Disturbance Attenuation for a Class of Nonlinear Systems by Output Feedback

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Abstract

This paper studies the problem of disturbance attenuation with internal stability via output feedback for a family of nonlinear systems. Using a feedback domination design which substantially differs from the separation principle, we explicitly construct a dynamic output compensator attenuating the disturbance's effect on the output to an arbitrary degree of accuracy in the L_2-gain sense, and achieving global asymptotic stability in the absence of disturbance.

1 Introduction and Discussion

In this paper we consider a class of single-input single-output uncertain nonlinear systems described by equations of the form

$$\begin{aligned} \dot{z} &= f_0(t,z,y) + g_0(t,z,y)w \\ \dot{x}_1 &= x_2 + f_1(t,z,x,u) + g_1(t,z,x,u)w \\ &\vdots \\ \dot{x}_{n-1} &= x_n + f_{n-1}(t,z,x,u) + g_{n-1}(t,z,x,u)w \\ \dot{x}_n &= u + f_n(t,z,x,u) + g_n(t,z,x,u)w \\ y &= x_1 \end{aligned} \tag{1.1}$$

where $(z,x) \in \mathbb{R}^m \times \mathbb{R}^n$ is the system state, $u \in \mathbb{R}$, $y \in \mathbb{R}$ and $w \in R^s$ are the system input, output and disturbance, respectively. The functions $f_0 : \mathbb{R}^{m+2} \to \mathbb{R}^m$ and $g_0 : \mathbb{R}^{m+2} \to \mathbb{R}^{m\times s}$ are C^1 with $f_0(t,0,0) = 0\ \forall t$, while $f_i : \mathbb{R}^{n+m+2} \to \mathbb{R}$ and $g_i : \mathbb{R}^{n+m+2} \to \mathbb{R}^{1\times s}$ are C^0 *uncertain* functions, with $f_i(t,0,0,0) = 0\ \forall t$, for $i = 1,\cdots,n$.

The purpose of this paper is to investigate the problem of achieving an arbitrary small level of disturbance attenuation in the sense of L_2-gain, via *output feedback,* for a class of nonlinear systems (1.1) under appropriate conditions.

The disturbance attenuation problem of this kind has attracted considerable attention since the original work [12], and there are many interesting results available in the literature. For nonlinear systems, the problem was first studied in [8], a solution was given in terms of the L_∞ induced norm from the disturbance inputs to the outputs. However, an important issue—internal stability—was not considered in the paper [8]. The stability issue was addressed later in [9], where a recursive design technique based on adding an integrator was proposed, resulting in a solution to the global disturbance attenuation problem with internal stability, for a class of nonlinear systems in a triangular form, i.e., $f_i(t,z,x,u) = f_i(z,x_1,\cdots,x_i)$ and $g_i(t,z,x,u) = g_i(z,x_1,\cdots,x_i)$ in (1.1). The disturbance attenuation result obtained in [9] was later extended to a larger class of minimum-phase and nonminimum-phase nonlinear systems [4, 2, 3]. Recently, *global inverse* L_2-gain design for nonlinear triangular systems was presented in [3], achieving global disturbance attenuation together with local optimality.

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Most of the results reviewed so far have been obtained based on the assumption that the system state, i.e. $(z, x_1, \cdots, x_n)$ in (1.1), is available for feedback design. In the case when only the output information is measurable, fewer results exist in the literature, which are devoted to the disturbance attenuation problem by *output feedback.* One of them is the paper [10], where a *polynomial gain* disturbance attenuation property with global stability was achieved by output feedback, for a class of nonlinear systems in the so-called output feedback form, i.e., the system's nonlinearities depend on the output only. In the work [5], the problem of disturbance attenuation characterized in terms of a *nonlinear gain* was considered for a subclass of nonlinear systems (1.1) in which $f_i(t, z, x, u) = f_i(z, y)$ and $g_i(t, z, x, u) = g_i(z, y)$. The problem was solved by using the concept of input to state stability (ISS) and the ISS Lyapunov function, together with the adding an integrator design. There is also an interesting result [1] devoted to the same subject but for a different class of nonlinear systems which can depend on the output nonlinearly but must be *linear in the unmeasurable states.*

In this paper we shall pursue a line of the work started in [1, 10, 5]. Our objective is to find, if possible, a linear *output feedback* controller of the form

$$\begin{aligned}\dot{\xi} &= M\xi + Ny, \qquad M \in \mathbb{R}^{n\times n}, \quad N \in \mathbb{R}^{n} \\ u &= K\xi, \qquad\qquad K \in \mathbb{R}^{1\times n}\end{aligned} \tag{1.2}$$

such that

i) when $w = 0$, the closed-loop system (1.1)-(1.2) is globally asymptotically stable at the equilibrium $(z, x, \xi) = (0, 0, 0)$, uniformly in t;

ii) for every disturbance $w(t) \in L_2$ and given real number $\gamma > 0$, the response of the closed-loop system (1.1)-(1.2) starting from the origin is such that

$$\int_0^t |y(\tau)|^2 d\tau \le \gamma^2 \int_0^t \|w(\tau)\|^2 d\tau, \qquad \forall t \ge 0. \tag{1.3}$$

To solve the disturbance attenuation problem by output feedback, we make the following assumptions throughout this paper.

