Multirate Periodic Systems, ν-Gap Metric and Robust Stabilization

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Abstract

A new discription of multirate systems, called multirate periodic system, is given using the concept of periodic time-varying input-output spaces. We then define the ν-gap metric of two multirate periodic systems and study the robust stabilization with this metric. The optimal robust stabilization margin is explicitly computed and an obsever-form suboptimal controller is given. The solution amounts to solving two discrete-time algebraic Riccati equations and an extended Parrot problem.

1 Introduction

Multirate and periodic systems are finding more and more applications in control, communication, signal processing, econometrics and numerical mathematics. The reason may be due to their power in modeling physical systems with inherent features like periodic behavior changes, seasonal operating environment, nonuniform information exchange pattern, multirate sampling, etc., or due to the fact that they can often achieve objectives that cannot be achieved by single rate LTI systems.

The study of periodic systems can be traced back to [8]. Examples of more recent studies are [14, 21, 27], the works of an Italian school well reflected in [2] and computational aspect surveyed in [30]. The study of multirate systems goes back to late 1950's, see for example [16, 17, 19]. A renaissance of research on multirate systems has occured since 1980 in signal processing, communication and control communities. The driving force for studying multirate systems in signal processing comes from the need for sampling rate conversion, subband coding, and their ability to generate wavelets. Multirate signal processing is now one of the most vibrant areas of research in signal processing, see recent book [29] and references therein. In communication systems, blind identification and equalization call for using multirate sampling [28]. In control community, two groups of research stand out: using multirate control to achieve what single rate control cannot as well as the limitation of doing this, see for example [7, 18], and the optimal design of multirate controllers [5, 10, 20, 23, 26]. We also notice the cross discipline fertilization between signal processing and control in using $\mathcal{H}_\infty$ optimization to design filter banks [4, 6].

Recently, there has been considerable research devoted to the problem of robust stabilization [12, 15, 33]. For LTI systems with gap and ν-gap metric uncertainty, it is now well-known that both the optimal robustness bound and the suboptimal controller can be easily obtained without the so-called γ-iteration and the suboptimal controller is an observer form. In this paper, we will extend these results to multirate periodic systems.

The paper is organized as follows. In section 2, we give the general setup on MP systems and the lifting technique. We will see that a general MP system can be converted to an LTI system with a structural constraint due to the causality requirement. In section 3, we introduce the ν-gap metric to MP systems and show that a robust stabilization problem of an MP system with ν-gap metric uncertainty can be converted to a constrained $\mathcal{H}_\infty$ optimization problem. Section 4 deals with the Nehari problem with structural constraint which is used to solve the robust stabiliztion problem of

MP systems. The optimal robust stability margin and an observer-based suboptimal controller are presented explicitly in section 5. Finally, this paper is concluded in section 6.

2 Setup of MP systems

In this paper, we model an MP system by a discrete time system, shown in Fig. 1, with periodic time-varying input and output spaces. This concept is used in [13] to define a general periodic system. Here, we show that various multirate systems can be viewed as special cases of the general periodic systems defined in [13]. Precisely, we assume that the input sequence $u = \{u(k)\}_{k=-\infty}^{\infty}$ takes values in $\bigoplus_{k=-\infty}^{\infty}\mathcal{U}(k)$, i.e., $u(k) \in \mathcal{U}(k)$, and the output sequence $y = \{y(k)\}_{k=-\infty}^{\infty}$ takes values in $\bigoplus_{k=-\infty}^{\infty}\mathcal{Y}(k)$, i.e., $y(k) \in \mathcal{Y}(k)$, where $\mathcal{U}(k)$ and $\mathcal{Y}(k)$ are M-periodic time-varying vector spaces, i.e., they satisfy $\mathcal{U}(k+M) = \mathcal{U}(k)$ and $\mathcal{Y}(k+M) = \mathcal{Y}(k)$. We further make the following assumptions:

1. Linearity. The system G_{mp} is a linear operator from $\bigoplus_{k=-\infty}^{\infty}\mathcal{U}(k)$ to $\bigoplus_{k=-\infty}^{\infty}\mathcal{Y}(k)$.
2. Periodicity. Let $\mathcal{X}(k)$ be vector space valued M-periodic functions. Define the M-step shift operator S^M on $\bigoplus_{k=-\infty}^{\infty}\mathcal{X}(k)$ as

   $$S^M\{\ldots,x(-1),|x(0),x(1),\ldots\} = \{\ldots,x(-M-1),|x(-M),x(-M+1),\ldots\}.$$

