b^2+c^2}} \end{pmatrix} \begin{pmatrix} \tau_D^1 \\ \tau_D^2 \end{pmatrix} \tag{3.11}$$

Here, $T_D = \{(\tau_D^1, \tau_D^2)' \in \mathbb{R}^2 | (a(\tau_D^1, \tau_D^2), b(\tau_D^1, \tau_D^2), c(\tau_D^1, \tau_D^2)) \text{ is minimal } \}$ and $\varphi_D(T_D)$ is a subset of the whole affine plane in $S(1)$ given by $(a,b,c)' + Q^\perp \tau_D$. Starting with an initial system $(a,b,c) = (a,b,\pm b)$, $\varphi_D(T_D)$ becomes a subset of the whole plane given by $b = \pm c$. In the sequel, we will call the affine subspace (plane) containing $\varphi_D(T_D)$ the *affine subspace (plane) corresponding to* T_D.

From figure 1 the following *geometrical and topological properties* of DDLC can be seen:

(i) The affine subspace corresponding to T_D intersects the planes given by $b = 0$ and $c = 0$ yielding two straight lines given by

$$\begin{pmatrix} a(\xi) \\ b(\xi) \\ c(\xi) \end{pmatrix} = \begin{pmatrix} a + \xi \\ b - \frac{c^2}{b} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a(\xi) \\ b(\xi) \\ c(\xi) \end{pmatrix} = \begin{pmatrix} a + \xi \\ 0 \\ c - \frac{b^2}{c} \end{pmatrix}, \quad \xi \in \mathbb{R} \tag{3.12}$$

In case of an initial system of the form $(a,b,\pm b)$, this intersection becomes the a-axis only. In any case, T_D *is seen to be an open and dense subset of* $\mathbb{R}^2$.

(ii) There exists a neighborhood T_D^{loc} of $(0,0) \in T_D$ (containing the initial system (a,b,c)), such that *each hyperbola entering a (sufficiently small) neighborhood of* (a,b,c) *in* $\mathbb{R}^3$

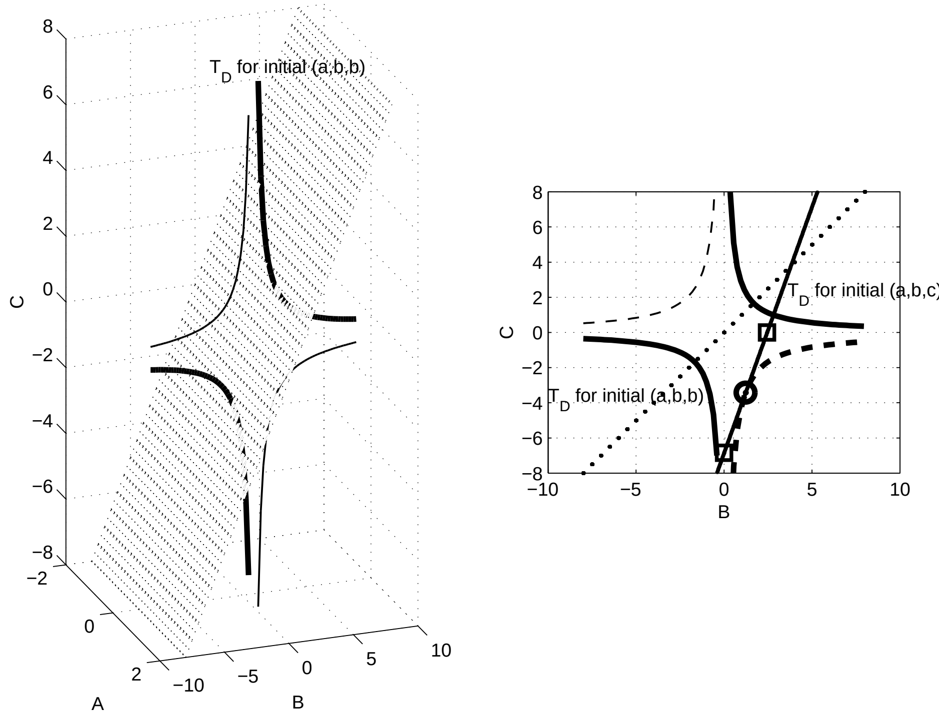

Figure 1: Equivalence classes and parametrized manifolds in $S(1) = \mathbb{R}^3$ and projection onto the (b, c)-plane. On the left hand side, T_D is shown for initial systems of the form (a, b, b). On the right hand side, T_D for initial systems (a, b, b) – see the dotted straight line – and T_D for an initial system (a, b, c) with $|b| \neq |c|$ – see the thick straight line – are indicated.

also intersects the affine plane corresponding to T_D and the intersection yields one single point in T_D^{loc}. Moreover, no non minimal system is described in T_D^{loc}.

