Hamiltonian structure of the Algebraic Riccati Equation and its Infinitesimal V-Stability

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Abstract

We will investigate the stability behavior of quadratic maps in higher dimensions. To check stability, we will use infinitesimal V-stability of critical points of the map; since the infinitesimal V-stability of a map at all of its critical points is equivalent to the stability of the map. We will establish the connection between infinitesimal V-stability of solutions to the Algebraic Riccati Equations, and the Hamiltonian eigenstructure of the solutions, by investigating the stability behavior of the corresponding Riccati map. Infinitesimal V-stability of critical points of the Riccati map is crucially related to stability of the Riccati map and characterizes the behavior of these solutions under perturbations of problem data. Gröbner Bases are used to implement the calculations.

1 Introduction

The study of behavior of maps under perturbation arises often in control theory. Often it is desired to know the effects on the solutions of perturbing the parameters of a system under study. This can usually be translated to the question of the effects of perturbations on a specific map. The maps of interest to us are often differentiable and, in fact, often polynomial. This allows us to use the tools of differential topology, algebraic geometry, and commutative algebra. Differential topology becomes relevant since the maps are differentiable and because we are interested in local behavior near critical points. Algebraic geometry and commutative algebra can be used since many of the maps are polynomial. A number of related notions of "stability" of maps exist in the differential topology literature [2, 3, 10]. Intuitively, a map is stable if a small perturbation does not drastically change the topological and differential properties of the map. Stability is a global property and in general, it is difficult to decide whether a map is stable. Locally, a differentiable map that does not have any critical points is differentiably equivalent to the identity map and hence the obstruction to stability is in the nature of the critical points. We thus turn to local notions of stability of a map at one of its critical points. One such notion is called the "infinitesimal V-stability" of a map at a critical point. This notion has its roots in the work of Malgrange, Mather, and Arnold. We use the formulation of Arnold, Gusein-Zade and Varchenko [2]. The importance of this notion is two-fold. On the one hand, combining theorems of Mather, Malgrange, Arnold, and others we see that under some mild conditions, the infinitesimal V-stability of a map at all of its critical points is equivalent to the stability of the map. On the other hand, infinitesimal V-stability is defined in algebraic terms and thus one can use methods from commutative algebra to decide infinitesimal V-stability of a map at a critical point.

Arnold, Gusein-Zade and Varchenko use ad hoc algebraic arguments to decide whether specific maps are infinitesimally V-stable. One of our contributions is to use the so-called Gröbner bases to simplify these arguments. We will show this simplification in an example. As an illustration of our methods, we will investigate the stability of the "Riccati Map". This is a natural map arising from the Algebraic Riccati Equation (ARE) whose zeros are the solutions to the ARE. Bucy [4] has defined a notion of structural stability of these solutions. It follows from his work that a solution to the ARE is structurally unstable if the corresponding zero of the Riccati Map is a critical point. For this reason, we can further classify the structurally unstable solutions to the ARE. Some such solutions are infinitesimally V-stable and some are infinitesimally V-unstable. We develop a number of conjectures relating the eigenstructure of the Hamiltonian matrix arising from ARE, the infinitesimal V-stability of the structurally unstable solutions, and the properties of the quadratic differential. We will prove for the 2×2 case that there is direct connection between the eigenstructure of the Hamiltonian matrix corresponding to that solution, the infinitesimal V-stability of those solutions, and a differentiable invariant called the quadratic differential. We will make this algebraic approach very practical by simplifying the calculations using Gröbner bases. Due to complexity of calculations and dimension-dependence, this approach has to be carried over one dimension at a time. We will prove our results in $\mathbf{R}^3$ i.e., for 2×2 AREs. However, we will present conjectures that generalize our results to higher dimensions. This dimension-dependence is not surprising in this kind of problems, as exemplified by Mather's "good dimensions" [14]. In this paper we will first give a brief background to infinitesimal V-stability, quadratic differential and Gröbner bases. This will be followed by a discussion of stability of quadratic maps, structural stability of solutions to ARE and Riccati maps. We will then state our theorems in low-dimensions and conjectures in higher-dimensions and due to space limitation will give proof to some theorems.

2 Infinitesimal V-Stability and the Quadratic Differential

We need to first introduce some basic definitions. We follow [2] closely, for more detailed description see [7].