   Then G_{mp} satisfies $G_{mp}S^M = S^M G_{mp}$. Notices that when $M > 1$, the 1-step shift S^1 is generally not defined.
3. Causality. Let P_k be a projection operator on $\bigoplus_{k=-\infty}^{\infty}\mathcal{X}(k)$ defined as

   $$P_k\{\ldots,x(k-1),x(k),x(k+1),\ldots\} = \{\ldots,x(k-1),x(k),0,\ldots\}.$$

   Then G_{mp} satisfies $P_k G_{mp}(I-P_k) = 0$.

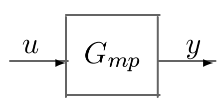

Figure 1: A general periodic and multirate system

The general class of MP systems defined here covers many familiar classes of systems as special cases. An MP system with $\mathcal{U}(k) = \mathcal{U}$ and $\mathcal{Y}(k) = \mathcal{Y}$ for all $k \in \mathbb{Z}$ is a usual M-periodic system, for which there is a vast literature [2]. The multirate feature arises when $\mathcal{U}(k)$ and $\mathcal{Y}(k)$ are truly time-varying. If

$$\mathcal{U}(k) = \begin{cases} \mathcal{U} & \text{if } m|k \\ \{0\} & \text{otherwise} \end{cases}, \qquad \mathcal{Y}(k) = \begin{cases} \mathcal{Y} & \text{if } n|k \\ \{0\} & \text{otherwise} \end{cases},$$

and M is a multiple of m and n, then such an MP system is a dual rate system considered in [6]. Let M be a multiple of integers $m_i, i = 1,\ldots,p$, and $n_j, j = 1,\ldots,q$. If

$$\mathcal{U}_i(k) = \begin{cases} \mathcal{U}_i & \text{if } m_i|k \\ \{0\} & \text{otherwise} \end{cases}, \qquad \mathcal{Y}_j(k) = \begin{cases} \mathcal{Y}_j & \text{if } n_j|k \\ \{0\} & \text{otherwise} \end{cases}$$

and

$$\mathcal{U}(k) = \bigoplus_{i=1}^{p} \mathcal{U}_i(k), \qquad \mathcal{Y}(k) = \bigoplus_{j=1}^{q} \mathcal{Y}_j(k),$$

then such an MP system becomes a general multirate system with uniform synchronized but different sampling in each input or output channel [1, 5, 23, 26, 32]. One advantage of modeling a multirate system as a periodic system with periodically varying input output spaces is that it better relates back to the rich theory on the usual periodic system, as surveyed in [2]. Other advantages are its generality: it allows for nonuniform and asynchronous sampling, and its convenience: the treatments using this model take similar forms than those using other models, such as the one in [5, 23].

A standard way for the analysis of such systems is to use lifting or blocking. Let $\mathcal{X}_l(r) = \bigoplus_{k=l+rM}^{(r+1)M+l-1} \mathcal{X}(k)$. Define a lifting operator $L_l : \bigoplus_{k=-\infty}^{\infty} \mathcal{X}(k) \to \bigoplus_{r=-\infty}^{\infty} \mathcal{X}_l(r)$ by

$$L_l : \{\cdots | x(0), x(1), \cdots\} \mapsto \left\{ \cdots \left| \left[\begin{array}{c} x(l) \\ x(l+1) \\ \vdots \\ x(M+l-1) \end{array} \right], \left[\begin{array}{c} x(M+l) \\ x(M+l+1) \\ \vdots \\ x(2M+l-1) \end{array} \right], \cdots \right. \right\}.$$