(iii) The boundary points of T_D (which do not belong to T_D) represent the trivial transfer function $k(z) = 0$: $\pi(\bar{T}_D)$ *contains lower degree transfer functions.*

(iv) These boundary points of T_D are given by the two straight lines in (3.12) or by the a-axis in case of an initial system of the form $(a, b, \pm b)$. They constitute one equivalence class, which is described by the nonlinear equation $bc = 0$ and another set of linear equations restricting the points to be in the affine plane corresponding to T_D.

Within T_D, the k-equivalence classes consist of two elements except for the points where T_D touches a hyperbola which gives a singleton. These touching points – see the circle on the right hand side of figure 1 – constitute a straight line and are given by:

$$\begin{pmatrix} a(\xi) \\ b(\xi) \\ c(\xi) \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \frac{b}{c} \end{pmatrix} \cdot \begin{pmatrix} \xi \\ -\frac{c^2+b^2}{2b} \end{pmatrix}, \quad \xi \in \mathbb{R} \tag{3.13}$$

The touching points do not occur if the initial system is of the form $(a, b, \pm b)$. In any case, we have a lack of global identifiability of T_D, but *every transfer function in* V_D *has just a finite number of representations within* T_D.

(v) V_D *is clearly open in* $\pi(\bar{T}_D)$. However, note that V_D *is not necessarily open in* $\mathbb{M}(1)$*:* The transfer functions corresponding to the points where hyperbolae touch the affine plane corresponding to T_D are boundary points of V_D which belong to V_D. This is a direct consequence of the fact that in any neighborhood of such a touching point there are hyperbolae which do not touch or intersect the affine plane corresponding to T_D. Hence, we can find a sequence of minimal realizations $(a_n, b_n, c_n) \notin \varphi_D(T_D), n \in \mathbb{N}$ converging to the touching point and continuity of π shows the statement. Note that *an inner point of* T_D *in this case corresponds to a boundary points of* V_D *and that at these points local identifiability is also violated.*

Remark 3.1. *Note that iterating the DDLC construction commencing from an initial* (a, b, b)*, (e.g. in course of a gradient-type search procedure) leaves* V_D *unchanged. This can be seen from the dotted straight line on the right hand side of figure 1: For any minimal system in the affine plane corresponding to* T_D*, the ortho-complement to its equivalence class would coincide with the "old" affine plane and therefore we could never leave the first and third quadrant and would miss all hyperbolae in the second and fourth quadrant. If, however, we start from an initial* (a, b, c) *where* $|b| \neq |c|$*, then we could reach any hyperbola in the second step, i.e. by applying the DDLC construction again for some system in the affine plane corresponding to the "old"* T_D. In any case, for an arbitrarily given initial $(a, b, c) \in S_m(1)$ it is guaranteed that we reach any transfer function $k(z)$ in the same pathwise connected component of $\mathbb{M}(1)$ as $\pi(a, b, c)$ by applying the DDLC construction finitely many times.

Remark 3.2. *It is evident that the affine plane corresponding to* T_D *does not intersect "almost all" hyperbolae. Hence, the corresponding* V_D leaves out more than just a thin subset of $\mathbb{M}(1)$*, i.e.* $\mathbb{M}(1) \setminus V_D$ *always contains an open set.*

Remark 3.3. *Note that* V_D *is strongly dependent on the particular initial system in the equivalence class where the DDLC construction is performed;* V_D *is smallest for the case of choosing a representative of the form* $(a, b, \pm b)$ *because we miss all hyperbolae corresponding to systems* (a, b, c) *with the opposite sign of* bc.

3.2 The main theorem

The following theorem holds true for arbitrary n, s and m:

Theorem 3.1. *Assume that the initial system* (A, B, C, D, K) *is* minimal. *The parametrization by DDLC as given in (2.10) has the following properties:*

(i) T_D *is an open and dense subset of* $\mathbb{R}^{2ns+m(n+s)}$.

(ii) There exist open neighborhoods T_D^{loc} of $0 \in T_D$ and V_D^{loc} of $\pi(A, B, C, D, K)$ in $\mathbb{M}(n)$ such that T_D^{loc} is identifiable, $V_D^{loc} = \pi(T_D^{loc})$ and the mapping $\psi_D^{loc} : V_D^{loc} \to T_D^{loc}$ defined by $\psi_D^{loc}(\pi(\tau_D)) = \tau_D$ is a homeomorphism.

