Linear Matrix Inequalities in Robust Control
A Brief Survey

Venkataramanan Balakrishnan
School of Electrical and Computer Engineering
Purdue University
West Lafayette, IN 47907-1285
USA

Abstract

Control system models must often explicitly incorporate in them uncertainties or perturbations. Robust control deals with the analysis of and design for such uncertain control system models. This paper provides a brief survey to some robust control techniques that are based on numerical convex optimization over Linear Matrix Inequalities (LMIs).

1 Introduction

Control system models must often explicitly incorporate in them "uncertainties", which model a number of factors, including: dynamics that are neglected to make the model tractable, as with large scale structures; nonlinearities that are either hard to model or too complicated; and parameters that are not known exactly, either because they are hard to measure or because of varying manufacturing conditions. Robust control deals with the analysis of and design for such control system models.

We will consider control system models of the form

$$\frac{d}{dt}x(t) = f(x,w,u,t), \quad z(t) = g(x,w,u,t), \quad y(t) = h(x,w,u,t), \tag{1.1}$$

where $x(t) \in \mathbf{R}^n$, $w(t) \in \mathbf{R}^{n_w}$, $u(t) \in \mathbf{R}^{n_u}$, $y(t) \in \mathbf{R}^{n_y}$ and $z(t) \in \mathbf{R}^{n_z}$. The function x is called the "state" of the system, while w and u are "inputs", and z and y are "outputs". w consists of exogenous inputs, i.e., inputs that we have no control over, such as noises, reference inputs etc. u consists of control inputs; we may set $u(t)$ to any value we wish, for every t. The outputs z are those of interest; these may consist, for instance, of components of x or even those of u. y consists of outputs that can be measured. In order to accommodate uncertainties, it is assumed that f, g and h are not known exactly, but only known to satisfy some properties. (We will be more specific shortly.) Equations (1.1) models so-called continuous-time systems. It is straightforward to extend the following discussion to discrete-time systems as well.

Robust control analysis problems consist of the study of the solutions of equations (1.1). Typical questions that arise in this context are "Are the solutions x of equations (1.1) bounded?" or "With $x(0) = 0$, how large can $\int_0^\infty z(t)^T z(t)\,dt$ be, over all w with $\int_0^\infty w(t)^T w(t)\,dt \le 1$?" Robust design problems consist of designing control laws $u(t) = \mathcal{K}(y,t)$, so that with the control law in place, desired answers are obtained for the analysis questions.

1.1 Linear fractional representation of uncertain systems

We now focus on a special instance of system (1.1), consisting of an interconnection of a linear time-invariant system and an "uncertainty" or "perturbation" in the feedback loop. This model

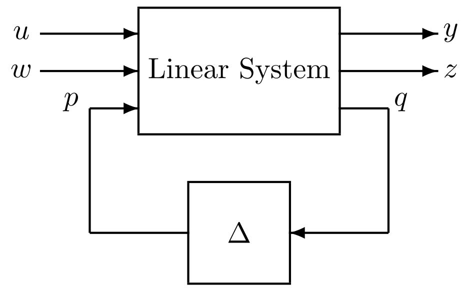

Figure 1: A common framework for robustness analysis and robust synthesis.

has found wide applicability in the analysis and design of control systems for which only imperfect models are available; see for example, [1]. The model is described by

$$\begin{array}{c}\frac{d}{dt}x(t) = Ax(t) + B_p p(t) + B_u u(t) + B_w w(t), \quad q(t) = C_q x(t) + D_{qp} p(t) + D_{qu} u(t) + D_{qw} w(t),\\ y(t) = C_y x(t) + D_{yp} p(t) + D_{yu} u(t) + D_{yw} w(t), \quad z(t) = C_z x(t) + D_{zp} p(t) + D_{zu} u(t) + D_{zw} w(t),\\ p(t) = \Delta(q,t),\end{array} \tag{1.2}$$

where $p \in \mathbf{R}^m$, $q \in \mathbf{R}^m$, A, B_p, B_u, B_w, C_q, C_y, C_z, D_{yp}, D_{yu}, D_{yw}, D_{qp}, D_{qu}, D_{qw}, D_{zp}, D_{zu} and D_{zw} are real matrices of appropriate sizes. $\Delta : \mathbf{L_2}^m[0,\infty) \to \mathbf{L_2}^m[0,\infty)$ is in general a nonlinear operator representing the "uncertainty" in modeling, and is known or assumed to lie in some set $\boldsymbol{\Delta}$. Often $\boldsymbol{\Delta}$ contains the origin, i.e., $\Delta = 0$; the linear time-invariant system that results is called the "nominal model". A block diagram of this system model is shown in Figure 1.