H1) There exist a C^1 Lyapunov function $U_0(t, z)$, class K_∞-functions $\underline{\beta}(\cdot), \overline{\beta}(\cdot)$ and positive constants a_0, b_0, c_0, satisfying

$$\begin{aligned}\underline{\beta}(||z||) &\le U_0(t, z) \le \overline{\beta}(||z||) \\ \dot{U}_0(t, z) &\le -a_0||z||^2 + b_0 y^2 + c_0||w||^2,\end{aligned}$$

H2) $|f_i(t, z, x, u)| \le c(||z|| + |x_1| + \cdots + |x_i|)$, $\forall i = 1, \cdots, n$, where $c > 0$ is a known constant;

H3) $||g_i(t, z, x, u)|| \le G$, $\forall i = 1, \cdots, n$, where $G > 0$ is a known constant.

Remark 1.1 If one treats (y, w) as an input and z a state of the z-subsystem of (1.1), the assumption H1) implies that the $z-$subsystem is input to state stable (ISS) and $U_0(t, z)$ is an ISS Lyapunov function for the z-subsystem of (1.1) [11]. A condition similar to H1) has also been used in [2, 5]. The assumption H2) requires basically the entire family of uncertain nonlinear systems (1.1) are dominated by a triangular system satisfying linear growth conditions.

It should be pointed out that the systems (1.1) characterized by the conditions H1)-H3) represent an important class of uncertain nonlinear systems that are not covered by the work [1, 10, 5]. Indeed, it is not difficult to see that the system below

$$\begin{aligned}
\dot{z} &= -z^{5/3} + z^{2/3}(y+w) \\
\dot{x}_1 &= x_2 + \ln(1+z^2) \\
\dot{x}_2 &= x_3 + x_2 \sin x_2 + w \cos x_1 \\
\dot{x}_3 &= u \\
y &= x_1
\end{aligned} \tag{1.4}$$

does satisfy H1)-H3) but does not belong to the class of nonlinear systems considered in [1, 10, 5]. As a such, global disturbance attenuation with stability does not seem to be solvable by any existing output feedback control scheme. When the system involves uncertainties, the problem becomes even more challenging, as illustrated by the following example.

$$\begin{aligned}
\dot{z} &= -z^3 + zy + zw \\
\dot{x}_1 &= x_2 + \frac{1}{4}z^2 + d_1(t)x_1 \sin z + \frac{1}{2}(1+\cos(x_2 z))w \\
\dot{x}_2 &= u + d_2(t)\frac{1}{3}\ln(1+x_2^4) \\
y &= x_1
\end{aligned} \tag{1.5}$$

where $|d_1(t)| \le 1$ and $|d_2(t)| \le 1$ are *unknown continuous* functions whose bounds are known. For this system, when $d_1(t) = d_2(t) = 0$, disturbance attenuation with global stability can be easily achieved by output feedback. However, the problem cannot be solved by existing output feedback control schemes including those in [1, 10, 5] in the presence of uncertain functions $d_1(t)$ and $d_2(t)$. Nevertheless, it is solvable by the approach proposed in Section 2.

Inspired by the recent work [6, 7], we shall present in the next section a feedback domination design method which provides a systematic procedure for the construction of *linear* dynamic output compensators (1.2) for a class of nonlinear systems (1.1), achieving global disturbance attenuation with internal stability. The novelty of our output feedback control scheme is in the explicit design of a dynamic compensator that is *not based on the separation principle.* Instead of constructing the observer and controller separately, we design a high-gain linear observer and a controller simultaneously. This is substantially different from most of the existing works where the designed observer itself can asymptotically recover the state of the controlled plant, regardless of the design of the controller, i.e., the controller design is independent of the observer design—known as the separation principle.

2 Output Feedback Design — A Non-Separation Principle Paradigm

In this section, we prove that the problem of global disturbance attenuation with internal stability can be solved by output feedback, for a class of nonlinear systems (1.1) under the assumptions H1)—H3). This will be accomplished by explicitly constructing a dynamic output compensator, via a feedback domination design method that is motivated by the technique of *adding a power integrator* [6, 7]. The main result of this paper is the following theorem.

Theorem 2.1 Under the conditions H1), H2) and H3), there is a dynamic output compensator of the form (1.2), such that the closed-loop system (1.1)–(1.2) is uniformly globally asymptotically stable when $w = 0$ and achieves global disturbance attenuation in the sense of L_2-gain, i.e., in the sense of (1.3).

Proof. The proof consists of three steps. First, we design a reduced-order, linear high-gain observer for partial-states $(x_1, \cdots, x_n)$, without involving the system nonlinearities $f_i(t,z,x,u)$ and $g_i(t,z,x,u)$, $i = 1, \cdots, n$, which are unknown and not available for the observer design. By doing so, the resulted error dynamics contains some extra terms that prevent convergence of the observer. We then construct a set of dummy output controllers step-by-step, to eliminate the extra terms arising in the error dynamics. Finally, we obtain a dynamic output compensator as well as a differential dissipation inequality, rendering the closed-loop system globally asymptotically stable at the origin when $w = 0$, and attenuating the effect of the disturbance on the output to an arbitrary degree of accuracy in the presence of w.

