# A Comparison Between Two Robust Adaptive Controllers w.r.t a Non–singular Transient Cost

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**Abstract**

A nonsingular performance comparison between two standard robust adaptive control designs based on the dead-zone and projection modifications is given. A worst case transient cost functional penalising the $\mathcal{L}^\infty$ norm of the state, control and control derivative has been chosen as the criterion of comparison. If a bound on the $\mathcal{L}^\infty$ norm of the disturbance is known, it is shown that the dead-zone controller outperforms the projection controller when the a-priori information on the unknown system parameter is sufficiently conservative. For clarity of the presentation, the result is restricted to scalar systems and the generalisations are only briefly discussed.

## 1 Introduction

Parameter drift is a known undesirable phenomenon in adaptive control design. It can happen even when small input disturbances are present. Dead-zone and projection modifications are two standard techniques to overcome such a problem.

Both methods require some a-priori knowledge and have their own advantages and drawbacks. For example, dead-zone modifications require a-priori knowledge of the disturbance level, and only achieve convergence of the output to some pre-specified neighbourhood of the origin (whilst keeping all signals bounded). In particular if the disturbance vanishes, then the dead-zone controller does not typically achieve convergence to zero: the convergence remains to the pre-specified neighbourhood of the origin. On the other hand, projection modifications generally achieve boundedness of all signals, and furthermore have the desirable property that if no disturbances are present, then the output converges to zero, however, an arbitrarily small $\mathcal{L}^\infty$ disturbance can completely destroy any convergence of the state.

This illustrates that in the case of asymptotic performance, there are some known characterisations of 'good' and 'bad' behaviour. However, there are many situations in which we cannot definitively state whether a projection or dead-zone controller is superior even when only considering asymptotic performance. Furthermore, the known results, as with most results in adaptive control, are confined to non-singular performances, ie. without any consideration of the control signal.

Following our previous work [7], we aim to compare dead-zone and projection based adaptive controllers with respect to transient performance. Furthermore, the transient performance measure will be nonsingular (ie. penalise both the state ($x$) and the input ($u$) of the plant); specifically we will consider cost functionals of the form:

$$\mathcal{P} = \|x(\cdot)\|_{\mathcal{L}^\infty} + \|u(\cdot)\|_{\mathcal{L}^\infty} + \|\dot{u}(\cdot)\|_{\mathcal{L}^\infty}.$$

We will identify circumstances in which the dead-zone controller is superior to the projection controller w.r.t. $\mathcal{P}$.

# 2 System Description and Control Design

Consider the following class of SISO nonlinear system:

$$\Sigma(\theta, \phi, d(\cdot)) : \ \dot{x}(t) = \theta\phi(x(t)) + u(t) + d(\cdot) \qquad x(0) = x_0, \tag{2.1}$$

where $x(\cdot), u(\cdot), \theta \in \mathbb{R}$ are the state, the control, and unknown constant parameter respectively. $\phi(\cdot) \in \mathbb{R}$ is a known basis, which is taken to be locally Lipschitz, and $d \in \mathcal{D}$ is a bounded disturbance. It is well known that the controller

$$\begin{aligned} \Xi : \ & u(t) = -ax(t) - \hat{\theta}(t)\phi(x(t)) \\ & \dot{\hat{\theta}}(t) = x(t)\phi(x(t)), \qquad \hat{\theta}(0) = 0 \end{aligned} \tag{2.2}$$

stabilises system (2.1) when $\mathcal{D} = \{0\}$ and $a > 0$. This can be easily proven by defining the Lyapunov function $V(x(t), \hat{\theta}(t)) = x(t)^2/2 + (\theta - \hat{\theta}(t))^2/2$ and observing that $\dot{V}(x(t), \hat{\theta}(t)) = -ax(t)^2 \leq 0$. We say $\hat{\theta}(\cdot)$ is the adaptive estimator of $\theta$.