Many commonly encountered systems with structured and/or parametric uncertainties can be represented by the system model (1.2) [2]. In the control literature, model (1.2) is also known as the "Linear Fractional Representation" of the uncertain system, or simply an LFR system. Usually, additional information about the size of the uncertainty (typically some bound on the norm of $\Delta \in \boldsymbol{\Delta}$), its structure (i.e., diagonal or block-diagonal), and nature (for instance, sector-bounded memoryless, linear time-invariant (LTI) or parametric, etc) is available. A very general framework for LFR systems is provided by Integral Quadratic Constraints (IQCs); see for example [3].

1.2 Polytopic systems

Polytopic systems form a special class of LFR systems. For these systems, there exists an extensive body of work on analysis and synthesis using quadratic Lyapunov functions [4]. These systems are described by

$$\begin{array}{c}\frac{d}{dt}x(t) = A(t)x(t) + B_u(t)u(t) + B_w(t)w(t), \quad y(t) = C_y(t)x(t) + D_{yu}(t)u(t) + D_{yw}(t)w(t),\\ z(t) = C_z(t)x(t) + D_{zu}(t)u(t) + D_{zw}(t)w(t), \quad \Sigma(t) = \begin{bmatrix} A(t) & B_u(t) & B_w(t) \\ C_y(t) & D_{yu}(t) & D_{yw}(t) \\ C_z(t) & D_{zu}(t) & D_{zw}(t) \end{bmatrix} \in \boldsymbol{\Xi}\end{array} \tag{1.3}$$

where

$$\mathbf{\Xi} = \mathbf{Co}\left\{\begin{bmatrix} A_1 & B_{u,1} & B_{w,1} \\ C_{y,1} & D_{yu,1} & D_{yw,1} \\ C_{z,1} & D_{zu,1} & D_{zw,1} \end{bmatrix}, \ldots, \begin{bmatrix} A_L & B_{u,L} & B_{w,L} \\ C_{y,L} & D_{yu,L} & D_{yw,L} \\ C_{z,L} & D_{zu,L} & D_{zw,L} \end{bmatrix}\right\}, \tag{1.4}$$

where **Co** denotes the convex hull. (The matrices $\begin{bmatrix} A_i & B_{u,i} & B_{w,i} \\ C_{y,i} & D_{yu,i} & D_{yw,i} \\ C_{z,i} & D_{zu,i} & D_{zw,i} \end{bmatrix}$, $i = 1, \ldots, L$ are given.)

2 Robust stability analysis and design problems

Problems of interest are that of stability analysis and stabilizing controller synthesis for both polytopic systems (1.3) and the more general LFR systems (1.2):

(P1) With w and u identically zero, does the state x of system (1.2) (respectively system (1.3)) satisfy $\lim_{t\to\infty} x(t) = 0$ for every initial condition $x(0)$? If so, we say that the system (1.2) (respectively system (1.3)) is "robustly stable over $\mathbf{\Delta}$ (respectively Ξ)".

(P2) With w identically zero, does there exist a control law u such that the state x of system (1.2) (respectively system (1.3)) satisfies $\lim_{t\to\infty} x(t) = 0$ for every initial condition $x(0)$? If so, we say that the system (1.2) (respectively system (1.3)) is "robustly stabilizable over $\mathbf{\Delta}$ (respectively Ξ)".

Each of these "robust stability" questions has a "robust performance" counterpart: For a robustly stable system, measures of performance—usually with smaller values being better—can be defined that quantify how good the map from the exogenous inputs w to the outputs of interest z is. (Simple examples are norms.) Robust performance analysis questions then ask how large these performance measures can be over $\mathbf{\Delta}$ or Ξ. Robust performance design questions concern the design of control laws that minimize the largest values of the performance measures over $\mathbf{\Delta}$ or Ξ. Design with multiple robust performance design constraints leads to the the so-called multi-objective design problem [5].

2.1 Robust stability and performance analysis

One approach towards answering question (P1) uses the notion of *quadratic stability*. A system is said to be quadratically stable if there exists a positive-definite quadratic Lyapunov function $V(\zeta) = \zeta^T P \zeta$ that decreases along every trajectory of the system. For system (1.3), a necessary and sufficient condition for quadratic stability can be directly formulated in terms of a finite number of linear matrix inequalities[1] (LMIs) [4]. To illustrate, consider the simple polytopic system $\dot{x} = (\lambda(t)A_1 + (1-\lambda(t))A_2)x$, where $\lambda(t) \in [0,1]$ for all t, and A_1 and A_2 are given real matrices. Then, it is easily shown that

$$\frac{d}{dt}x(t)^T P x(t) < 0 \text{ along every trajectory} \quad \Longleftrightarrow \quad A_1^T P + P A_1 < 0,\ A_2^T P + P A_2 < 0.$$

[1] We assume that the reader is familiar with linear matrix inequalities or LMIs, which are constraints that require an affine combination of given Hermitian matrices to be positive semidefinite. For an introduction, see for example, [4].

