# A Linear Controller for a Multifrequency Model of a Pulse-Width-Modulated Ćuk Converter

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**Abstract**

This paper presents a linear controller for a multifrequency model of a pulse-width-modulated Ćuk converter. The controller is based on an equivalent discrete-time version of the linearized multifrequency model and uses the continuous-time output over one switching period. Using this combination we determine the duty ratio at the beginning of every switching period. We discuss how this controller, designed for an arbitrary number of harmonics, can be applied in a realistic situation. Simulations are given to demonstrate the influence of the controller on the response of the system.

# 1 Introduction

Analysis of DC-DC converters is often based on their switching behavior. Having a pulse-width-modulator, the switching points are basically determined by two strategies, namely fixed pulse-width modulation and running pulse-width modulation. A previous work [1], using the Boost converter as an example, analyzed the system based on the assumption that the whole state of the system is known. This is not so realistic in applications since then the state variables of the system are often not available. Furthermore, in the closed loop situation the switching points were determined based on a running pulse-width-modulator.

This paper presents a realistic approach to determine a switching point based on the output at the beginning of every switching period, which is the fixed pulse-width-modulator. We are inspired by the results of [2], [5] to exploit the behavior of the multifrequency model.

Based on the stationary periodic condition and all the output information we control the system by computing the duty ratio at the beginning of every switching period. This leads us to employ the discrete-time aspects of the linearized multifrequency model to design a linear controller for an arbitrary number of harmonics. Hence, we combine both discrete-time aspects and continuous-time aspects of the system in every switching period. This combination may be regarded as a hybrid system. As a result, we control the equivalent discrete-time version of the linearized multifrequency model by means of an output feedback to obtain the next duty ratio. Application of the obtained duty ratio to the system provides the continuous-time output for the next switching period. Our simulations confirm that the controller can produce responses which are much faster than the open loop responses in reaching the stationary periodic situation.

## 2 A multifrequency model of the Ćuk converter

Consider the Ćuk converter which is depicted in Fig. 1. We assume that all components of

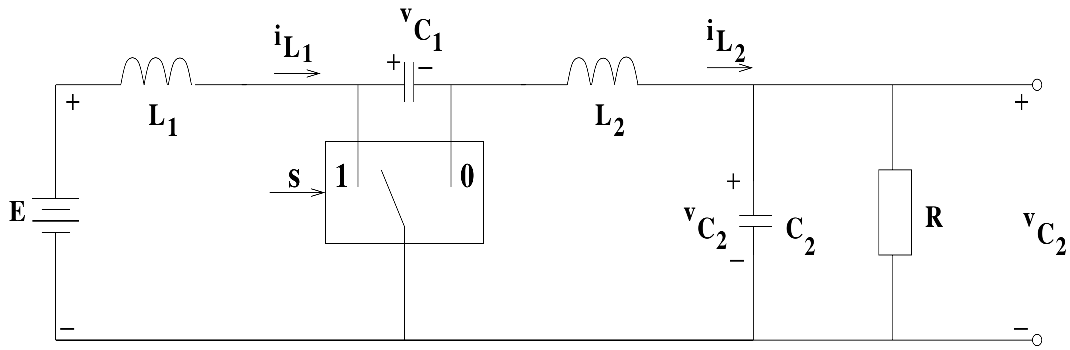


Fig. 1: Open-loop Ćuk converter.

the Ćuk converter are ideal and the circuit is in continuous conduction mode (CCM). Let $x_1(t) = i_{L_1}(t)$, $x_2(t) = v_{C_1}(t)$, $x_3(t) = i_{L_2}(t)$, and $x_4(t) = v_{C_2}(t)$ be the inductor currents and capacitor voltages. Then a continuous-time model of PWM Ćuk converter is given by

$$\begin{aligned} \dot{x}(t) &= A(s)x(t) + b, \\ y(t) &= Cx(t), \end{aligned} \tag{2.1}$$