We begin by designing the following reduced-order, linear high-gain observer

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + La_1(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \hat{x}_3 + L^2a_2(x_1 - \hat{x}_1) \\ &\vdots \\ \dot{\hat{x}}_{n-1} &= \hat{x}_n + L^{n-1}a_{n-1}(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_n &= u + L^n a_n(x_1 - \hat{x}_1) \end{aligned} \tag{2.1}$$

for the partial-states $(x_1, \cdots, x_n)$ of (1.1), where $L \geq 1$ is a constant gain to be determined later, and $a_j > 0,\ j = 1, \cdots, n$, are coefficients of the Hurwitz polynomial $p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$.

Define $\varepsilon_i = \frac{x_i - \hat{x}_i}{L^{i-1}}, \quad i = 1, \cdots, n$. A direct calculation yields

$$\dot{\varepsilon} = LA\varepsilon + \begin{bmatrix} f_1(t,z,x,u) + g_1(t,z,x,u)w \\ \frac{1}{L}(f_2(t,z,x,u) + g_2(t,z,x,u)w) \\ \vdots \\ \frac{1}{L^{n-1}}(f_n(t,z,x,u) + g_n(t,z,x,u)w) \end{bmatrix}, \tag{2.2}$$

where

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix} \quad A = \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix}.$$

Since A is a Hurwitz matrix, there is a positive definite symmetric matrix $P = P^T > 0$ such that

$$A^T P + PA = -I.$$

Consider the Lyapunov function $V_0(t,z,\varepsilon) = U_0(t,z) + (n+2)\varepsilon^T P\varepsilon$. Using the assumption H2), it is easy to show the existence of a real constant $c_1 > 0$ (independent of L), such that

$$\begin{aligned} \dot{V}_0 &\leq -a_0||z||^2 + b_0y^2 + c_0||w||^2 - (n+2)L\|\varepsilon\|^2 + 2(n+2)\varepsilon^T P \begin{bmatrix} f_1(t,z,x,u) + g_1(t,z,x,u)w \\ \frac{f_2(t,z,x,u)}{L} + \frac{g_2(t,z,x,u)w}{L} \\ \vdots \\ \frac{f_n(t,z,x,u)}{L^{n-1}} + \frac{g_n(t,z,x,u)w}{L^{n-1}} \end{bmatrix} \\ &\leq -a_0||z||^2 + b_0y^2 + c_0||w||^2 - (n+2)L\|\varepsilon\|^2 + c_1\|\varepsilon\|\left(||z|| + |x_1| + \frac{1}{L}|x_2| + \cdots + \frac{1}{L^{n-1}}|x_n|\right) \\ &\quad + 2(n+2)\varepsilon^T P\left[g_1(t,z,x,u)w,\ \frac{g_2(t,z,x,u)w}{L},\ \cdots,\ \frac{g_n(t,z,x,u)w}{L^{n-1}}\right]^T \end{aligned} \tag{2.3}$$

By H3) and the fact that $L \geq 1$, there is a constant $\bar{c}_0 > 0$ (independent of L) satisfying

$$2(n+2)\varepsilon^T P\left[g_1(t,z,x,u)w,\ \frac{g_2(t,z,x,u)w}{L},\ \cdots,\ \frac{g_n(t,z,x,u)w}{L^{n-1}}\right]^T \leq ||\varepsilon||^2 + \bar{c}_0||w||^2. \tag{2.4}$$

Substituting (2.4) into (2.3), we have

$$\dot{V}_0 \leq -a_0||z||^2 + b_0y^2 - ((n+2)L - 1)\|\varepsilon\|^2 + c_1\|\varepsilon\|\left(||z|| + |x_1| + \frac{1}{L}|x_2| + \cdots + \frac{1}{L^{n-1}}|x_n|\right) + \hat{c}_0||w||^2 \tag{2.5}$$

where $\hat{c}_0 := \bar{c}_0 + c_0$ does not depend on L.

Recall $x_i = \hat{x}_i + L^{i-1}\varepsilon_i$. It is easy to see that

$$\begin{aligned}
& c_1\|\varepsilon\|\left(\|z\| + |x_1| + \frac{1}{L}|x_2| + \cdots + \frac{1}{L^{n-1}}|x_n|\right) \\
& \le c_1||\varepsilon||\left(\|z\| + |\hat{x}_1| + |\varepsilon_1| + \cdots + \frac{1}{L^{n-1}}|\hat{x}_n| + |\varepsilon_n|\right) \\
& \le c_1||z||||\varepsilon|| + \sqrt{n}c_1||\varepsilon||^2 + c_1\|\varepsilon\|\left(|\hat{x}_1| + \frac{1}{L}|\hat{x}_2| + \cdots + \frac{1}{L^{n-1}}|\hat{x}_n|\right) \\
& \le \frac{a_0}{2}||z||^2 + \left(\frac{c_1^2}{2a_0} + \sqrt{n}c_1 + \frac{n}{2}c_1\right)\|\varepsilon\|^2 + c_1\left(\frac{1}{2}\hat{x}_1^2 + \frac{1}{2L^2}\hat{x}_2^2 + \cdots + \frac{1}{2L^{2n-2}}\hat{x}_n^2\right) \qquad (2.6)
\end{aligned}$$