Definition 1 *Given a function $f:\mathbf{R}^n\to\mathbf{R}^n$, $x\in\mathbf{R}^n$ is a* critical point *of f if the Jacobian of f at x, J_xf, is not invertible.*

Definition 2 *Two differentiable maps are topologically conjugate or differentiably equivalent if one can transform one map into the other by means of smooth changes of the independent* *and dependent variables.*

Definition 3 *A differentiable map f is* stable *if every map sufficiently close to it (in the Whitney topology) is differentiably equivalent to it.*

Definition 4 *A differentiable map f is said to be* infinitesimally stable *if, for any differentiable deformation field u, the equation*

$$u(x) = -\frac{\partial f}{\partial x} h(x) + k(f(x))$$

can be solved for differentiable vector fields h and k.

Stability and infinitesimal stability are global concepts. We now introduce a more easily tractable local concept.

Definition 5 *If two functions agree in some neighborhood of a point x, then we consider them* equivalent*. The collection of equivalent functions at the point x is called a* map-germ *at the point x.*

Definition 6 *[2] Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a differentiable map germ with $f(0) = 0$. Let $f = (f_1, f_2, \ldots, f_m)$ where each $f_i : \mathbf{R}^n \longrightarrow \mathbf{R}$ is a coordinate map of f. Let A_x be the commutative algebra of formal power series in the variables $x_1, x_2, \ldots, x_n$. Let I be the submodule of $(A_x)^m$ generated by $\frac{\partial f}{\partial x_j}$ and $f_i e_r$ for $j = 1, \ldots n$; and $i, r = 1, \ldots m$. The map germ f is said to be* (infinitesimally) V-stable *if the images of the basis vectors $e_1, \ldots, e_m$ generate over $\mathbf{R}$ the quotient module $T = (A_x)^m / I$.*

The following theorems motivates our interest in infinitesimal V-stability:

Theorem 7 *([2, page 129],[10]) The infinitesimal V-stability of a germ of a differentiable map is equivalent to its infinitesimal stability.*

Theorem 8 (Mather's Theorem,[2],[10]) *Infinitesimally stable maps are stable and vice versa.*

We will further make connections between infinitesimal V-stability and a differentiable invariant, the **quadratic differential**. The quadratic differential is, roughly, the quadratic part of the Taylor expansion of a map at a point where the linear term vanishes.

Definition 9 *Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a differentiable map with $f(0) = 0$. The Jacobian of f at 0 is a linear transformation $J_f(0) : \mathbf{R}^n \to \mathbf{R}^m$. Let Im denote the image of $J_f(0)$ in $\mathbf{R}^m$. The quadratic differential of f at 0 is the map*

$$d^2 f_0 : \ker(J_f(0)) \longrightarrow coker(J_f(0)) = \mathbf{R}^m / Im$$

defined by

$$d^2 f_0(v) = \lim_{t \to 0} \frac{f(tv)}{t^2} / Im \in \mathbf{R}^m / Im,$$

for $v \in \ker(J_f(0))$.

If two differentiable germs give rise to distinct quadratic differentials of different rank then the two germs will not be differentiably equivalent. Now $d^2 f_0$ is a non-linear map between equidimensional vector spaces. Thus its image may not (in fact, usually will not) be a subspace of the cokernel. However, in the case of polynomial maps, the image will be a variety and hence we can associate with it a dimension (the dimension of a tangent space at a non-critical point).

Definition 10 *If the dimension of the image of $d^2 f_0$ is less than the dimension of the kernel of $J_f(0)$, then we say that $d^2 f_0$ has a* rank drop *or is* singular*.*

3 Gröbner Bases

In general, it is difficult to check if a differentiable map is stable. Infinitesimal V-stability provides an algebraic method for deciding the stability of a map. However, it is not easy to check the infinitesimal V-stability criterion due to complexity of algebraic calculations. However, the calculations are much simplified by using Gröbner bases. Gröbner bases were introduced by Bruno Buchberger in 1965. The terminology acknowledges the influence of Wolfgang Gröbner, his thesis advisor, on Buchberger's work. We assume that the reader is familiar with the basic definitions of rings and ideals. Let k be a field (for our purposes the field is usually the real numbers or the complex numbers) and let $k[x]$ be the algebra of polynomials in one variable. Let I be an ideal in $k[x]$. An example of an ideal is the set of polynomials $p(x)$ such that $p(A) = 0$ for some given $n \times n$ matrix A. Ideals in $k[x]$ behave very nicely. If $f(x), g(x) \in k[x]$, then we can divide f by g and get a unique quotient and remainder:

$$f(x) = g(x)q(x) + r(x), \qquad \text{with } r(x) = 0 \text{ or } \deg(r(x)) < \deg(q(x)).$$