Then the lifted systems $G_l = L_l G_{mp} L_l^{-1}$ are LTI systems in the sense that $G_l S^1 = S^1 G_l$, where S^1 is the unit shift on $\bigoplus_{r=-\infty}^{\infty} \mathcal{X}_l(r)$. Hence they have transfer functions in the λ-transform $(\lambda = \frac{1}{z})$:

$$\hat{G}_l(\lambda) = \left[\begin{array}{ccc} \hat{G}_{l,11}(\lambda) & \cdots & \hat{G}_{l,1M}(\lambda) \\ \vdots & \ddots & \vdots \\ \hat{G}_{l,M1}(\lambda) & \cdots & \hat{G}_{l,MM}(\lambda) \end{array} \right].$$

Assume that $\dim \mathcal{U}(k) = p(k)$ and $\dim \mathcal{Y}(k) = q(k)$. Then $\hat{G}_l$ takes values in the set of $\sum_{k=l}^{M+l-1} q(k) \times \sum_{k=l}^{M+l-1} p(k)$ complex matrices. The LTI system G_l is not an arbitrary LTI system, instead its direct feedthrough term $\hat{G}_l(0)$ is subject to a constraint that results from the causality of G_{mp} :

$$\hat{G}_{l,ij}(0) = 0 \quad \text{for } i < j,$$

i.e., $\hat{G}_l(0)$ is a block lower triangular matrix. Notice that the form of the causality here is simpler than that in [5, 23] due to the new form of the model. It can be easily shown that $\hat{G}_l$ are not all independent. They are related by

$$\hat{G}_{l+1}(\lambda) = \left[\begin{array}{cc} 0 & \mathrm{diag}(I_{q(l+1)} \cdots I_{q(l+M-1)}) \\ \lambda^{-1} I_{q(l)} & 0 \end{array} \right] \hat{G}_l(\lambda) \left[\begin{array}{cc} 0 & \lambda I_{p(l)} \\ \mathrm{diag}(I_{p(l+1)} \cdots I_{p(l+M-1)}) & 0 \end{array} \right].$$

Hence, any one of the $G_l, l = 0, \ldots, M-1$, can be defined as the LTI equivalent of the MP system G_{mp}. In the rest of this paper, we choose G_0 as the LTI equivalent of the MP system G_{mp} without loss of generality.

3 ν-gap Metric of MP Systems

The first issue in robust control is the description of the uncertainty. The most natural way to describe system uncertainty is by using a metric in the set of all systems under consideration and an uncertain system is then simply a ball defined by this metric centered at a nominal system with certain radius. There are several metrics in the literature for this very purpose: gap metric [11], pointwise gap metric [24], ν-gap metric [31]. In this paper, the ν-gap metric is studied for MP systems. We generalize the treatment of [34] to define the ν-gap metric for MP systems because the treatment in [31] is based on the transfer function, which is not appropriate to extend to MP systems.

Given two M-periodic MP systems G_{mp} and $\tilde{G}_{mp}$, the graphs of G_{mp} and $\tilde{G}_{mp}$ are defined as

$$\begin{aligned} \mathcal{G}(G_{mp}) &= \left\{ \begin{bmatrix} u \\ G_{mp}u \end{bmatrix}; u \in \ell_+^2 \text{ and } G_{mp}u \in \ell_+^2 \right\}, \\ \mathcal{G}(\tilde{G}_{mp}) &= \left\{ \begin{bmatrix} \tilde{u} \\ \tilde{G}_{mp}\tilde{u} \end{bmatrix}; \tilde{u} \in \ell_+^2 \text{ and } \tilde{G}_{mp}\tilde{u} \in \ell_+^2 \right\}, \end{aligned}$$

respectively. Here ℓ_+^2 means the direct sum $\oplus_{k=0}^{\infty}\mathcal{X}(k)$ with $\sum_{n=0}^{\infty} x^2(k+nM) < \infty$ for any $k = 0, 1, \dots, M-1$. Clearly $\mathcal{G}(G_{mp})$ and $\mathcal{G}(\tilde{G}_{mp})$ are subspaces of $\ell_+^2 \oplus \ell_+^2$. A subspace $\mathcal{G}$ of $\ell_+^2 \oplus \ell_+^2$ is called M-shift-invariant if $S^M\mathcal{G} \subset \mathcal{G}$. It is easy to see that the graph of G_{mp} is M-shift-invariant. A subgraph of an M-periodic MP system is defined as an M-shift-invariant subspace of its graph. We denote the set of all subgraphs as $\mathcal{S}_{\mathcal{G}}(G_{mp})$. To define the $\nu-$gap between two MP systems, we need the notion of the index of a subgraph $\mathcal{V}$ with respect to $\mathcal{G}(G_{mp})$, defined as [34]

$$\text{ind}(\mathcal{V}) := \dim\left(\mathcal{G}(G_{mp}) \ominus \mathcal{V}\right).$$