The last inequality is due to the fact that

$$c_1||z||||\varepsilon|| \le \frac{a_0}{2}||z||^2 + \frac{c_1^2}{2a_0}\|\varepsilon\|^2, \quad \frac{1}{L^{i-1}}||\varepsilon|||\hat{x}_i| \le \frac{1}{2}||\varepsilon||^2 + \frac{1}{2L^{2i-2}}\hat{x}_i^2.$$

Putting (2.5) and (2.6) together, we have,

$$\begin{aligned}
\dot{V}_0 + \frac{\hat{c}_0}{\gamma^2}y^2 - \hat{c}_0||w||^2 \quad \le \quad & -\frac{a_0}{2}||z||^2 + \hat{b}_0 y^2 - \left((n+2)L - \frac{c_1^2}{2a_0} - \sqrt{n}c_1 - \frac{n}{2}c_1 - 1\right)\|\varepsilon\|^2 \\
& + c_1\left(\frac{1}{2}\hat{x}_1^2 + \frac{1}{2L^2}\hat{x}_2^2 + \cdots + \frac{1}{2L^{2n-2}}\hat{x}_n^2\right), \qquad \hat{b}_0 := \left(b_0 + \frac{\hat{c}_0}{\gamma^2}\right). \qquad (2.7)
\end{aligned}$$

Initial Step: Construct the Lyapunov function $V_1(t, z, \varepsilon, \hat{x}_1) = V_0(t, z, \varepsilon) + \frac{\hat{x}_1^2}{2}$. Then,

$$\begin{aligned}
\dot{V}_1 + \frac{\hat{c}_0}{\gamma^2}y^2 - \hat{c}_0||w||^2 \quad \le \quad & -\frac{a_0}{2}||z||^2 + \hat{b}_0 y^2 - \left((n+2)L - \frac{c_1^2}{2a_0} - (\sqrt{n} + \frac{n}{2})c_1 - 1\right)\|\varepsilon\|^2 \\
& + c_1\left(\frac{1}{2}\hat{x}_1^2 + \frac{1}{2L^2}\hat{x}_2^2 + \cdots + \frac{1}{2L^{2n-2}}\hat{x}_n^2\right) + \hat{x}_1(\hat{x}_2 + La_1\varepsilon_1)
\end{aligned}$$

Choosing $L \ge 2\left(\frac{c_1^2}{2a_0} + (\sqrt{n} + \frac{n}{2})c_1 + 1\right)$ gives

$$\begin{aligned}
\dot{V}_1 + \frac{\hat{c}_0}{\gamma^2}y^2 - \hat{c}_0||w||^2 \quad \le \quad & -\frac{a_0}{2}||z||^2 + \hat{b}_0 y^2 - (n + \frac{3}{2})L\|\varepsilon\|^2 + \frac{c_1}{2}\hat{x}_1^2 \\
& + c_1\left(\frac{1}{2L^2}\hat{x}_2^2 + \cdots + \frac{1}{2L^{2n-2}}\hat{x}_n^2\right) + \hat{x}_1\hat{x}_2 + \frac{1}{2}La_1^2\hat{x}_1^2 + \frac{L}{2}\varepsilon_1^2 \qquad (2.8)
\end{aligned}$$

Define $\xi_2 = \hat{x}_2 - \hat{x}_2^*$ with $\hat{x}_2^*$ being the virtual control. Thus,

$$\frac{1}{2L^2}\hat{x}_2^2 \le \frac{1}{L^2}\xi_2^2 + \frac{1}{L^2}\hat{x}_2^{*2}.$$

This, together with the fact that $L > c_1$ and $\|\varepsilon\|^2 \ge \varepsilon_1^2$, implies

$$\begin{aligned}
\dot{V}_1 + \frac{\hat{c}_0}{\gamma^2}y^2 - \hat{c}_0||w||^2 \quad \le \quad & -\frac{a_0}{2}||z||^2 + \hat{b}_0 y^2 - (n+1)L\|\varepsilon\|^2 + L\left(\frac{1}{2}a_1^2 + \frac{1}{2}\right)\hat{x}_1^2 + c_1\frac{1}{L^2}\xi_2^2 + c_1\frac{1}{L^2}\hat{x}_2^{*2} \\
& + c_1\left(\frac{1}{2L^4}\hat{x}_3^2 + \cdots + \frac{1}{2L^{2n-2}}\hat{x}_n^2\right) + \hat{x}_1\xi_2 + \hat{x}_1\hat{x}_2^*
\end{aligned}$$