It has been shown that even a small $\mathcal{L}^\infty$ disturbance may cause a drift of the parameter estimates $\hat{\theta}(\cdot)$, see eg. [1]. In following we briefly describe two common robust adaptive controllers based on modifying the adaptive law, namely dead-zone modification and parameter projection modification. More details can be found in most adaptive control text books (see e.g. [5]). Let

$$\dot{\hat{\theta}}(t) = \tau(x(t), \hat{\theta}(t)), \qquad \hat{\theta}(0) = 0 \tag{2.3}$$

be the unmodified adaptation law. The idea of dead-zone [1, 5, 6] is to modify the parameter estimator so that the adaptive mechanism is 'switched off' when system trajectory $x(\cdot)$ lies inside a region $\Omega_0$ where the disturbance has a destabilising effect on the dynamics. In scalar case, the dead-zone region $\Omega_0$ is defined by $\Omega_0 = [-\eta, \eta]$ and the modified adaptive law is taken to be

$$\dot{\hat{\theta}}(t) = D_{[-\eta,\eta]}(x)\ \tau(x(t), \hat{\theta}(t)), \qquad \hat{\theta}(0) = 0, \tag{2.4}$$

where

$$D_{[-\eta,\eta]}(x) = \begin{cases} 0, & |x| \leq \eta \\ 1, & |x| > \eta \end{cases} \tag{2.5}$$

The size of the disturbance is necessary a-priori knowledge in order to define the region $[-\eta, \eta]$. Let $d_{max} > 0$ be the upper bound on disturbance $d(\cdot)$, and define the dead-zone controller

$$\begin{aligned} \Xi_D(d_{\max}):\ & u(t) = -ax(t) - \hat{\theta}(t)\phi(x(t)) \\ & \dot{\hat{\theta}} = D_{[-\eta,\eta]}(x)\ x(t)\phi(x(t)) \qquad \hat{\theta}(0) = 0, \qquad \eta = \frac{d_{max}}{a}. \end{aligned} \tag{2.6}$$

It has been shown (see e.g. [5]) that the closed loop $(\Sigma(\theta, \phi, d(\cdot)), \Xi_D(d_{\max}))$ is stable in the sense that all loop signals are bounded and $x(t) \to \Omega_0$ as $t \to \infty$.

The projection modification [1, 3] is an alternative method to eliminate parameter drift by keeping the parameter estimates within some a priori defined bounds. The general definition of the projection can be found in e.g. [4]. A simplified version of parameter projection can be obtained by defining $\Pi = [-\theta_{\max}, \theta_{\max}]$ where $\theta_{\max} > 0$ is an upper bound of $|\theta|$, and modifying the adaptive law as follows

$$\dot{\hat{\theta}} = \text{Proj}\ (\tau), \qquad \hat{\theta}(0) = 0, \tag{2.7}$$

where

$$\text{Proj}\ (\tau) = \begin{cases} \tau, & |\hat{\theta}| < \theta_{\max} \ \text{ or }\ \hat{\theta}\,\tau \leq 0 \\ 0, & |\hat{\theta}| = \theta_{\max} \ \text{ and }\ \hat{\theta}\,\tau > 0. \end{cases} \tag{2.8}$$

Consequently the projection controller $\Xi_P(\theta_{\max})$ is defined as follows:

$$\begin{aligned} \Xi_P(\theta_{\max}):\ & u(t) = -ax(t) - \hat{\theta}(t)\phi(x(t)) \\ & \dot{\hat{\theta}}(t) = \text{Proj}\ (x(t)\phi(x(t))) \qquad \hat{\theta}(0) = 0 \end{aligned} \tag{2.9}$$

In the presence of bounded disturbances, the projection controller (2.9) guarantees the boundedness of all signals in the closed loop $(\Sigma(\theta, \phi, d(\cdot)), \Xi_P(\theta_{\max}))$ (see e.g. [?]).

# 3 Statement of the Main Result

The goal of this paper is to establish a comparison between dead-zone and projection methods. We will compare the performances of the controllers with respect to the following worst case non-singular transient cost functional $\mathcal{P}(\Sigma(\theta, \phi, d(\cdot)), \Xi)$

$$\mathcal{P}(\Sigma(\theta, \phi, d(\cdot)), \Xi) = \sup_{\|d(\cdot)\|_{\mathcal{L}^\infty} \leq \epsilon}\ \sup_{|\theta| \leq \delta} \left( \|x(\cdot)\|_{\mathcal{L}^\infty} + \|u(\cdot)\|_{\mathcal{L}^\infty} + \|\dot{u}(\cdot)\|_{\mathcal{L}^\infty} \right), \tag{3.10}$$

where $\delta > 0$ . We also let $\phi$ to be taken such that

$$\begin{cases} \text{a)} \quad x = 0 \Longleftrightarrow \phi = 0, \\ \text{b)} \quad \left.\dfrac{\partial \phi}{\partial x}\right|_{x=0} \neq 0. \end{cases} \tag{3.11}$$