The $\nu-$gap between two plants G_{mp} and $\tilde{G}_{mp}$ is then defined by

$$\delta_\nu(G_{mp}, \tilde{G}_{mp}) = \inf_{\substack{\mathcal{V} \in \mathcal{S}_{\mathcal{G}}(G_{mp}) \\ \tilde{\mathcal{V}} \in \mathcal{S}_{\mathcal{G}}(\tilde{G}_{mp}) \\ \text{ind}(\mathcal{V}) = \text{ind}(\tilde{\mathcal{V}})}} \|\Pi_{\mathcal{V}} - \Pi_{\tilde{\mathcal{V}}}\|$$

where $\Pi_{\mathcal{V}}$ and $\Pi_{\tilde{\mathcal{V}}}$ are the orthogonal projections from $\ell_+^2 \oplus \ell_+^2$ onto $\mathcal{V}$ and $\tilde{\mathcal{V}}$ respectively. The ν-gap metric ball centered at G_{mp} with radius r is defined by

$$\mathcal{B}_\nu(G_{mp}, r) = \{\tilde{G}_{mp} : \delta_\nu(G_{mp}, \tilde{G}_{mp}) < r\}.$$

By the following lemma, the ν-gap between two M-periodic MP systems can be computed from that between their equivalent LTI systems, where efficient methods are available [31].

Lemma 3.1. Let G_{mp} and $\tilde{G}_{mp}$ be two M-periodic MP systems and their equivalent LTI systems are G and $\tilde{G}$ respectively, that is

$$G = L_0 G_{mp} L_0^{-1}, \quad \tilde{G} = L_0 \tilde{G}_{mp} L_0^{-1}.$$

Then we have $\delta_\nu(G_{mp}, \tilde{G}_{mp}) = \delta_\nu(G, \tilde{G})$.

Proof. Noting that $\mathcal{V}$ a subgraph of G_{mp} if and only if $\mathcal{V}_L = \begin{bmatrix} L_0 & 0 \\ 0 & L_0 \end{bmatrix} \mathcal{V}$ is a subgraph of G. Similar result holds to a subgraph $\tilde{\mathcal{V}}$ of $\tilde{G}_{mp}$. Denote

$$\tilde{\mathcal{V}}_L = \begin{bmatrix} L_0 & 0 \\ 0 & L_0 \end{bmatrix} \tilde{\mathcal{V}}.$$

Since the lifting operator L_0 is unitary, we then have

$$\delta_\nu(G_{mp}, \tilde{G}_{mp}) = \inf \|\Pi_{\mathcal{V}} - \Pi_{\tilde{\mathcal{V}}}\| = \inf \left\|\Pi_{\mathcal{V}_L} - \Pi_{\tilde{\mathcal{V}}_L}\right\| = \delta_\nu(G, \tilde{G}).$$

This completes the proof. □

In the following, we will discuss the robust stabilization problem for MP uncertain systems with ν-gap metric. First, some notation is needed. Define the set of block matrices:

$$\mathcal{M}(\mathbb{R}^{m\times n}) := \left\{ T = \left[\begin{array}{ccc} T_{11} & \cdots & T_{1M} \\ \vdots & & \vdots \\ T_{M1} & \cdots & T_{MM} \end{array} \right] : \quad T \in \mathbb{R}^{m\times n} \right\}.$$

The block lower triangular subset of $\mathcal{M}(\mathbb{R}^{m\times n})$, denoted by $\mathcal{T}(\mathbb{R}^{m\times n})$, consists of all matrices with $T_{ij} = 0$, $i < j$, and the strictly block lower triangular subset, $\mathcal{T}_s(\mathbb{R}^{m\times n})$, consists of matrices with $T_{ij} = 0$, $i \leq j$.