(iii) $\pi(\bar{T}_D)$ contains transfer functions of lower McMillan degree.

(iv) For "almost every" $(l, k) \in V_D$, the (l, k)-equivalence class in T_D consists of a finite number of isolated points.

(v) V_D is open (and trivially dense) in $\pi(\bar{T}_D)$, but not necessarily open in $\mathbb{M}(n)$.

We give the following remarks:

(i) This means that the parameters are really *free* in the sense that they are not restricted to a thin subset of $\mathbb{R}^{2ns+m(n+s)}$. More specifically, *"almost any"* point in $\mathbb{R}^{2ns+m(n+s)}$ corresponds to a transfer function in $\mathbb{M}(n)$. Note that this is an important requirement for many numerical optimization procedures to work properly.

(ii) The set V_D contains an open subset V_D^{loc} of $\mathbb{M}(n)$ which contains the transfer function $\pi(A, B, C, D, K)$. Additionally, (ii) assures bijectivity and continuity of the parametrization on the piece V_D^{loc}: The estimation problem is *locally well posed* in the sense that consistency of the transfer function estimates in V_D^{loc} (coordinate free consistency; see [7]) implies consistency of the parameter estimates in T_D^{loc}.

(iii) In the closure of the parameter space T_D – note that $\bar{T}_D = \mathbb{R}^{2ns+m(n+s)}$ by (i) – we can always describe transfer functions of equal and lower McMillan degree.

(iv) First, note that the set $\bar{T}_D \setminus T_D$ only consists of parameters corresponding to lower degree transfer functions (l, k). The corresponding (l, k)-equivalence classes are not easy to describe. This can be illustrated by considering, e.g. the equivalence class of the trivial transfer function $(l, k) = (D, I)$. Here, $0 = C\tilde{B} = CA\tilde{B} = \cdots = CA^{2n-1}\tilde{B}$ must hold true, and the corresponding subset of $S(n)$ additionally has to be intersected with the affine plane corresponding to T_D. Clearly, the first restrictions are nonlinear.

For the case of parameters in T_D, the corresponding (l, k)-equivalence classes are in almost any case guaranteed to consist of a finite number of isolated points. For criteria functions, which are constant along (l, k)-equivalence classes and have a unique minimum, it is thus obvious that this minimum will be attained at any of the corresponding (finitely many) points in the parameter space. The fact that these points are separated is of importance, e.g. when using gradient type algorithms, and for aymptotic theory.

(v) states that the set V_D is always open in $\pi(\bar{T}_D)$. This is important in connection with coordinate free consistency: If a sequence of, e.g. maximum likelihood estimates $(l_t, k_t) \in \pi(\bar{T}_D)$ satisfies $(l_t, k_t) \to (l, k) \in V_D$, then $(l_t, k_t) \in V_D$ from a certain $t \geq T_0$ onwards, i.e. for $t \geq T_0$ the transfer functions (l_t, k_t) do have representatives within T_D. However, V_D need not be open in $\mathbb{M}(n)$.

4 Proof of the main theorem

Before we give the proof of theorem 3.1, let us introduce the mapping

$$\begin{array}{rcl} F: GL(n) \times T_D & \rightarrow & S_m(n) \\ (T, \tau_D) & \mapsto & \begin{pmatrix} vec(TA(\tau_D)T^{-1}) \\ vec(T\tilde{B}(\tau_D)) \\ vec(C(\tau_D)T^{-1}) \\ vec(D(\tau_D)) \end{pmatrix} = T_{vec} \left[\begin{pmatrix} vec(A) \\ vec(\tilde{B}) \\ vec(C) \\ vec(D) \end{pmatrix} + Q^{\perp}\tau_D \right] \end{array} \quad (4.14)$$

Note that $\frac{\partial F}{\partial T}(T, \tau_D)$ describes the tangent space to the equivalence at $(A(\tau_D), \tilde{B}(\tau_D), C(\tau_D), D(\tau_D))$ and has already been computed in (2.7). Moreover, $\frac{\partial F}{\partial \tau_D}(T, \tau_D) = T_{vec} \cdot Q^{\perp}$ is straightforward to see. Hence,

$$\frac{\partial F}{\partial (T, \tau_D)}(T, \tau_D) = T_{vec} \cdot \underbrace{\begin{pmatrix} A(\tau_D)' \otimes I_n - I_n \otimes A(\tau_D) & \\ B(\tau_D)' \otimes I_n & \vdots \;\; Q^{\perp} \\ -I_n \otimes C(\tau_D) & \\ 0_{sm \times n^2} & \end{pmatrix}}_{X(\tau_D)} \cdot \begin{pmatrix} I_n \otimes T^{-1} & 0 \\ 0 & I_{2ns+m(n+s)} \end{pmatrix} \quad (4.15)$$