This simple property allows one to answer many questions about ideals easily. Now if we have a polynomial and we want to know whether it is in the ideal or not we just have to divide it by the generator. If the remainder is zero then the polynomial belongs to the ideal and otherwise it does not. Thus answering an ideal membership question is an easy one for polynomials of one variable. The situation becomes more complicated for polynomials of several variables. Let k be a field and let $k[x_1, \ldots, x_n]$ be the algebra of polynomials in n variables over k. If I is an ideal in this ring then it is not necessary generated by one element, and we do not have an exact analogue of the division algorithm. By the celebrated Hilbert Basis Theorem, I is generated by a finite number of polynomials. However, not all sets of generators have the same usefulness in answering questions about ideals. A Gröbner basis is a special type of a generating set for an ideal that allows us to answer many questions (e.g., ideal membership) easily. For a detailed development of this material see [1]. In order to check the infinitesimal V-stability of solutions to ARE we use Gröbner basis for modules. In particular, we will find a Gröbner basis for the submodule I generated by $\frac{\partial f}{\partial x_j}$ and $f_i e_r$ as was stated in definition 6. We will use Maple to compute Gröbner basis for our examples.

4 Stability of Quadratic Maps

It is often hard to show that a map is stable. The stability of real valued differentiable functions from $\mathbf{R}^n$ to R is well understood by Morse theory [15]. However, other than real valued differentiable maps, only some other special classes of maps have been classified according to their stability. Whitney [18] solved the problem for maps from a *two* dimensional manifold to a *two* dimensional manifold. For control theory interpretation and our particular interest on Riccati maps we are interested in quadratic maps between equi-dimensional spaces from $\mathbf{R}^n$ to $\mathbf{R}^n$. In this section we give number of theorems and conjectures that we have for quadratic maps from $\mathbf{R}^n$ to $\mathbf{R}^n$. For $n > 3$, it is hard to find stable classes of maps in n-dimensional manifolds, since as mentioned earlier, stable maps are *not* dense in higher dimensions. We will find a class of stable quadratic maps in $\mathbf{R}^n$. In this paper, we are particularly interested in quadratic maps in higher dimensions, since the Riccati map is a quadratic map and we would like to study it in high dimensions. The following theorems, and conjectures represent different forms of quadratic Riccati maps from $\mathbf{R}^n$ to $\mathbf{R}^n$.

Theorem 11 *Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}^n$, be defined by:*

$$f(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ y \end{bmatrix}) = \begin{bmatrix} a_{11}x_1 + \ldots + a_{1n-1}x_{n-1} + b_1y^2 \\ a_{21}x_1 + \ldots + a_{2n-1}x_{n-1} + b_2y^2 \\ \vdots \\ a_{n-11}x_1 + \ldots + a_{n-1n-1}x_{n-1} + b_{n-1}y^2 \\ by^2 \end{bmatrix}.$$

Let

$$F = \begin{bmatrix} a_{11} & \ldots & a_{1n-1} \\ \vdots & & \\ a_{n-11} & \ldots & a_{n-1n-1} \end{bmatrix}$$

and assume $b \neq 0$ and F is invertible. Then f is infinitesimally V-stable with a non-singular quadratic differential.

Theorem 12 *Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}^n$, be defined by:*

$$f(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ y \end{bmatrix}) = \begin{bmatrix} a_{11}x_1 + \ldots + a_{1n-1}x_{n-1} \\ a_{21}x_1 + \ldots + a_{2n-1}x_{n-1} \\ \vdots \\ a_{n-11}x_1 + \ldots + a_{n-1n-1}x_{n-1} \\ by^2 \end{bmatrix}.$$

Assume $b \neq 0$ and F is invertible and assumed to be as in theorem 11. Then f is infinitesimally V-stable with a non-singular quadratic differential.

Conjecture 1 *Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}^n$, be defined by:*

$$f(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ y \end{bmatrix}) = \begin{bmatrix} a_{11}x_1 + \ldots + a_{1n-1}x_{n-1} + b_1 y^2 + Q(x_1, x_2, \ldots, x_{n-1}, y) \\ a_{21}x_1 + \ldots + a_{2n-1}x_{n-1} + b_2 y^2 + Q(x_1, x_2, \ldots, x_{n-1}, y) \\ \vdots \\ a_{n-11}x_1 + \ldots + a_{n-1n-1}x_{n-1} + b_{n-1} y^2 + Q(x_1, x_2, \ldots, x_{n-1}, y) \\ by^2 + Q(x_1, x_2, \ldots, x_{n-1}, y) \end{bmatrix}.$$

Let $Q(x_1, x_2, \ldots, x_{n-1}, y)$ be any quadratic polynomial. Assume $b \neq 0$ and F is invertible and assumed to be as in theorem 11. Then f is infinitesimally V-stable with a non-singular quadratic differential.