Clearly, the virtual controller

$$\hat{x}_2^* = -Lb_1\hat{x}_1, \qquad b_1 := n + 1 + \frac{1}{2}a_1^2 + \frac{1}{2}, \qquad \text{which is independent of } L,$$

is such that

$$\dot{V}_1 + \frac{\hat{c}_0}{\gamma^2}y^2 - \hat{c}_0||w||^2 \leq -\frac{a_0}{2}||z||^2 + \hat{b}_0 y^2 - (n+1)L\|\varepsilon\|^2 - ((n+1)L - c_1 b_1^2)\hat{x}_1^2 + \frac{c_1}{L^2}\xi_2^2 + c_1\left(\frac{1}{2L^4}\hat{x}_3^2 + \cdots + \frac{1}{2L^{2n-2}}\hat{x}_n^2\right) + \hat{x}_1\xi_2.$$

Note that $\|\varepsilon\|^2 + \hat{x}_1^2 \geq \frac{1}{2}y^2$. By letting $L \geq max\ \{2\hat{b}_0,\ 2\left(\frac{c_1^2}{2a_0} + (\sqrt{n} + \frac{n}{2})c_1 + 1\right)\}$, we have

$$\dot{V}_1 + \frac{\hat{c}_0}{\gamma^2}y^2 - \hat{c}_0||w||^2 \leq -\frac{a_0}{2}||z||^2 - nL\|\varepsilon\|^2 - (nL - c_1 b_1^2)\hat{x}_1^2 + \frac{c_1}{L^2}\xi_2^2 + c_1\left(\frac{1}{2L^4}\hat{x}_3^2 + \cdots + \frac{1}{2L^{2n-2}}\hat{x}_n^2\right) + \hat{x}_1\xi_2.$$

Inductive Step: Suppose at step k, there exist a set of virtual controllers $\hat{x}_1^*, \cdots, \hat{x}_{k+1}^*$, defined by $x_1^* = 0,\ \xi_1 = \hat{x}_1 - x_1^*$ and

$$\hat{x}_i^* = -Lb_{i-1}\xi_{i-1}, \qquad \xi_i = \hat{x}_i - \hat{x}_i^*, \qquad i = 2, \cdots, k+1,$$

with $b_i > 0$ being *independent of* the gain constant L, and a smooth Lyapunov function $V_k(t, z, \varepsilon, \xi_1, \cdots, \xi_k)$ of the form

$$V_k(t, z, \varepsilon, \xi_1, \cdots, \xi_k) = V_0(t, z, \varepsilon) + \sum_{j=1}^{k} \frac{\xi_j^2}{2L^{2(j-1)}}$$

such that

$$\dot{V}_k + \frac{\hat{c}_0}{\gamma^2}y^2 - \hat{c}_0||w||^2 \leq -\frac{a_0}{2}||z||^2 - (n+1-k)L\|\varepsilon\|^2 - \sum_{j=1}^{k}\frac{1}{L^{2j-2}}\left((n+1-k)L - c_1 b_j^2\right)\xi_j^2 + c_1\left(\frac{1}{2L^{2k+2}}\hat{x}_{k+2}^2 + \cdots + \frac{1}{2L^{2n-2}}\hat{x}_n^2\right) + \frac{c_1}{L^{2k}}\xi_{k+1}^2 + \frac{1}{L^{2(k-1)}}\xi_k\xi_{k+1}. \quad (2.9)$$

Now construct the Lyapunov function

$$V_{k+1}(t, z, \varepsilon, \xi_1, \cdots, \xi_{k+1}) = V_k(t, z, \varepsilon, \xi_1, \cdots, \xi_k) + \frac{1}{2L^{2k}}\xi_{k+1}^2, \qquad \xi_{k+1} := \hat{x}_{k+1} - \hat{x}_{k+1}^*.$$

Since

$$\xi_{k+1} = \hat{x}_{k+1} + Lb_k\hat{x}_k + L^2 b_k b_{k-1}\hat{x}_{k-1} + \cdots + L^k b_k b_{k-1}\cdots b_1\hat{x}_1,$$

it is clear that

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2L^{2k}}\xi_{k+1}^2\right) &= \frac{1}{L^{2k}}\xi_{k+1}\left(\hat{x}_{k+2} + L^{k+1}a_{k+1}\varepsilon_1 + Lb_k\sum_{i=1}^{k}\frac{\partial\xi_k}{\partial\hat{x}_i}(\hat{x}_{i+1} + L^i a_i\varepsilon_1)\right) \\
&= \frac{1}{L^{2k}}\xi_{k+1}\left(\hat{x}_{k+2} + L^{k+1}a_{k+1}\varepsilon_1 + \sum_{i=1}^{k}L^{k-i+1}b_k\cdots b_i(\xi_{i+1} - Lb_i\xi_i + L^i a_i\varepsilon_1)\right) \\
&= \frac{1}{L^{2k}}\xi_{k+1}\left(\hat{x}_{k+2} + L^{k+1}d_0\varepsilon_1 + L^{k+1}d_1\xi_1 + L^k d_2\xi_2 + \cdots + Ld_{k+1}\xi_{k+1}\right) \qquad (2.10)
\end{aligned}$$

where $d_0, \cdots, d_{k+1}$ are suitable constants independent of gain L.