Note that F is linear in τ_D and rational in T, and thus a real analytic function. In fact, it is a real analytic mapping between real analytic manifolds: Its domain of definition is an open (and dense) subset of $\mathbb{R}^{n \times n} \times \mathbb{R}^{2ns+m(n+s)} \equiv \mathbb{R}^{n^2+2ns+m(n+s)}$ because the set $GL(n)$ is open (and dense) in $\mathbb{R}^{n \times n}$ and T_D is open (and dense) in $\mathbb{R}^{2ns+m(n+s)}$ by statement (i). The image space $S_m(n)$ can also be shown to be open (and dense) in $S(n) = \mathbb{R}^{n^2+2ns+m(n+s)}$. Trivially, open sets of Euclidean spaces are real analytic manifolds. Note, however, that not every point in $S_m(n)$ is an image point of some (T, τ_D); see figure 2.

Let us call a point $(\bar{T}, \bar{\tau}_D) \in GL(n) \times T_D$ a *regular point* if the Jacobian $\frac{\partial F}{\partial (T, \tau_D)}(\bar{T}, \bar{\tau}_D)$ has full rank. We will call $b = (vec(A)', vec(\tilde{B})', vec(C)', vec(D)')' \in S_m(n)$ a *regular value* of F if every point in $F^{-1}(b)$ is regular (and, in particular, if $F^{-1}(b) = \varnothing$). Non regular points and non regular values are said to be singular points and singular values, respectively.

We are now ready to prove theorem (3.1):

Proof. (i) We consider the mapping $\Delta : \mathbb{R}^{2ns+m(n+s)} \rightarrow \mathbb{R}$ attaching $det(W_o^n(\tau_D) W_c^n(\tau_D))$ to τ_D where $W_o^n(\tau_D) = \mathcal{O}_n(\tau_D)' \mathcal{O}_n(\tau_D) \in \mathbb{R}^{n \times n}$ with

$$\mathcal{O}_n(\tau_D) = \mathcal{O}_n(A(\tau_D), B(\tau_D), C(\tau_D), D(\tau_D), K(\tau_D)) = \mathcal{O}_n(\varphi_D(\tau_D))$$

being the corresponding (finite) observability matrix. The Gramian $W_c^n(\tau_D)$ is given analogously. Note that $W_o^n(\tau_D)$ and $W_c^n(\tau_D)$ have full rank if and only if $\varphi_D(\tau_D)$ is

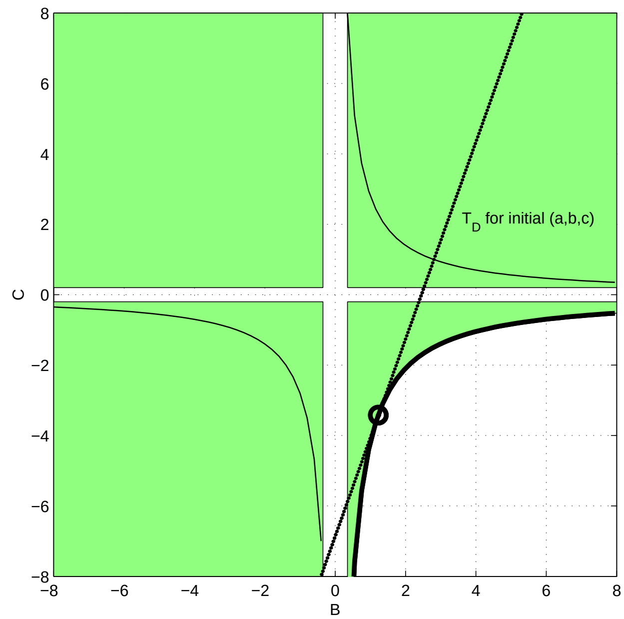

Figure 2: The dark area indicates the projection of the image set $F(GL(n) \times T_D)$ onto the (b, c)-plane. Clearly, the b-axis and the c-axis are not included in this set. Note that $F(GL(n) \times T_D)$ is not open in this case.

a minimal state-space realization. Due to the fact that φ_D is affine, the determinant of $W_o^n(\tau_D)W_c^n(\tau_D)$ is a polynomial in the parameters $\tau_D^i, i = 1, \ldots, 2ns + m(n+s)$ and thus analytic (and trivially continuous). Openness of T_D in $\mathbb{R}^{2ns+m(n+s)}$ follows from the fact that $T_D = \Delta^{-1}(\mathbb{R} \setminus \{0\})$ is the inverse image of an open set and Δ is continuous. Denseness of T_D in $\mathbb{R}^{2ns+m(n+s)}$ follows from a well known result for analytic functions: $\Delta(\tau_D) = 0$ can only hold true on a thin subset of $\mathbb{R}^{2ns+m(n+s)}$ (Δ cannot vanish everywhere in $\mathbb{R}^{2ns+m(n+s)}$).