We are not concerned in this paper with the comparison of asymptotic performance, this has been studied previously, see eg. [5] and the references therein. The following theorem is the main result of this paper:

**Theorem 3.1.** *Let $\phi$ be such that* (3.11) *hold. Consider the system $\Sigma(\theta, \phi, d(\cdot))$ defined by* (2.1)*, and the controllers $\Xi_D(d_{\max}), \Xi_P(\theta_{\max})$ defined by equations* (2.6)*,* (2.9)*. Consider the transient performance cost functional* (3.10)*. Then $\forall d_{\max} \geq \epsilon \quad \exists \theta^*_{\max} \geq \delta$ such that $\forall \theta_{\max} \geq \theta^*_{\max} > 0$,*

$$\mathcal{P}(\Sigma(\theta, \phi, d(\cdot)), \Xi_P(\theta_{\max})) \;>\; \mathcal{P}(\Sigma(\theta, \phi, d(\cdot)), \Xi_D(d_{\max})). \tag{3.12}$$

In order to proof Theorem 3.1, firstly we show that $\mathcal{P} = \infty$ for the basic design (2.2) From this we can show that the projection modification design, $\Xi_P$, has the property that $\mathcal{P} \to \infty$ as $\theta_{\max} \to \infty$. Finally we show that $\mathcal{P} < \infty$ for the dead-zone design, $\Xi_D$, and $\mathcal{P}$ is independent of $\theta_{\max}$. This suffices to establish Theorem 3.1. In fact by this, we will prove the stronger result that the ratio between the two costs can be made arbitrarily large, i.e

$$\frac{\mathcal{P}(\Sigma(\theta, \phi, d(\cdot)), \Xi_P(\theta_{\max}))}{\mathcal{P}(\Sigma(\theta, \phi, d(\cdot)), \Xi_D(d_{\max}))} \to \infty \quad \text{as} \quad \theta_{\max} \to \infty, \qquad \forall d_{\max} \geq \epsilon.$$

The proof of the theorem uses the following propositions:

**Proposition 3.1.** *Suppose $\phi$ satisfies* (3.11)*. Consider the closed loop $(\Sigma(\theta, \phi, d(\cdot)), \Xi)$ defined by* (2.1)*,* (2.2)*, where $d(\cdot) = \epsilon$, for some $\epsilon > 0$. Then*

$$x(t) \to 0 \quad \text{as} \quad t \to \infty \Longleftrightarrow \hat{\theta}(t) \to \infty \quad \text{as} \quad t \to \infty. \tag{3.13}$$

*Proof.* $\to$ ) Suppose for contradiction $\hat{\theta}(t) \not\to \infty$. Then $\hat{\theta}(t) \to \hat{\theta}^* < \infty$ since $\hat{\theta}$ is monotonic by (3.11). Therefore $(x, \hat{\theta}) = (0, \hat{\theta}^*)$ is an equilibrium point of the closed loop. It contradicts the fact that by (3.11-a) the closed loop differential equation system:

$$\begin{aligned} -ax(t) + (\theta - \hat{\theta}(t))\phi(x(t)) + \epsilon = 0 \\ x(t)\phi(x(t)) = 0, \end{aligned} \tag{3.14}$$

has no solution. Therefore $\hat{\theta}(t) \to \infty$ as $t \to \infty$.