Conjecture 2 *Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, and $f : \mathbf{R}^n \longrightarrow \mathbf{R}^n$, be defined by:*

$$f(X) = J_f(0)X + \begin{bmatrix} X^T K_1 X \\ X^T K_2 X \\ \vdots \\ X^T K_n X \end{bmatrix};$$

where K_i is any quadratic term and the linear term $J_f(0)$ is the Jacobian matrix of the map f at 0;

$$J_f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

Let 0 be the critical point of f, and rank$(J_f(0)) = n - 1$.

Assume Ker$(J_f(0))\oplus$ Im$(J_f(0)) = \mathbf{R}^n$. Let $a = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$ be a basis for Ker$(J_f(0))$.

If $\begin{bmatrix} a^t K_1 a \\ a^t K_2 a \\ \vdots \\ a^t K_n a \end{bmatrix} \notin$ Im$(J_f(0))$, then f is infinitesimally V-stable with a non-singular quadratic differential.

Conjecture 3 *Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}^n$, be any quadratic map. If f is infinitesimally V-stable then the corresponding quadratic differential is non-singular.*

Conjecture 4 *Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}^n$, be a quadratic map. Let J_f be the Jacobian matrix of f. If rank $(J_f(0)) < n - 1$ then f is infinitesimally V-unstable.*

The above theorems, and conjectures represent Riccati maps with different associated Hamiltonian eigenstructure. We have many examples to support our conjectures.

5 ARE and Structural Stability

The Algebraic Riccati Equation is

$$A^T P + PA + Q - PBR^{-1}B^T P = 0. \tag{5.1}$$

There are three kinds of symmetric solutions to the ARE. The stabilizing solution P_+, the anti-stabilizing solution P_-, and the mixed solutions P_θ with both negative and positive eigenvalue real parts. All three solution types are needed to know the complete phase portrait of the Riccati differential equations. Given a symmetric solution for the ARE, Bucy in [4] calls a solution *structurally stable* iff it is continuously deformed and keeps the same closed-loop stability properties (inertia) under data perturbation. It can easily be shown that P_+ and P_- are structurally stable. However, P_θ may or may not be structurally stable depending on how the RHP and LHP eigenvalues of the corresponding Hamiltonian are combined. Bucy's main result in [4] relates the invertibility of Jacobian matrix J to the structural stability of the solutions to ARE:

Theorem 13 (Bucy) *A solution P of the ARE is structurally stable if and only if J is invertible.*

Structurally unstable solutions are good candidates for bifurcation. Depending on the eigenstructure of the associated Hamiltonian matrix, a mixed solution may be structurally unstable. In fact from Bucy's work, it can be shown that repeated eigenvalues of the Hamiltonian matrix is a necessary condition for the mixed solution of the ARE to be structurally unstable. Therefore, for these solutions we can define a Riccati map such that the solution is a critical point of the Riccati map and we can further analyze the solution to decide its infinitesimal V-stability. That criterion will determine whether that particular mixed solution bifurcates under data perturbation. Mixed solutions, $P_\theta s$, have an application in smoothing problems when we need to estimate the state of the system using both past and future measurements. The potential instability of the smoothing solution can be justified from the fact that smoothing requires a choice (of past and future observations) and that it might not be possible to make that choice continuously under data perturbation.