The latter matrix inequalities are LMIs. For system (1.2), in general, only sufficient conditions for quadratic stability are known; these are stated in terms of a finite number of LMIs. The underlying quadratic Lyapunov functions can be used to derive bounds on robust performance measures; see for example [4].

A system can be robustly stable without being quadratically stable, and more general Lyapunov functions can be employed to derive weaker sufficient conditions for robust stability. For instance, when the state-space matrices of the polytopic system (1.3) vary slowly with time, stability analysis using parameter-dependent Lyapunov functions—for example, $V(\zeta) = \zeta^T P(\theta)\zeta$, where θ denotes the vector of measurable parameters, and $P(\cdot)$ is some specified function—usually leads to less conservative robust stability conditions than the analysis based on quadratic Lyapunov functions [6]. For the LFR system (1.2), the framework of integral quadratic constraints (IQCs) [3] provides a systematic method for deriving sufficient conditions for robust stability that are weaker than quadratic stability. These sufficient conditions can be reduced to LMIs either exactly or approximately. In some cases, the framework can be interpreted as searching for more general Lyapunov functionals [7]. In these cases, bounds on robust performance measures can be derived, and computed using LMI optimization.

2.2 Robust control synthesis

In addressing the problem of controller synthesis (P2), there are several possibilities for generating the control input $u(t)$. Perhaps the simplest control law is that of constant state-feedback, $u(t) = Kx(t)$, where K is a real matrix. Of course, in order to implement a state-feedback scheme, the state $x(t)$ has to be measurable at every time t. If only the measured output y is available for generating u, output feedback control laws of the form $u = \mathcal{K}(y, t)$ can be envisioned; a simple example of such a control law is constant output feedback $u = Ky(t)$. If in addition to the measured output, the uncertainty Σ in a polytopic system (or Δ in an LFR system) is measurable in real time [8, 9, 10, 11, 12, 13], a control law $u = \mathcal{K}(y, \Sigma, t)$ (or $u = \mathcal{K}(y, \Delta, t)$) that explicitly depends on the uncertainty can be implemented. This is the so-called gain-scheduled controller.

The problem of synthesizing robustly stabilizing constant state-feedback for both polytopic and LFR systems, using quadratic Lyapunov functions, can be formulated as LMI feasibility problems [4]. However, no convex reformulation is known for the problem of even constant output feedback synthesis for even polytopic systems. It is worthy of note that a number of results are available for the LMI-based synthesis of LTI controllers for LTI systems (i.e., a model with no uncertainties); a sampling is provided by [5, 14, 15, 16].

For robust control, gain-scheduled controllers appear to hold promise: Designing gain-scheduled output feedback controllers for polytopic systems using quadratic Lyapunov functions can be reduced to the solution of an optimization problem with a finite number of LMIs [10, 17]. For LFR systems, conditions for the existence of robustly stabilizing gain-scheduled output feedback controllers, derived using quadratic Lyapunov functions, result in a finite number of LMIs [18, 19, 20, 21, 22]. As for the stability criteria derived using parameter-dependent Lyapunov functions or in the IQC framework, although they may yield less conservative conditions for robust stability, the corresponding conditions for the existence of robustly stabilizing controllers (gain-scheduled or otherwise) turn out to be nonconvex. However in special cases, parameter-dependent Lyapunov functions do yield a LMI-based parametrization of stabilizing gain-scheduled controllers; see for example [17, 23, 6, 22].

3 Conclusion

It is fair to say the advent of LMI optimization has significantly influenced the direction of research in robust control. A widely-accepted technique for "solving" robust control problems now is to simply reduce them to LMI problems. This paper represents an attempt at providing a brief survey of such techniques. For details and further references, see [24]. Other comprehensive accounts can be found, for example, in [25, 26].

While it true in principle that the reduction of a robust control problem to an LMI problem provides a solution, it is also now recognized that in many practical applications, the resulting LMI problems are so large as to test the limits of currently available software. Thus, much remains to be gained with the development of special purpose LMI solvers that take advantage of the underlying problem structure and information.

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