With (2.9) and (2.10) in mind, one has

$$\begin{aligned}
&\dot{V}_{k+1} + \frac{\hat{c}_0}{\gamma^2}y^2 - \hat{c}_0||w||^2 \\
&\leq -\frac{a_0}{2}||z||^2 - (n+1-k)L\|\varepsilon\|^2 - \sum_{j=1}^{k}\frac{1}{L^{2j-2}}\left((n+1-k)L - c_1 b_j^2\right)\xi_j^2 + \frac{c_1}{L^{2k+2}}\xi_{k+2}^2 + \frac{c_1}{L^{2k+2}}\hat{x}_{k+2}^{*2} \\
&\quad + c_1\left(\frac{1}{2L^{2k+4}}\hat{x}_{k+3}^2 + \cdots + \frac{1}{2L^{2n-2}}\hat{x}_n^2\right) + \frac{1}{L^{2k-2}}\xi_k\xi_{k+1} + \frac{c_1}{L^{2k}}\xi_{k+1}^2 + \frac{1}{L^{2k}}\xi_{k+1}\xi_{k+2} + \frac{1}{L^{2k}}\xi_{k+1}\hat{x}_{k+2}^* \\
&\quad + \xi_{k+1}\left(\frac{d_0}{L^{k-1}}\varepsilon_1 + \frac{d_1}{L^{k-1}}\xi_1 + \frac{d_2}{L^k}\xi_2 + \cdots + \frac{d_{k-1}}{L^{2k-3}}\xi_{k-1} + \frac{d_k}{L^{2k-2}}\xi_k + \frac{d_{k+1}}{L^{2k-1}}\xi_{k+1}\right). \qquad (2.11)
\end{aligned}$$

By construction, $L \geq c_1$. Hence

$$\frac{c_1}{L^{2k}} \xi_{k+1}^2 \leq \frac{1}{L^{2k-1}} \xi_{k+1}^2. \tag{2.12}$$

Using the completion of square, it is straightforward to show that

$$-L||\varepsilon||^2 + \frac{d_0}{L^{k-1}} \xi_{k+1}\varepsilon_1 \leq \frac{1}{4L^{2k-1}} d_0^2 \xi_{k+1}^2, \qquad -\frac{L}{L^{2k-2}} \xi_k^2 + \frac{d_k+1}{L^{2k-2}} \xi_{k+1}\xi_k \leq \frac{(d_k+1)^2}{4L^{2k-1}} \xi_{k+1}^2 \tag{2.13}$$

$$-\frac{L}{L^{2i-2}} \xi_i^2 + \frac{d_i}{L^{k+i-2}} \xi_{k+1}\xi_i \leq \frac{d_i^2}{4L^{2k-1}} \xi_{k+1}^2, \qquad i = 1, \cdots, k-1. \tag{2.14}$$

Substituting (2.12)-(2.14) into (2.11) yields,

$$\begin{aligned} \dot{V}_{k+1} + \frac{\hat{c}_0}{\gamma^2} y^2 - \hat{c}_0||w||^2 \quad \leq \quad & -\frac{a_0}{2}||z||^2 - (n-k)L\|\varepsilon\|^2 - \sum_{j=1}^{k} \frac{1}{L^{2j-2}} \left((n-k)L - c_1 b_j^2\right) \xi_j^2 + \frac{c_1}{L^{2k+2}} \xi_{k+2}^2 \\ & + \frac{1}{L^{2k}} \xi_{k+1}\xi_{k+2} + c_1 \left(\frac{1}{2L^{2k+4}} \hat{x}_{k+3}^2 + \cdots + \frac{1}{2L^{2n-2}} \hat{x}_n^2 \right) + \frac{1}{L^{2k}} \xi_{k+1} \hat{x}_{k+2}^* \\ & + \frac{c_1}{L^{2k+2}} \hat{x}_{k+2}^{*2} + \frac{\xi_{k+1}^2}{L^{2k-1}} \left[\frac{d_0^2 + d_1^2 + \cdots + d_{k-1}^2 + (d_k+1)^2}{4} + d_{k+1} + 1 \right] \end{aligned} \tag{2.15}$$

Then, it follows from the inequality above that

$$\hat{x}_{k+2}^* = -Lb_{k+1}\xi_{k+1}, \qquad b_{k+1} = n - k + \frac{d_0^2}{4} + \frac{d_1^2}{4} + \cdots + \frac{d_{k-1}^2}{4} + \frac{(d_k+1)^2}{4} + d_{k+1} + 1$$

with b_{k+1} being independent of L, renders

$$\begin{aligned} \dot{V}_{k+1} + \frac{\hat{c}_0}{\gamma^2} y^2 - \hat{c}_0||w||^2 \quad \leq \quad & -\frac{a_0}{2}||z||^2 - (n-k)L\|\varepsilon\|^2 - \sum_{j=1}^{k+1} \frac{1}{L^{2j-2}} \left((n-k)L - c_1 b_j^2\right) \xi_j^2 \\ & + c_1 \left(\frac{1}{2L^{2k+4}} \hat{x}_{k+3}^2 + \cdots + \frac{1}{2L^{2n-2}} \hat{x}_n^2 \right) + \frac{c_1}{L^{2k+2}} \xi_{k+2}^2 + \frac{1}{L^{2k}} \xi_{k+1}\xi_{k+2} \end{aligned} \tag{2.16}$$