$\leftarrow$ ) Defining the Lyapunov function $V(x(t)) = x(t)^2/2$, we have that

$$\dot{V}(x(t)) = -ax(t)^2 + \epsilon\, x(t) + \gamma(t), \quad \gamma(t) = (\theta - \hat{\theta}(t))x(t)\phi(x(t)). \tag{3.15}$$

Note that since $\hat{\theta}(t) \to \infty$ as $t \to \infty$, it follows that for sufficiently large $t > 0$, $\gamma(t) < 0$ for all $x(t) \neq 0$ by (2.2) and (3.11). Applying Young's inequality, we observe that

$$\dot{V}(x(t)) \leq -\left(\frac{ax(t)^2}{2} - \gamma(t)\right) + \frac{\epsilon^2}{2a}. \tag{3.16}$$

Therefore $V(x(t))$ is non-increasing if

$$\frac{ax(t)^2}{2} - \gamma(t) \; > \; \frac{\epsilon^2}{2a}. \tag{3.17}$$

Now suppose for contradiction $x(t) \not\to 0$, then either 1. $\liminf_{t\to\infty} x(t) > 0$ or 2. $\liminf_{t\to\infty} x(t) = 0$:

1. Suppose $\liminf_{t\to\infty} x(t) > 0$. Then there exists $\epsilon' > 0$ s.t. $x(t) > \epsilon'$ $\forall t$. Since $\gamma(t) \to -\infty$ as $\hat{\theta}(t) \to \infty$, It follows by (3.15) that $\dot{V}(x(t)) \to -\infty$ as $t \to \infty$, i.e.

$$\forall M > 0 \quad \exists T > 0 \;\; s.t. \;\; \forall t \geq T \quad \dot{V}(x(t)) \leq -M, \tag{3.18}$$

   which implies that $V(x(t)) \to -\infty$. This contradicts the positive definiteness of $V(x(t))$.

2. If $\liminf_{t\to\infty} x(t) = 0$, then there must exists $\epsilon' > 0$, and a positive divergent sequence $\{t_k\}_{k\geq 1}$ such that $\dot{V}(x(t)) > 0$ and $x(t_k) > \epsilon'$. Since $\gamma(t_k) \to -\infty$ as $k \to \infty$, it follows that (3.17) holds at time $t_k$, hence contradiction.

Therefore $x \to 0$ as $t \to \infty$. Thus completing the proof.

□

**Proposition 3.2.** *Let $\phi$ satisfies* (3.11) *and consider the closed loop* $(\Sigma(\theta, \phi, d(\cdot)), \Xi)$ *defined by* (2.1)*,* (2.2)*, where $d(\cdot) = \epsilon$. Suppose $x(t)$ is bounded and uniformly continuous. Then*

$$x(t) \to 0, \;\; \hat{\theta}(t) \to \infty \;\; as \; t \to \infty.$$

*Proof.* Suppose for contradiction $x(t) \not\to 0$ as $t \to \infty$. So there exists a sequence $\{t_k\}$ for which $x(t_k) \geq M$ for some $M > 0$ i.e.

$$\exists M > 0 \qquad \exists \{t_k\}_{k\geq 1}, \; t_k \to \infty \qquad s.t. \qquad |x(t_k)| \geq M. \tag{3.19}$$

Since $x(t)$ is uniformly continuous, choosing $\epsilon = M/2$, we have that

$$\exists \delta \; s.t. \; \forall \tau \in [0, \delta], \; \forall t > 0, \qquad |x(t) - x(t + \tau)| < \frac{M}{2}. \tag{3.20}$$

So $|x(t_k) - x(t_k + \tau)| < M/2$, and since $x(t_k) \geq M$, we have $x(t_k + \tau) > M/2$ i.e.

$$x(t) \geq \frac{M}{2}, \qquad \forall t \in [\, t_k, t_k + \delta\,]. \tag{3.21}$$

It follows by (3.11) and the boundedness of $x(\cdot)$ that

$$\phi(x(t) \geq \alpha > 0 \qquad \forall t \in [\, t_k, t_k + \delta\,] \tag{3.22}$$

where $\alpha > 0$. Therefore

$$\exists N > 0 \quad s.t. \quad x(t)\phi(x(t)) \geq N, \qquad \forall t \in [\, t_k, t_k + \delta\,]. \tag{3.23}$$

It follows that

$$\int_{t_k}^{t_k+\delta} x(\tau)\phi(x(\tau))d\tau \geq N\delta. \tag{3.24}$$

Now let us define $\{S_n\}$ as follows:

$$\{S_n\}_{n\geq 1}, \qquad S_{2k-1} = t_k,\ S_{2k} = t_{k+\delta},\ k \geq 1, \tag{3.25}$$