with

$$A(s) = \begin{pmatrix} 0 & -\frac{1-s}{L_1} & 0 & 0 \\ \frac{1-s}{C_1} & 0 & \frac{s}{C_1} & 0 \\ 0 & -\frac{s}{L_2} & 0 & -\frac{1}{L_2} \\ 0 & 0 & \frac{1}{C_2} & -\frac{1}{RC_2} \end{pmatrix}, \; b = \begin{pmatrix} \frac{E}{L_1} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; C = (0 \quad 0 \quad 0 \quad 1),$$

where $E$ and $R$ denote the source voltage and load resistor value, respectively. We assume that $E$ is constant in time. The output of the system is denoted by $y$ and $s$ denotes the switch of the converter. When the switch is in the ON-position then $s = 1$ and $s = 0$ denotes the switch in the OFF-position. The position of the switch does depend on the time $t$ and is defined as

$$s(t) = \begin{cases} 1 & \text{if} \quad iT \leq t < (i + D_i)T, \\ 0 & \text{if} \quad (i + D_i)T \leq t < (i+1)T, \end{cases} \tag{2.2}$$

where $T$ denotes the switching period and $D_i$, $D_i \in (0,1)$, denotes the duty ratio in the $i^{th}$ switching interval, $\forall i \geq 0$. If $D_i = D$, $\forall i \geq 0$, where $D$ is fixed, then the switching function $s$ is periodic with period $T$. By an open loop situation we mean that system (2.1) has a fixed duty ratio $D$, $0 < D < 1$.

We recall that on the interval $[t - T,\ t]$ the real-valued function $f$ can be written as

$$f(\tau) = \sum_{k=-\infty}^{\infty} < f >_k (t)\, e^{jk\omega_s \tau}, \tag{2.3}$$

where

$$< f >_k (t) = \frac{1}{T} \int_{t-T}^{t} f(\tau)\, e^{-jk\omega_s \tau}\, d\tau, \tag{2.4}$$

where $\omega_s = \frac{2\pi}{T}$, $j^2 = -1$ and $< f >_k (t)$ are the Fourier coefficients of $f$, assuming the function to be periodic with period $T$. In this paper we use also the conventions and properties of [5] where the reader can find the details. Applying (2.4) to the original system (2.1) and using the above mentioned conventions and properties, we obtain an infinite set of differential equations that can be approximated by a finite set of the differential equations, by neglecting all indexes larger than some chosen nonnegative integer $N$, where $N$ denotes the number of "harmonics" taken into account. Let the vector $\hat{x}(t)$ and $\hat{y}(t)$ be defined as follows

$$\hat{x}(t) = \begin{pmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \\ \hat{x}_4(t) \end{pmatrix}, \ \hat{y}(t) = \begin{pmatrix} < y >_{-N} (t) \\ \vdots \\ < y >_0 \\ \vdots \\ < y >_N (t) \end{pmatrix},$$

$$\hat{x}_i(t) = \begin{pmatrix} < x_i >_{-N} (t) \\ \vdots \\ < x_i >_0 \\ \vdots \\ < x_i >_N (t) \end{pmatrix}, \ i = 1, 2, 3, 4.$$

As in [2], with $D_i = D$, $\forall i \geq 0$, we use the following notation

$$\hat{W} = \begin{pmatrix} +jN\omega_s & \dots & 0 \\ & \ddots & \\ \vdots & 0 & \vdots \\ & \ddots & \\ 0 & \dots & -jN\omega_s \end{pmatrix}, \ \hat{b} = \begin{pmatrix} 0 \\ \vdots \\ \frac{E}{L_1} \\ \vdots \\ 0 \end{pmatrix},$$

$$\hat{S} = \begin{pmatrix} < s >_0 & \dots & < s >_{-2N} \\ & \ddots & \\ \vdots & < s >_0 & \vdots \\ & \ddots & \\ < s >_{2N} & \dots & < s >_0 \end{pmatrix}.$$