6 The Stability of the Riccati map and the eigenstructure of the Hamiltonian matrix: Results

Given a solution P_{sol} to the ARE (5.1), let $\overline{P} = P - P_{sol}, M = BR^{-1}B^T, F = A^T - P_{sol}M$. Define the matrix Riccati map as $Ric(\overline{P}) = -\overline{P}M\overline{P}^T + F\overline{P}^T - \overline{P}F^T$. Ric is a map on the set of $n \times n$ symmetric matrices with a linear part $(F\overline{P}^T - \overline{P}F^T)$ and a quadratic part $(-\overline{P}M\overline{P}^T)$. $Ric(0) = 0$ corresponds to $P = P_{sol}$. Rewriting Ric in vector format and considering only the entries on or above the diagonal we get the Riccati map $f : \mathbf{R}^{\frac{n(n+1)}{2}} \longrightarrow \mathbf{R}^{\frac{n(n+1)}{2}}$. Now, by Bucy's work [4], 0 is a critical point of the Riccati map if and only if P_{sol} is a structurally unstable solution of the ARE. The Riccati map captures all the information about the ARE and hence understanding the nature of its behavior near zero amounts to understanding the behavior of the solution to the ARE. What allows us to proceed and develop detailed theorems in low dimensions and conjectures in higher dimensions about the stability and infinitesimal V-stability of solutions to ARE is the fact that much information about the stability of solutions to ARE can be gleaned from the eigenstructure of the Hamiltonian matrix. Here we will briefly discuss the connection between infinitesimal V-stability of solution to ARE and the Hamiltonian eigenstructure of that particular solution. Define the Hamiltonian of the ARE as $\mathcal{H} = \left[\begin{array}{c|c} A & -M \\ \hline -Q & -A^T \end{array}\right]$. $\mathcal{H}$ can be written in terms of its eigenvectors and eigenvalues, $\mathcal{H} = VDV^{-1}$. We pick V to be a symplectic matrix of generalized eigenvectors of $\mathcal{H}$, and D is either a diagonal matrix of eigenvalues of $\mathcal{H}$ or a Jordan matrix, depending on the structure of the Hamiltonian matrix [13]. Let $J = \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right]$. Since we chose V to be symplectic we have $V^{-1} = -JV^TJ$ and therefore $\mathcal{H} = VDV^{-1} = -VDJV^TJ$. Using this we can proceed to write M, F (and thus the linear and quadratic part of the matrix Riccati map) in terms of eigenvalues and eigenvectors of $\mathcal{H}$. Having rewritten the Riccati map in terms of the eigenstructure of the Hamiltonian matrix, we can proceed to analyze the stability of this Riccati map and relate it to the quadratic differential of the map. Due to complexity of calculations, this approach is conducted on a dimension by dimension basis. However, we will present conjectures that generalize our results to higher dimensions. We will prove our conjectures in $\mathbf{R}^3$ i.e., for 2×2 AREs. The results are as follows: if the associated Hamiltonian matrix $\mathcal{H}$ is diagonalizable with multiple eigenvalues then the Riccati map has a singular quadratic differential and moreover, the critical point is infinitesimally V-unstable. On the other hand when char($\mathcal{H}$) = min($\mathcal{H}$)(char($\mathcal{H}$) and min($\mathcal{H}$) are respectively the characteristic and minimal polynomials of the Hamiltonian matrix), the quadratic differential is nonsingular and the critical point is infinitesimally V-stable. Here, we will present the theorems that we have for $f : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ relating the quadratic differential and infinitesimal V-stability of the Riccati map to the eigenstructure of $\mathcal{H}$.

Theorem 14 *Let $f : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ be a Riccati map. Assume $\mathcal{H}$ the associated Hamiltonian matrix has same repeated eigenvalues, zero is not an eigenvalue of $\mathcal{H}$, and $\mathcal{H}$ is diagonalizable. Let P_{sol} be a mixed solution to the ARE constructed by the Hamiltonian construction and from eigenvalues $\lambda, -\lambda$ of $\mathcal{H}$. Then the quadratic differential of the Riccati map is singular, furthermore, the mixed solution is structurally unstable and infinitesimally V-unstable.*

Theorem 15 *Let $f : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ be a Riccati map. Assume $\mathcal{H}$ the associated Hamiltonian matrix has same repeated eigenvalues, zero is not an eigenvalue of $\mathcal{H}$, and min($\mathcal{H}$)=char($\mathcal{H}$). Let P_{sol} be a mixed solution to the ARE constructed by the Hamiltonian construction and* *from eigenvalues $\lambda, -\lambda$ of $\mathcal{H}$. Then the mixed solution is structurally unstable and infinitesimally V-stable, furthermore, the quadratic differential of the Riccati map is non-singular.*