Clearly $S_n \to \infty$ as $n \to \infty$. By (2.2), we have that

$$\hat{\theta}(t) = \int_0^\infty \dot{\hat{\theta}}(\tau)d\tau = \int_0^\infty x(\tau)\phi(x(\tau))d\tau = \int_{S_n} x(\tau)\phi(x(\tau))d\tau \ \to \infty, \tag{3.26}$$

which by Proposition 3.1, implies that $x(t) \to 0$ as $t \to \infty$; hence contradiction. It follows that as $t \to \infty$, $x(t) \to 0$, and by Proposition 3.1, $\hat{\theta}(t) \to \infty$ , thus completing the proof. □

**Proposition 3.3.** *Suppose $\phi$ satisfies* (3.11)*. Consider the closed loop* $(\Sigma(\theta, \phi, d(\cdot)), \Xi)$ *defined by equations* (2.1),(2.2) *and the transient performance cost functional* (3.10)*. Then*

$$\mathcal{P}(\Sigma(\theta, \phi, d(\cdot)), \Xi) = \infty$$

*Proof.* For ease of the notation let us denote $\sup\limits_{\|d(\cdot)\|_{\mathcal{L}^\infty} \leq \epsilon} \sup\limits_{|\theta| \leq \delta}$ by $\sup\limits_{\bullet}$ and $\limsup\limits_{t\to\infty}$ by $\overline{\lim}$.

We choose $d(\cdot) = \epsilon > 0$. Suppose for contradiction $\mathcal{P} < \infty$. Consider $\dot{x}(t)$. There are two cases either 1. $\overline{\lim}\, \dot{x}(t) = \infty$ or 2. $\overline{\lim}\, \dot{x}(t) < \infty$:

1. Suppose $\overline{\lim}\, \dot{x}(t) = \infty$, i.e. $\overline{\lim}\, (-ax(t) + (\theta - \hat{\theta}(t))\phi(x(t)) + \epsilon) = \infty$. Therefore either
   (a) $\overline{\lim}\, x(t) = \infty$, which implies that $\|x(\cdot)\|_{\mathcal{L}^\infty} = \infty$, hence contradiction, or
   (b) $\overline{\lim}\, x(t) < \infty$, therefore $\overline{\lim}\ \hat{\theta}(t)\phi(x(t)) = \infty$. It follows that

$$\sup_{\bullet} \|u(\cdot)\|_{\mathcal{L}^\infty} \geq \|u(\cdot)\|_{\mathcal{L}^\infty} \geq \left| \overline{\lim}\ \hat{\theta}(t)\phi(x(t)) - \overline{\lim}\, x(t) \right| = \infty, \tag{3.27}$$

   which is a contradiction.

2. Suppose $\overline{\lim}\, \dot{x}(t) < \infty$ i.e. $x(t)$ is uniformly continuous. Again there are two possibilities: either a) $\overline{\lim}\, x(t) = \infty$, or b) $\overline{\lim}\, x(t) < \infty$:

   (a) $\overline{\lim}\, x(t) = \infty$ implies that $\sup_{\bullet} \|x(\cdot)\|_{\mathcal{L}^\infty} = \infty$, hence contradiction.

   (b) $\overline{\lim}\, x(t) < \infty$ implies that $x(t)$ is bounded. Therefore by Proposition 3.2

$$x(t) \to 0, \quad \hat{\theta}(t) \to \infty \quad \text{as} \;\; t \to \infty. \tag{3.28}$$

   Considering $\overline{\lim}\, \dot{u}(t)$, and by applying (3.28), we observe that

$$\overline{\lim}\, \dot{u}(t) = \left(a + \hat{\theta}(t)\frac{\partial \phi(x)}{\partial x}\right)\left(\hat{\theta}(t)\phi(x(t)) - \epsilon\right). \tag{3.29}$$

   Now there are two possibilities: either i) $\hat{\theta}(t)\phi(x(t)) \not\to \epsilon$ (including the possibility that $\lim_{t\to\infty} \hat{\theta}(t)\phi(x(t))$ does not exists), or ii) $\lim_{t\to\infty} \hat{\theta}(t)\phi(x(t)) = \epsilon$ :

   i. Suppose $\lim_{t\to\infty} \hat{\theta}(t)\phi(x(t))$ does not exist, or $\hat{\theta}(t)\phi(x(t)) \not\to \epsilon$. Therefore by (3.11–b) we have that $\sup_{\bullet} \|\dot{u}(\cdot)\|_{\mathcal{L}^\infty} = \infty$; hence contradiction.