Then the open loop multifrequency model of the Ćuk converter is given by

$$\begin{array}{rcl} \dot{\hat{x}}(t) & = & \hat{A}_c\, \hat{x}(t) + \hat{B}_c, \\ \hat{y}(t) & = & \hat{C}_c\, \hat{x}(t), \end{array} \tag{2.5}$$

with

$$\hat{A}_c = \begin{pmatrix} \hat{W} & -\frac{1}{L_1}(\hat{I}-\hat{S}) & \hat{0} & \hat{0} \\ \frac{1}{C_1}(\hat{I}-\hat{S}) & \hat{W} & \frac{1}{C_1}\hat{S} & \hat{0} \\ \hat{0} & -\frac{1}{L_2}\hat{S} & \hat{W} & -\frac{1}{L_2}\hat{I} \\ \hat{0} & \hat{0} & \frac{1}{C_2}\hat{I} & \hat{W}-\frac{1}{RC_2}\hat{I} \end{pmatrix},$$

$$\hat{B}_c = \begin{pmatrix} \hat{b} \\ \hat{0} \\ \hat{0} \\ \hat{0} \end{pmatrix}, \; \hat{C}_c = (\hat{0} \quad \hat{0} \quad \hat{0} \quad \hat{I}),$$

where the index "$c$" indicates the continuous-time version of the model. For $n = 2N+1$, $\hat{A}_c$ is a $(4n \times 4n)$-matrix, $\hat{B}_c$ is a $(4n \times 1)$-matrix, $\hat{I}$ is a $(n \times n)$ identity matrix and $\hat{0}$ represents zero matrix (vector) with suitable dimensions. The stationary state of (2.5), denoted $\hat{X}$, is obtained by setting the derivative of (2.5) is equal to zero. Given the number of harmonics $N$ and the stationary periodic duty ratio $D_{sp}$, the stationary output of (2.5), denoted $\hat{Y}$, can be obtained. Conversely, if the stationary output $\hat{Y}$ is known, then the corresponding duty ratio $D_{sp}$ can also be computed from the stationary state $\hat{X}$ of (2.5), since the only unknown parameter is $D_{sp}$.

In the open loop situation we can easily prove that system (2.5) is observable and asymptotically stable [2]. It follows that the system is also detectable.

# 3 Linearized of the multifrequency model

By assuming the duty ratio $D$ acts as a time-varying parameter, we define small deviations from the stationary periodic values $\hat{X}$, $\hat{Y}$ and $D_{sp}$ by $\Delta x$, $\Delta y$ and $\Delta D$, respectively, i.e.

$$\begin{array}{rcl} \Delta\hat{x} & = & \hat{x} - \hat{X}, \\ \Delta\hat{y} & = & \hat{y} - \hat{Y}, \\ \Delta D & = & D - D_{sp}. \end{array} \tag{3.6}$$

Then the linearized multifrequency model is given by

$$\begin{array}{rcl} \frac{d}{dt}\Delta\hat{x}(t) & = & \hat{\hat{A}}_c\, \Delta\hat{x}(t) + \hat{\hat{B}}_c\, \Delta D(t), \\ \Delta\hat{y}(t) & = & \hat{\hat{C}}_c\, \Delta x(t), \end{array} \tag{3.7}$$

where $\hat{\hat{A}}_c = \hat{A}_c|_{D_{sp}}$, $\hat{\hat{B}}_c = \frac{\partial \hat{A}_c}{\partial D}|_{D_{sp}}\hat{X}$ and $\hat{\hat{C}}_c = \hat{C}_c$.

It follows that (3.7) is observable and asymptotically stable. Note that (3.7) is a linear system and can be proved to be reachable.