Sketch of the Proof: For brevity, we will only show the first part of the proof for theorem (15). For the proof of the other part and other theorems in this paper we refer the reader to [6], [7]. The proof has two parts; we first show that the map $\overline{Ric}$ is infinitesimally V-stable and then we prove that the quadratic differential map is non-singular. We know by Bucy's result that when $\mathcal{H}$ has repeated real eigenvalues then the mixed solutions to the ARE are structurally unstable and therefore 0 is a critical point of the associated Riccati map.
Part I Proof: First we will show the mixed solutions to the ARE are infinitesimal V-stable. We will prove directly using infinitesimal V-stability criterion. The key feature, which generalizes to higher dimensions, is that the Hamiltonian eigenstructure provides a canonic form for the map $\overline{f}$ endowing the Gröbner basis with an identifiable structure. We will first define $\mathcal{H}$ in terms of its eigenstructure. Let $V = \left[\begin{array}{c|c|c} v_{11} & v_{12} & v_{13} \\ \hline v_{21} & v_{22} & v_{23} \end{array}\right]$ be any matrix of eigenvectors of $\mathcal{H}$. V can be chosen so that it is symplectic and therefore, $V^{-1} = -JV^TJ$. We know that $\mathcal{H} = VJ_dV^{-1}$, where $J_d = \left[\begin{array}{c|c} Jd_1 & 0 \\ \hline 0 & Jd_2 \end{array}\right]$, where J_d is a Jordan form and $Jd_1 = -{Jd_2}^T$ and λ is the eigenvalue of $\mathcal{H}$, using the result in [13] we have; $Jd_1 = \begin{bmatrix} -\lambda & 0 \\ 1 & -\lambda \end{bmatrix}, Jd_2 = \begin{bmatrix} \lambda & -1 \\ 0 & \lambda \end{bmatrix}$. Let $P_{sol} = v_{22}v_{12}^{-1}$ be the mixed solution to the ARE. We can rewrite the Riccati map in terms of the eigenstructure of the Hamiltonian matrix. Rewriting the original map, using the result in [7], we have the following:(with M being the quadratic part and F being the linear part of the map) $M \sim \overline{M} = \begin{bmatrix} \alpha\lambda & 1 \\ 1 & \gamma\lambda \end{bmatrix}$ for $\alpha, \gamma, \lambda \in \mathbf{R}$, and $F \sim \begin{bmatrix} -\lambda & 0 \\ 0 & \lambda \end{bmatrix}$.

$$\overline{Ric} : Sym_{2\times 2} \longrightarrow Sym_{2\times 2}.$$

By further work, with α, γ to be any constant and λ being the eigenvalue of the Hamiltonian matrix, we can show that,

$$\overline{Ric} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} \alpha\lambda x^2 + \gamma\lambda y^2 + 2xy - 2\lambda x \\ y^2 + \alpha\lambda xy + xz + \gamma\lambda yz \\ \gamma\lambda z^2 + \alpha\lambda y^2 + 2yz + 2\lambda z \end{bmatrix}.$$

The Jacobian matrix is:

$$J = \begin{bmatrix} 2\alpha\lambda x + 2y - 2\lambda & 2x + 2\gamma\lambda y & 0 \\ \alpha\lambda y + z & \alpha\lambda x + 2y + \gamma\lambda z & x + \gamma\lambda y \\ 0 & 2\alpha\lambda y + 2z & 2y + 2\gamma\lambda z + 2\lambda \end{bmatrix}.$$

Clearly $J(0) = 0$ and therefore 0 is a critical point to the above Riccati map. Let I denote the submodule generated over $\mathbf{R}$[[x,y,z]] by the columns

$$\begin{bmatrix} \alpha\lambda x^2 + \gamma\lambda y^2 + 2xy - 2\lambda x \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \alpha\lambda x^2 + \gamma\lambda y^2 + 2xy - 2\lambda x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \alpha\lambda x^2 + \gamma\lambda y^2 + 2xy - 2\lambda x \end{bmatrix},$$

$$\begin{bmatrix} y^2 + \alpha\lambda xy + xz + \gamma\lambda yz \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y^2 + \alpha\lambda xy + xz + \gamma\lambda yz \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ y^2 + \alpha\lambda xy + xz + \gamma\lambda yz \end{bmatrix},$$

$$\begin{bmatrix} \gamma\lambda z^2 + \alpha\lambda y^2 + 2yz + 2\lambda z \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \gamma\lambda z^2 + \alpha\lambda y^2 + 2yz + 2\lambda z \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \gamma\lambda z^2 + \alpha\lambda y^2 + 2yz + 2\lambda z \end{bmatrix},$$

$$\begin{bmatrix} 2\alpha\lambda x + 2y - 2\lambda \\ \alpha\lambda y + z \\ 0 \end{bmatrix}, \begin{bmatrix} 2x + 2\gamma\lambda y \\ \alpha\lambda x + 2y + \gamma\lambda z \\ 2\alpha\lambda y + 2z \end{bmatrix}, \begin{bmatrix} 0 \\ x + \gamma\lambda y \\ 2y + 2\gamma\lambda z + 2\lambda \end{bmatrix}.$$

We first find a Gröbner basis for I. The Gröbner basis G, (using Lexicographic ordering with $(x > y > z)$) consists of the following:

$$G = \{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ z \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix} \}.$$

Hence $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + I$ generates $T = (\mathbf{R}[[x,y,z]])^3/I$ over $\mathbf{R}$, and thus the critical point to the above Riccati map, 0, is infinitesimally V-stable. Therefore, the mixed solution to the ARE is infinitesimally V-stable when $\text{char}(\mathcal{H}) = \min(\mathcal{H})$.