   ii. Suppose $\lim_{t\to\infty} \hat{\theta}(t)\phi(x(t) = \epsilon$. By (3.28) we have that

$$\forall \hat{\theta}^* > 0 \;\; \exists T > 0 \;\; s.t. \;\; \forall t > T \quad \hat{\theta}(t) > \hat{\theta}^*. \tag{3.30}$$

   Now we choose $d_2(\cdot)$ as follows

$$d_2(t) = \begin{cases} \epsilon & t \le T \\ -\epsilon & t > T \end{cases} \tag{3.31}$$

   Note that $d_2(t) = d(t)$ for all $t \le T$. With this choice, by continuity and causality, we have that

$$\lim_{t\to T^+} x(t) = x(T), \quad \lim_{t\to T^+} \hat{\theta}(t) = \hat{\theta}(T), \quad \lim_{t\to T^+} \phi(x(t)) = \phi(x(T)). \tag{3.32}$$

   where $\lim_{t\to T^+}$ denotes $\lim_{t\to T, t>T}$. It follows that

$$\left(\lim_{t\to T^+} \dot{u}(t)\right) - \dot{u}(T) = 2\epsilon\left(a + \hat{\theta}(t)\frac{\partial \phi(x)}{\partial x}\right). \tag{3.33}$$

   The difference (3.33) can be made arbitrarily large by (3.28), (3.11) and choosing a suitable $\hat{\theta}^*$. Then either $\dot{u}(T)$ is large or $\lim_{t\to T^+} \dot{u}(t)$ is large, therefore $\sup_{\bullet} \|\dot{u}(\cdot)\|_{\mathcal{L}^\infty}$ can be made arbitrarily large hence contradiction[1] .

Therefore at least one component of (3.10) diverges, hence $\mathcal{P} = \infty$. □

[1]Note that the proof of 2b.ii in [7] is erroneous, and the argument given here corrects the proof in [7].

**Proposition 3.4.** *Let $\phi$ satisfy* (3.11) *and consider the closed loop* $(\Sigma, \Xi_P(\theta_{\max}))$ *defined by equations* (2.1), (2.9) *and the transient performance cost functional* (3.10). *Then*

$$\mathcal{P}\left(\Sigma(\theta, \phi, d(\cdot)), \Xi_P(\theta_{\max})\right) \to \infty \text{ as } \theta_{\max} \to \infty.$$

*Proof.* It is convenient to define

$$\mathcal{P}_{[0,T]}(\Sigma, \Xi) = \left( \|x(\cdot)\|_{\mathcal{L}^\infty[0,T]} + \|u(\cdot)\|_{\mathcal{L}^\infty[0,T]} + \|\dot{u}(\cdot)\|_{\mathcal{L}^\infty[0,T]} \right). \tag{3.34}$$

Now let $M > 0$. By Proposition 3.3 there exists $d(\cdot) \in \mathcal{D}$, $\|d(\cdot)\|_{\mathcal{L}^\infty} \leq \epsilon$ s.t.

$$\mathcal{P}_{[0,\infty)}(\Sigma(\theta, \phi, d(\cdot)), \Xi) \geq 2M. \tag{3.35}$$

It follows that $\exists T > 0$ s.t. $\mathcal{P}|_{[0,T]}(\Sigma(\theta, \phi, d(\cdot)), \Xi) \geq M$ and also $\hat{\theta}(T) \geq M$. Let

$$\theta_{\max} = \max\{M, 2\hat{\theta}(T)\}. \tag{3.36}$$

Then the unmodified and the projection design are identical on $[0, T]$, hence

$$\mathcal{P}_{[0,T]}(\Sigma(\theta, \phi, d(\cdot)), \Xi_P(\theta_{\max})) = \mathcal{P}_{[0,T]}(\Sigma(\theta, \phi, d(\cdot)), \Xi) \geq M. \tag{3.37}$$

Therefore

$$\mathcal{P}(\Sigma(\theta, \phi, d(\cdot)), \Xi_P(\theta_{\max})) \geq \mathcal{P}_{[0,T]}(\Sigma(\theta, \phi, d(\cdot)), \Xi_P(\theta_{\max})) \geq M. \tag{3.38}$$