# 4 Equivalent discrete-time linearized multifrequency model

We restrict our interest by letting $t = iT,\ i \in \{0, 1, 2, \dots\}$. In fact, $i = 0$ indicates that the model is at the initial condition with the duty ratio $D_0$ given. The situation of our interest is depicted in Fig. 2. At the beginning of the $(i+1)^{th}$ switching interval, i.e. $[(i+1)T,\ (i+2)T)$, we start to compute the duty ratio $D_{i+1}$ using $D_i$ and output $y$ on the interval $[iT,\ (i+1)T)$. We may control the system so that the duty ratio $D_i$ converges to $D_{sp}$ for $i \longrightarrow \infty$. This leads us to work for only one switching period in every computation step and to work with

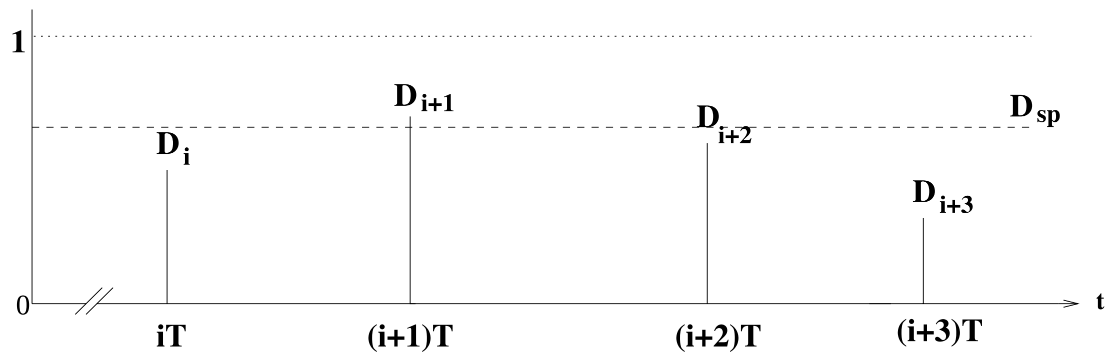


Fig. 2: Situation of the duty ratio at the $(i+1)^{th}$ switching interval.

the so-called discrete-time version of the system (3.7). Then we may replace (3.7) by the equivalent discrete-time version of the model of the form

$$\begin{aligned} \Delta\hat{x}_{i+1} &= \hat{\hat{A}}_d\,\Delta\hat{x}_i + \hat{\hat{B}}_d\,\Delta D_i, \\ \Delta\hat{y}_i &= \hat{\hat{C}}_d\,\Delta x_i, \end{aligned} \tag{4.8}$$

where $\hat{\hat{A}}_d = e^{\hat{\hat{A}}_c T},\ \hat{\hat{B}}_d = \hat{\hat{A}}_c^{-1}(e^{\hat{\hat{A}}_c T} - \hat{I})\hat{\hat{B}}_c,\ \hat{\hat{C}}_d = \hat{\hat{C}}_c,\ \Delta x_i = \Delta x(iT),\ \Delta y_i = \Delta y(iT)$ and $\Delta D_i = D_i - D_{sp}$.

The index "$d$" indicates the discrete-time version of the model. It is realistic to assume that the state $x$ in (2.1) is unknown and the output $y$ is constantly measured. Then we have enough information to compute the Fourier coefficients of the output $y$ at time instant $t = (i+1)T$, from the output $y$ over the interval $[iT,\ (i+1)T)$, with

$$< y >_k ((i+1)T) = \frac{1}{T}\int_{iT}^{(i+1)T} y(\tau)e^{-jk\omega_s\tau}d\tau. \tag{4.9}$$

## 5 A linear controller based on the output feedback

Consider

$$\begin{aligned} \Delta\hat{x}_{i+1} &= \hat{\hat{A}}_d \, \Delta\hat{x}_i + \hat{\hat{B}}_d \, \Delta D_i, \\ \Delta\hat{y}_i &= \hat{\hat{C}}_d \, \Delta\hat{x}_i, \end{aligned} \tag{5.10}$$

and we want to obtain the gain-matrix $K$ such that the control law

$$\Delta D_i = -K \Delta\hat{x}_i \tag{5.11}$$

minimizes the criterion

$$\mathcal{J}_K = \sum_{i=0}^{\infty} \left( \Delta\hat{x}_i^{\mathsf{T}} Q_K \Delta\hat{x}_i + (\Delta D_i)^2 \right), \tag{5.12}$$

where $Q_K = Q_K^{\mathsf{T}} \geq 0$.