Using the inductive argument above, we concludes that at the $n-th$ step, there are constants $b_1, b_2, \cdots, b_n$, all independent of the gain L, such that the linear controller

$$u = -Lb_n\xi_n = -Lb_n(\hat{x}_n + Lb_{n-1}(\hat{x}_{n-1} + \cdots + Lb_2(\hat{x}_2 + Lb_1\hat{x}_1)\cdots), \tag{2.17}$$

(where $\xi_i = \hat{x}_i + Lb_{i-1}\xi_{i-1}, i = 2, \cdots, n$) renders

$$\dot{V}_n + \frac{\hat{c}_0}{\gamma^2} y^2 - \hat{c}_0||w||^2 \leq -\frac{a_0}{2}||z||^2 - L\|\varepsilon\|^2 - \left(L - c_1 b_1^2\right) \xi_1^2 - \cdots - \frac{(L - c_1 b_{n-1}^2)}{L^{2n-4}} \xi_{n-1}^2 - \frac{L}{L^{2n-2}} \xi_n^2 \tag{2.18}$$

where $V_n(t, z, \varepsilon, \xi_1, \cdots, \xi_n) = V_0(t, z, \varepsilon) + \sum_{i=1}^n \frac{1}{2L^{2(i-1)}} \xi_i^2$. Clearly, by construction and H1), $V_n(t, z, \varepsilon, \xi_1, \cdots, \xi_n)$ is positive definite and proper.

Let $L > L^* = \max\{\frac{c_1^2}{2a_0} + (\sqrt{n} + \frac{n}{2})c_1 + 1,\ 2\hat{b}_0,\ c_1 b_1^2, \cdots,\ c_1 b_{n-1}^2\}$. As a consequence, the right-hand side of inequality (2.18) is negative definite. Therefore, the equilibrium of the closed-loop system is uniformly globally asymptotically stable when $w = 0$.

Moreover, observe that $V_n(t, 0, \cdots, 0) = 0, \forall t$ and $V_n(t, z, \varepsilon, \xi_1, \cdots, \xi_n) \geq 0$. Then, it is deduced from the dissipation inequality (2.18) that

$$\int_0^t |y(\tau)|^2 d\tau \leq \gamma^2 \int_0^t ||w(\tau)||^2 d\tau, \quad \forall t \geq 0, \quad \text{when} \quad (z(0), x(0), \hat{x}(0)) = (0, 0, 0). \qquad \blacksquare$$

This completes the proof of Theorem 2.1.

Remark 2.2 When $\underline{\beta}(\cdot)$ and $\overline{\beta}(\cdot)$ are quadratic functions, Theorem 2.1 includes Corollary 2 of [5] as a special case. However, the class of nonlinear systems considered in this paper is much more general than the one in [5].

Remark 2.3 From the design procedure above, one may notice that the observer gain L and controller gains $b_i, i = 1, \cdots, n$, cannot be small numbers. Therefore, the dynamic output compensator proposed in this paper is a high-gain controller. This may be a drawback for a practical implementation or application.

Without much effort, one can prove the following result which is an extension of Theorem 2.1.

Corollary 2.4 If H1) and H2) in Theorem 2.1 are replaced by the following conditions:

A1) There exist a C^1 ISS Lyapunov function $U_0(t, z)$, class $\mathcal{K}_\infty$-functions $\underline{\beta}(\cdot), \overline{\beta}(\cdot)$, and class $\mathcal{K}$ function $\alpha(\cdot)$, and positive constants b_0, c_0, such that

$$\begin{aligned} \underline{\beta}(||z||) &\leq U_0(t, z) \leq \overline{\beta}(||z||) \\ \dot{U}_0(t, z) &\leq -\alpha(||z||) + b_0 y^2 + c_0 ||w||^2, \end{aligned}$$

A2) $|f_i(t, z, x, u)| \leq c(\alpha^{1/2}(||z||) + |x_1| + \cdots + |x_i|), \ \forall i = 1, \cdots, n$, where $c > 0$ is a known constant,

then the L_2-gain disturbance attenuation problem with global asymptotically stability is still solvable by output feedback under A1), A2), H3).

The proof of this corollary can be carried out by using an argument similar to the proof of Theorem 2.1 and is therefore omitted for the sake of space.