Now we will state our conjectures about Riccati maps in higher dimensions:

Conjecture 5 *Let $f : \mathbf{R}^{\frac{n(n+1)}{2}} \to \mathbf{R}^{\frac{n(n+1)}{2}}$ be a Riccati map. Assume $\mathcal{H}$ the associated Hamiltonian matrix has same repeated eigenvalues, zero is not an eigenvalue of $\mathcal{H}$, and $\underline{\mathcal{H}\text{ is diagonalizable}}$. Let P_{sol} be a $n \times n$ mixed solution to the ARE constructed by the Hamiltonian construction and from eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathcal{H}$. Assume there is a pair (i,j) with $\lambda_i + \lambda_j = 0$. Then P_{sol} is a <u>structurally unstable</u> and <u>infinitesimally V-unstable</u> mixed solution for the ARE.*

Conjecture 6 *Let $f : \mathbf{R}^{\frac{n(n+1)}{2}} \to \mathbf{R}^{\frac{n(n+1)}{2}}$ be a Riccati map. Assume $\mathcal{H}$ the associated Hamiltonian matrix has same repeated eigenvalues, zero is not an eigenvalue of $\mathcal{H}$, and $\underline{\text{char}(\mathcal{H}) = \min(\mathcal{H})}$. Let P_{sol} be a $n \times n$ mixed solution to the ARE constructed by the Hamiltonian construction and from eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathcal{H}$. Assume there is a pair (i,j) with $\lambda_i + \lambda_j = 0$. Then P_{sol} is a <u>structurally unstable</u> mixed solution and under some condition for the quadratic term of the Riccati map, P_{sol} is <u>infinitesimally V-stable</u> mixed solution for the ARE.*

7 Example

As an illustration of the above, we will give an example. In this example, $\mathcal{H}$ has repeated eigenvalues and $\mathcal{H}$ is diagonalizable which results in family of solutions for ARE. The mixed solution is structurally unstable and therefore 0 is a critical point of the Riccati map. Here, we will look at the quadratic differential of the associated Riccati map and we will show that, it has a rank drop. Furthermore, we will use Arnold's criteria for infinitesimal V-stability combined with Gröbner bases techniques to show that the mixed solutions are infinitesimally V-unstable and therefore by Mather's theorem, the Riccati map associated with any of these mixed solutions is unstable.

Consider the 2×2 Riccati equation $P^2 = I$, where

$$A = A^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M = BR^{-1}B^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The stabilizing solution is I, the antistabilizing solution is $-I$, and the mixed solutions form a continuum, $P_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$. The existence of the *continuum* of mixed solutions can easily be seen from the eigenstructure of the Hamiltonian matrix, $\mathcal{H} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$. This Hamiltonian matrix has repeated eigenvalues at 1, -1. There are two eigenvectors associated to each repeated eigenvalue and therefore $\mathcal{H}$ is diagonalizable or $\mathrm{char}(\mathcal{H}) > \min(\mathcal{H})$. The symplectic V, matrix of eigenvectors for $\mathcal{H}$ is $V = \left[\begin{array}{cc|cc} 1 & 0 & -.5 & 0 \\ 0 & .5 & 0 & 1 \\ \hline 1 & 0 & .5 & 0 \\ 0 & -.5 & 0 & 1 \end{array}\right]$ and $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$, where $d_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $d_2 = -d_1$. We have $\mathcal{H} = VDV^{-1}, V^TJV = J$, for $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. Let $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ be a mixed solution.

The closed loop matrix $F(P) = A^T - PM = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ has eigenvalues at 1, and -1. This linear part of the Riccati map is similar to d_2. Under an arbitrarily small perturbation of the coefficient matrices the set of mixed solutions goes from a continuum to a finite set. We define x, y, y', and z, where

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} x & y \\ y' & z \end{bmatrix}$$

and we focus our analysis on a neighborhood of this particular mixed solution. According to Bucy's result, the mixed solution is structurally unstable since $J_P f$ is singular and under perturbation of the coefficient matrices the mixed solution will bifurcate. Consider the following perturbation of the coefficient matrices where, A and $BR^{-1}B^T$ are as before and $Q = \begin{bmatrix} 1+\epsilon & 0 \\ 0 & 1 \end{bmatrix}$. For $\epsilon > 0$, we see that number of mixed solutions (P_θ) will decrease:

$$P_+ = \begin{bmatrix} \sqrt{\epsilon} & 0 \\ 0 & 1 \end{bmatrix}, P_- = \begin{bmatrix} -\sqrt{\epsilon} & 0 \\ 0 & -1 \end{bmatrix}, P_{\theta_1} = \begin{bmatrix} -\sqrt{\epsilon} & 0 \\ 0 & 1 \end{bmatrix}, P_{\theta_2} = \begin{bmatrix} \sqrt{\epsilon} & 0 \\ 0 & -1 \end{bmatrix},$$

and for $\epsilon < 0$ there will be no solution. We now reinterpret these odd features in the context of infinitesimal V-instability of the Riccati map $P \mapsto P^2 - I$. In the (x, y, z) variables(enforcing symmetric $y = y'$ property) , the Riccati map becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x^2 + (y+1)^2 - 1 \\ xy + x + zy + z \\ (y+1)^2 + z^2 - 1 \end{bmatrix}.$$

The Jacobian of the Riccati map is

$$J_f = \begin{bmatrix} 2x & 2(y+1) & 0 \\ y+1 & x+z & y+1 \\ 0 & 2(y+1) & 2z \end{bmatrix}.$$

This Jacobian is clearly rank deficient at $(x, y, z) = (0, 0, 0)$. As stated and proved by theorem 14 we can show that the H-map which in this case is identical to the quadratic differential of the above Riccati map is singular.

$$Ker(J_f(0)) = \{\begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} \mid x \in \mathbf{R}\}, Im(J_f(0)) = \{\begin{bmatrix} x \\ y \\ x \end{bmatrix} \mid x, y \in \mathbf{R}\}.$$

The quadratic differential is:

$$d^2 f_0(\begin{bmatrix} x \\ 0 \\ -x \end{bmatrix}) = \lim_{t \to 0} \begin{bmatrix} \frac{t^2 x^2}{t^2} \\ 0 \\ \frac{t^2 x^2}{t^2} \end{bmatrix} / \mathrm{Im}(J_f(0)) = \begin{bmatrix} x^2 \\ 0 \\ x^2 \end{bmatrix} / \mathrm{Im}(J_f(0)) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} / \mathrm{Im}(J_f(0)).$$

Thus the quadratic differential is singular. Now our concern is infinitesimal V-stability of the above map around $0 \mapsto 0$. Let I denote the submodule generated over the ring $\mathbf{R}[[x, y, z]]$ by the columns

$$\begin{bmatrix} x^2 + (y+1)^2 - 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x^2 + (y+1)^2 - 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ x^2 + (y+1)^2 - 1 \end{bmatrix}, \begin{bmatrix} 0 \\ xy + x + zy + z \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} xy + x + zy + z \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ xy + x + zy + z \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ (y+1)^2 + z^2 - 1 \end{bmatrix}, \begin{bmatrix} 0 \\ (y+1)^2 + z^2 - 1 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} (y+1)^2+z^2-1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2x \\ 2(y+1) \\ 0 \end{bmatrix}, \begin{bmatrix} (y+1) \\ x+z \\ (y+1) \end{bmatrix}, \begin{bmatrix} 0 \\ 2(y+1) \\ 2z \end{bmatrix}.$$

To check stability of the Riccati map, we have to check whether $T = (\mathbf{R}[[x,y,z]])^3/I$ is generated over $\mathbf{R}$ by the images of the basis vectors of $\mathbf{R}^3$. We first find a Gröbner basis for I. The Gröbner basis G, (using Lexicographic ordering with $(x > y > z)$) consists of the following:

$$G = \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -z^2+1 \\ zy+z \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ y^2+2y+z^2 \end{bmatrix}, \begin{bmatrix} 0 \\ y+1 \\ z \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ x+z \end{bmatrix}, \begin{bmatrix} 0 \\ x+z \\ 0 \end{bmatrix} \}.$$

By closer observation, we can show that T is *not* generated over $\mathbf{R}$ by the images of the basis vectors of $\mathbf{R}^3$. Therefore the Riccati map is infinitesimally V-unstable at 0 and as we stated in our theorem the mixed solutions will bifurcate under perturbation of coefficient matrices.

8 Conclusion

We have investigated the stability of quadratic maps in higher dimensions using infinitesimal V-stability criterion. We have further classified the structurally unstable solutions to the ARE into infinitesimal V-stable and infinitesimal V-unstable solutions. The latter are known to bifurcate under data perturbation. There is a direct connection between the eigenstructure of the corresponding Hamiltonian matrix and stability of the corresponding Riccati map. The implementation to check infinitesimal V-stability of the critical points of the Riccati map can be substantially simplified by means of Gröbner bases. It can be observed that dimension plays an important role on stability of above Riccati maps.

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