Taking into account that only the output $y$ can be measured, a state observer has to be employed so that we may accurately reconstruct all unmeasured state components. We use the well-known Kalman filter to obtain such a reconstruction. This filter has the following form

$$\begin{aligned} \Delta\hat{x}^e_{i+1} &= \hat{\hat{A}}_d \, \Delta\hat{x}^e_i + \hat{\hat{B}}_d \, \Delta D_i + L(\Delta\hat{y}_i - \Delta\hat{y}^e_i), \\ \Delta\hat{y}^e_i &= \hat{\hat{C}}_d \, \Delta\hat{x}^e_i, \end{aligned} \tag{5.13}$$

where $\Delta\hat{x}^e_i$ denotes an estimate of the state $\Delta\hat{x}_i$. The gain-matrix $L$, which is associated with the dual of system (5.10) of the form

$$\begin{aligned} \Delta\hat{z}_{i+1} &= \hat{\hat{A}}_d^{\mathsf{T}} \, \Delta\hat{z}_i + \hat{\hat{C}}_d^{\mathsf{T}} \, \Delta v_i, \\ \Delta\hat{w}_i &= \hat{\hat{B}}_d^{\mathsf{T}} \, \Delta\hat{z}_i, \end{aligned} \tag{5.14}$$

such that the associated state feedback minimizes the criterion

$$\mathcal{J}_L = \sum_{i=0}^{\infty} \left( \Delta\hat{z}_i^{\mathsf{T}} Q_L \Delta\hat{z}_i + \Delta v_i^{\mathsf{T}} R_L \Delta v_i \right), \tag{5.15}$$

where $Q_L = Q_L^{\mathsf{T}} \geq 0$ and $R_L = R_L^{\mathsf{T}} > 0$. The control law $\Delta D_i = -K\Delta\hat{x}_i$ will be replaced by

$$\Delta D_i = -K \Delta\hat{x}^e_i. \tag{5.16}$$

The control situation is depicted in Fig. 3. Note that $R_L$ used in (5.15) is a $(n \times n)$ real matrix. The symmetric weighting matrices $Q_K$ and $Q_L$ may be chosen based on design rules for the linear-quadratic-regulator (LQR) techniques for discrete-time systems.

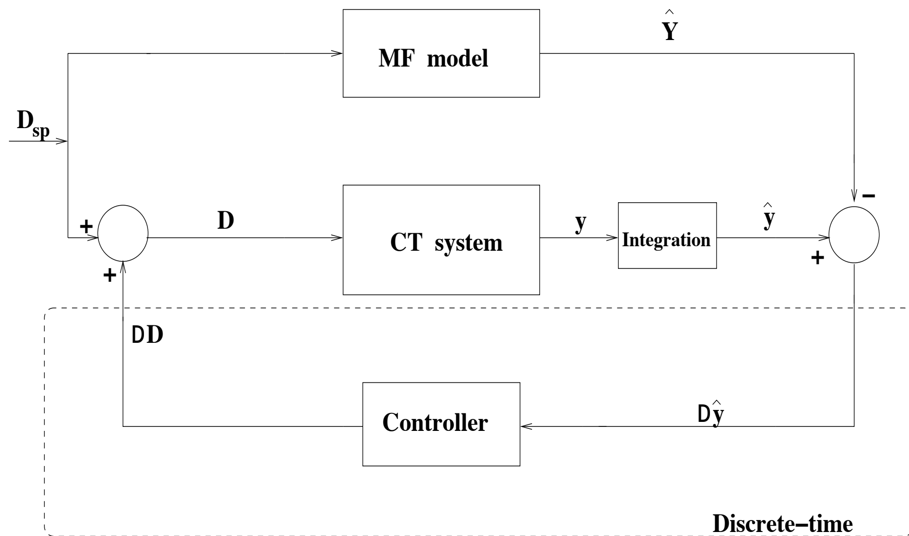


Fig. 3: The block diagram of the combined model.