Since this holds for all $M > 0$, this completes the proof. □

**Proposition 3.5.** *Consider the closed loop* $(\Sigma(\theta, \phi, d(\cdot)), \Xi_D(d_{\max}))$ *defined by equations* (2.1), (2.6) *and the transient performance cost functional* (3.10). *Then* $\forall d_{\max} \geq \epsilon$,

$$\mathcal{P}(\Sigma(\theta, \phi, d(\cdot)), \Xi_D(d_{\max})) < \infty.$$

*Proof.* Due to switching nature of the dead-zone, all our differential equations have a discontinuous right hand sides, for which the classical definition of solution is not valid, we therefore consider solutions in a Filippov sense. A complete proof of stability can be found in [2]. To establish performance bounds, we define the Lyapunov function

$$V(x(t), \hat{\theta}(t)) = \frac{1}{2}x(t)^2 + \frac{1}{2}\left(\theta - \hat{\theta}(t)\right)^2, \tag{3.39}$$

and we let

$$V_0 = \frac{1}{2}\max(\, x_0^2, \epsilon^2 \,) + \frac{1}{2}\theta^2. \tag{3.40}$$

It has been shown [2] that

$$V(x(t), \hat{\theta}(t)) \leq V_0 \qquad \forall t > 0. \tag{3.41}$$

Hence by (3.39), $x(\cdot)$ and $\hat{\theta}(\cdot)$ are uniformly bounded in terms of $V_0, \theta$ as follows:

$$|x(t)| \leq \sqrt{2V_0}\,, \qquad |\hat{\theta}(t)| \leq |\theta| + \sqrt{2V_0}. \tag{3.42}$$

It follows that by (2.1), (2.6) that $\dot{x}(\cdot), u(\cdot)$ are uniformly bounded in terms of $V_0, \theta$. Then the uniformly boundedness of $\dot{u}(\cdot)$ follows by continuity of $\phi(\cdot)$:

$$\dot{u}(t) = -a\dot{x}(t) - \hat{\theta}(t)\frac{\partial\phi(x}{\partial x}\dot{x}(t) - D_{[-\eta,\eta]}x(t)\phi(x(t))^2. \tag{3.43}$$

Therefore $(\|x(\cdot)\|_{\mathcal{L}^\infty} + \|u(\cdot)\|_{\mathcal{L}^\infty} + \|\dot{u}(\cdot)\|_{\mathcal{L}^\infty}) < \infty$, thus completing the proof. □

We can now prove the main result, which we repeat for convenience of the reader:

**Theorem 3.1.** *Let $\phi$ be such that* (3.11) *hold. Consider the system $\Sigma(\theta, \phi, d(\cdot))$ defined by* (2.1)*, and the controllers $\Xi_D(d_{\max}), \Xi_P(\theta_{\max})$ defined by equations* (2.6)*,* (2.9)*. Consider the transient performance cost functional* (3.10)*. Then $\forall d_{\max} \geq \epsilon \quad \exists\theta^*_{\max} \geq \delta$ such that $\forall\theta_{\max} \geq \theta^*_{\max} > 0$,*

$$\mathcal{P}(\Sigma(\theta, \phi, d(\cdot)), \Xi_P(\theta_{\max})) \;>\; \mathcal{P}(\Sigma(\theta, \phi, d(\cdot)), \Xi_D(d_{\max})). \tag{3.44}$$

*Proof.* This is a simple consequence of Proposition 3.4 and Proposition 3.5. □

# 4 Conclusions

This paper has demonstrated an analytical comparison between dead-zone and projection based robust adaptive control designs. We have shown that if the a-priori knowledge of the parametric uncertainty level is sufficiently conservative then the dead-zone based design will out-perform the projection based design. The result of this paper can be extended to systems in the form of integrator chains and also to minimum phase, relative degree one, linear systems [8]. Similarly we have developed results to demonstrate the contrary relationship between the controllers, ie. establishing results which show when the projection controllers outperform the dead-zone controllers. Establishing whether the same results can be given for the alternative costs, for example $\mathcal{P} = \|x(\cdot)\|_{\mathcal{L}^\infty} + \|u(\cdot)\|_{\mathcal{L}^\infty}$, is future work.

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