Remark 2.5 Corollary 2.4 indicates that the condition H1) of Theorem 2.1 can be relaxed, i.e., $-a||z||^2$ in H1) can be replaced by any K_∞ function $-\alpha(||z||)$ (e.g. $\alpha(||z||) = z^4$, which is indeed the case when considering a nonlinear system like (1.5)). As a trade off, H2) must be, however, replaced by a stronger growth condition such as A2), in which $f_i(\cdot), \ i = 1, \cdots, n$, are dominated by $\alpha^{1/2}(||z||)$ and a linear growth function $(|x_1| + \cdots + |x_i|)$.

We conclude this section with a simple example which illustrates an interesting application of Corollary 2.4. In particular, we demonstrate how to design a dynamic output controller explicitly, achieving global disturbance attenuation with internal stability for the uncertain nonlinear system (1.5).

Obviously, system (1.5) fails to satisfy H1) and H2) of Theorem 2.1 but does satisfy A1), A2), H3) of Corollary 2.4, with $U_0(z) = \frac{z^2}{2}, \alpha(||z||) = \frac{z^4}{2}, b_0 = c_0 = 1, G = 1$. By Corollary 2.4, the problem of global disturbance attenuation is solvable by output feedback. Indeed, an output feedback control law that solves the problem can be constructed step by step, as shown below.

First, design a high-gain observer of the form

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + L(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 &= u + L^2(x_1 - \hat{x}_1) \end{aligned} \tag{2.19}$$

where

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

are such that

$$A^T P + PA = -I.$$

The error dynamic is given by

$$\begin{aligned}\dot{e}_1 &= e_2 - Le_1 + \frac{1}{4}z^2 + d_1(t)x_1 \sin z + \frac{1}{2}(1+\cos(x_2 z))w \\ \dot{x}_2 &= -L^2 a_2 e_1 + d_2(t)\frac{1}{3}\ln(1+x_2^4)\end{aligned}$$

Let $\varepsilon_1 = e_1 = x_1 - \hat{x}_1, \varepsilon_2 = \frac{e_2}{L} = \frac{x_2 - \hat{x}_2}{L}$. Then

$$\begin{aligned}\dot{\varepsilon}_1 &= L\varepsilon_2 - L\varepsilon_1 + \frac{1}{4}z^2 + d_1(t)x_1 \sin z + \frac{1}{2}(1+\cos(x_2 z))w \\ \dot{\varepsilon}_2 &= -L\varepsilon_1 + \frac{d_2(t)\frac{1}{3}\ln(1+x_2^4)}{L}\end{aligned} \tag{2.20}$$

Consider the Lyapunov function $V_0(z,\varepsilon) = U_0(z) + 4\varepsilon^T P\varepsilon$. Its derivative along the trajectories of (1.5)–(2.20) is

$$\dot{V}_0 + \frac{9}{\gamma^2}y^2 - 9w^2 \le -\frac{z^4}{4}\left(1+\frac{9}{\gamma^2}\right)y^2 - (4L - 40 - 16\sqrt{2})||\varepsilon||^2 + 16\left(\frac{\hat{x}_1^2}{2} + \frac{\hat{x}_2^2}{2L^2}\right)$$

where $\gamma > 0$ is a given real number.

Next we define

$$V_1 = V_0 + \frac{\hat{x}_1^2}{2}, \qquad \hat{x}_2^* = -4L\hat{x}_1, \qquad \xi_2 = \hat{x}_2 - \hat{x}_2^*,$$

and choose the observer gain

$$L \ge \max\{80 + 32\sqrt{2},\ 2 + \frac{18}{\gamma^2}\}.$$

A simple calculation gives

$$\dot{V}_1 + \frac{9}{\gamma^2}y^2 - 9w^2 \le -\frac{z^4}{4} - 2L||\varepsilon||^2 - (2L - 256)\hat{x}_1^2 + \frac{16\xi_2^2}{L^2} + \hat{x}_1\xi_2.$$

At the last step, let $V_2 = V_1 + \frac{\xi_2^2}{2L^2}$. Then, it is easy to show that

$$u = -(6 + \frac{125}{2})L\xi_2$$

leads to

$$\dot{V}_2 + \frac{9}{\gamma^2}y^2 - 9w^2 \le -\frac{z^4}{4} - L||\varepsilon||^2 - (L - 256)\hat{x}_1^2 - \frac{1}{L}\xi_2^2.$$

From the dissipation inequality above, it is concluded that the output feedback controller

$$u = -(6 + \frac{125}{2})L(\hat{x}_2 + 4L\hat{x}_1) \tag{2.21}$$

with (2.19) solves the problem of global disturbance attenuation for the uncertain nonlinear system (1.5). Note that the gain L in (2.19)–(2.21) can be any positive constant satisfying $L > L^* = \max\{256,\ 80 + 32\sqrt{2},\ 2 + \frac{18}{\gamma^2}\}$.

3 Conclusion

We have addressed the problem of L_2-gain disturbance attenuation with stability by *output feedback* for a family of uncertain nonlinear systems, which cannot be dealt with by existing methods. The main contribution of the paper is a systematic construction of a linear dynamic output compensator in which the design of the high-gain observer and controller must be coupled in a delicate manner and processed simultaneously. This is substantially different from existing output feedback control schemes—most of them are based on the separation principle.

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