# 6 Simulation

In our simulations, we used the parameters $L_1 = L_2 = 100\ \mu$H, $C_1 = C_2 = 25\ \mu$F, $R = 12$ $\Omega$, $E = 1.5$ Volt, $f_s = 10$ kHz, $D_{sp} = 0.5$, $x(0) = (0,0,0,0)^{\mathsf{T}}$, the number of harmonics $N = 5$ and $R_L = \hat{I}$, the $n \times n$ identity matrix. The weighting matrices $Q_K$ and $Q_L$ are chosen so that the converter with the controller behaves in a desired way

The responses of the system with and without the linear controller are depicted in Fig. 4. In Fig. 4(a) the open loop response has not reached the stationary periodic solution in 30 periods. In fact, it requires for the open loop response about 170 periods to coincide with the stationary periodic solution. In contrast to the open loop response, the closed loop response of the system is shown in Fig. 4(b) which already coincides with the stationary periodic solution after 10 periods. Therefore, our simulations demonstrate that the response using the linear controller is much faster in reaching the stationary periodic solution.

# 7 Conclusions

We have described a realistic approach of designing a linear controller for a PWM Ćuk converter. The controller design is based on the discrete-time version of the linearized multifrequency model and knowledge of the output of the converter. The controller is to be applicable for any number of harmonics of the multifrequency model. From our simulations, the response using the linear controller is much faster in reaching the stationary periodic solution than the response in open loop situation. Finally we note that the approach of this paper can also be applied to other converters.

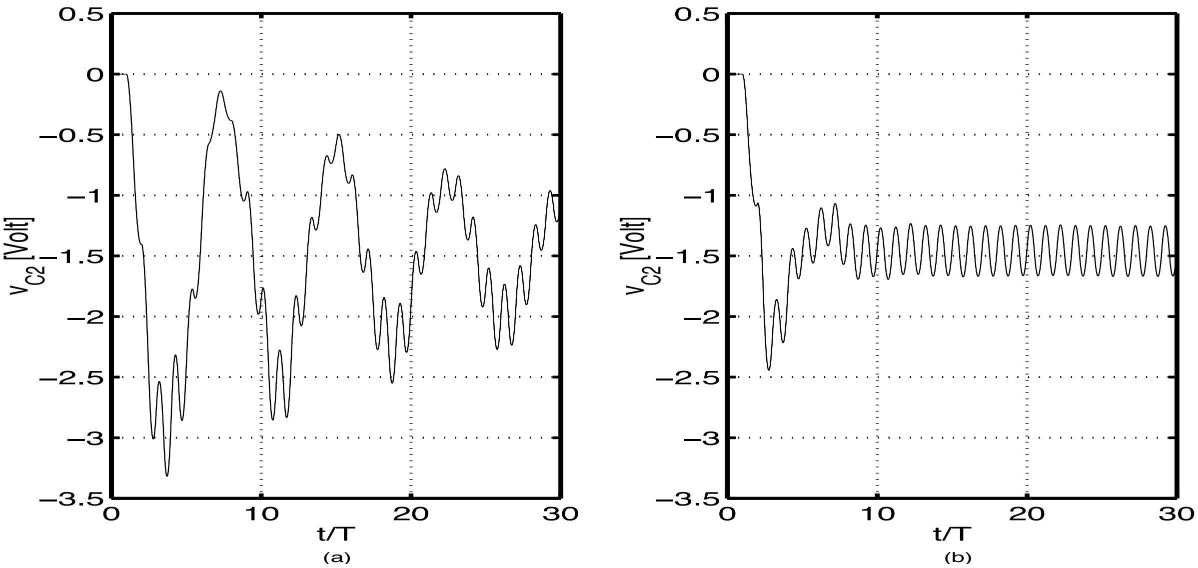


Fig. 4: Comparison of the responses of the system with and without the linear controller. ($a$) The open loop response. ($b$) The response of the system using the linear controller.

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