(ii) We show that for a sufficiently small open neighborhood T_D^{loc} of $0 \in T_D$ the corresponding $\pi(T_D^{loc}) = V_D^{loc}$ is a nonvoid open subset of $\mathbb{M}(n)$ and the mapping $\pi|_{T_D^{loc}}$ is a homeomorphism. This implies, of course, that $\psi_D^{loc} = \left(\pi|_{T_D^{loc}}\right)^{-1}$ is a homeomorphism from V_D^{loc} onto T_D^{loc}, too.

A detailed proof of this statement is given in [11]. The main idea is to show local injectivity of the mapping F in (4.14) at the point $(I, 0)$: The Jacobian of the mapping F is given in (4.15) and has *constant and full rank* in a neighborhood of $(I, 0)$. This is easy to see as $\frac{\partial F}{\partial(T,\tau_D)}(I, 0) = [Q \vdots Q^\perp]$ (which has full rank), the derivative depends continuously on both T and τ_D and the determinant is a continuous function of the matrix entries. Note that $\pi|_{T_D^{loc}} = \pi \circ F|_{GL(n)\times T_D^{loc}}$ is, *independent of the choice of* $T \in GL(n)$

– clearly continuous.

– an open mapping because both $F|_{GL(n)\times T_D^{loc}}$ and π are open mappings. This is shown in [11], however, note that F is not globally open as has been erroneously stated there. Thus $\pi(T_D^{loc}) = V_D^{loc}$ is open in $\mathbb{M}(n)$.

– bijective when considered as a function from T_D^{loc} to V_D^{loc} because of the injectivity of $F|_{(GL(n)\times T_D^{loc})}$.

Hence, $\pi|_{T_D^{loc}}$ is a homeomorphism. This is valid for all parametrizations (not necessarily affine ones) around a minimal (A, B, C, D, K) fulfilling the "constant and full rank condition" above.

(iii) We have proved in [12] that the trivial system $(l, k) = (0, 0)$ can always be represented in $\bar{T}_D$. The proof is rather unelegant and lengthy, and is therefore not included here due to limitations of space.

(iv) Let us consider the mapping F in (4.14).

First, note the following: From a well known result (see, e.g. lemma 5.9 in [3]) we know that for any regular value $b \in S_m(n)$ of F the corresponding set $F^{-1}(b)$ is a real analytic submanifold of $GL(n) \times T_D$ of dimension zero. Note that this submanifold is given by the set of equations $F(T, \tau_D) - b = 0$ which can be transformed into a set of polynomial equations by multiplication by $det(T)$, where $det(T) \neq 0$ by assumption. Thus, $F^{-1}(b)$ is in fact a (semi-) algebraic set, and the dimension is zero. Clearly, from the discussion in the introductory section (1.4), $F^{-1}(b)$ consists of a finite number of points in $GL(n) \times T_D$.

Second, from Sard's theorem (see, for instance, theorem 6.1 in [3]) we know that the set of singular values of F has Lebesgue measure zero in $S_m(n)$. It follows that for "almost all" points in $S_m(n)$ – and therefore, because of continuity of π, for "almost all" $(l, k) \in \mathbb{M}(n)$ – the (l, k)-equivalence classes in T_D consist of (at most) finitely many points in T_D. The set V_D is known to contain an open subset of $\mathbb{M}(n)$ by statement (ii), and thus it is also true that the (l, k)-equivalence classes of "almost all" $(l, k) \in V_D$ consist of (at most) finitely many points in T_D.

(v) Openness of V_D in $\pi(\bar{T}_D)$ follows from the definition of relative openness. Here $V_D = \pi(\bar{T}_D) \cap \mathbb{M}(n)$ and $\mathbb{M}(n)$ is known to be open in $\bar{\mathbb{M}}(n)$; see [7], for instance. Denseness is trivial. The fact that V_D need not necessarily be open in $\mathbb{M}(n)$ has been discussed in detail for the special case $m = 0$ and $n = s = 1$ in (v) of section (3.1).

□

5 Conclusions

In this paper a theorem summarizing the main topological and geometrical properties of DDLC is given and its relevance in connection with actual calculations (and maximum likelihood estimation) using DDLC is discussed in detail.

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