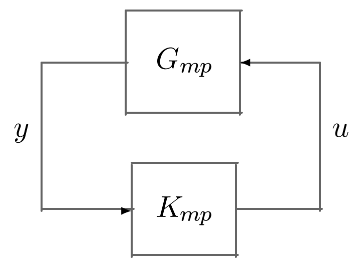

Figure 2: A general MP feedback control system.

Now consider the feedback system shown in Fig. 2. Here we assume that G_{mp} and K_{mp} are M-periodic MP systems with M-periodic time-varying signal spaces $\oplus_{k=-\infty}^{\infty}\mathcal{U}(k)$ and $\oplus_{k=-\infty}^{\infty}\mathcal{Y}(k)$. Assume that $\dim\mathcal{U}(k) = p(k)$ and $\dim\mathcal{Y}(k) = q(k)$. Denote $p = \sum_{k=0}^{M-1} p(k)$ and $q = \sum_{k=0}^{M-1} q(k)$. Let $G = L_0 G_{mp} L_0^{-1}$ and $K = L_0 K_{mp} L_0^{-1}$, then G and K are LTI and hence have transfer functions $\hat{G}(\lambda)$ and $\hat{K}(\lambda)$ respectively. Due to causality constraint, $\hat{G}(0)$ and $\hat{K}(0)$ are block lower triangular, that is, $\hat{G}(0) \in \mathcal{T}(\mathbb{R}^{q\times p})$ and $\hat{K} \in \mathcal{T}(\mathbb{R}^{p\times q})$. For fixed G_{mp} and K_{mp}, the stability robustness of the feedback system is given by the following lemma:

Lemma 3.2. ([31, 25]) Given a norminal plant G_{mp} and a stabilizing controller K_{mp}. Let G and K be the LTI equivalence of G_{mp} and K_{mp} respectively. For any positive real numbers r_1 and r_2, the feedback system with plant $\tilde{G}_{mp}$ and controller $\tilde{K}_{mp}$ is stable for all $\tilde{G}_{mp} \in \mathcal{B}_\nu(G_{mp}, r_1)$ and all $\tilde{K}_{mp} \in \mathcal{B}_\nu(K_{mp}, r_2)$ if and only if

$$\arcsin r_1 + \arcsin r_2 + \arccos b_{G,K} \leq \frac{1}{2}\pi,$$

where

$$b_{G,K} = \| \left[\begin{array}{c} I \\ \hat{G} \end{array} \right] (I - \hat{K}\hat{G})^{-1} \left[\begin{array}{cc} I & -\hat{K} \end{array} \right] \|_\infty^{-1}.$$

The proof is straightforward by slightly modifying the procedure in [31]. The quantity $b_{G,K}$ is defined as the robust stability margin. The robust stabilization problem is to find the optimal robust stability margin

$$b_{opt} = \sup_{K,\ \hat{K}(0)\in\mathcal{T}(\mathbb{R}^{p\times q})} b_{G,K} \tag{3.1}$$

for a given G and also find a K with $\hat{K}(0) \in \mathcal{T}(\mathbb{R}^{p\times q})$, called a suboptimal controller, such that $b_{G,K} \geq \gamma$ for any $\gamma < b_{opt}$.

Hence our robust stabilization problem becomes a special discrete-time $\mathcal{H}_\infty$ optimal control problem. Since the causality of G_{mp} and K_{mp} is equivalent to that $\hat{G}(0) \in \mathcal{T}(\mathbb{R}^{q\times p})$ and $\hat{K}(0) \in \mathcal{T}(\mathbb{R}^{p\times q})$, we need to respect the structural constraint $\hat{K}(0)$ and possibly to utilize the structural constraint $\hat{G}(0)$ in solving the special discrete-time $\mathcal{H}_\infty$ optimal control problem.

4 Constrained Nehari Extension Problems

In this section, we study a constrained Nehari extension problem which is used to solve the robust stabilization problem of MP sytems with ν-gap metric uncertainty.

Given $G(\lambda) \in \mathcal{L}_\infty(\mathcal{U}, \mathcal{Y})$ satisfying $G(\frac{1}{\lambda}) \in \mathcal{H}_\infty(\mathcal{U}, \mathcal{Y})$ and $\alpha > \|G\|_H$, where $\|G\|_H$ denotes the Hankel norm of G, the suboptimal Nehari extension problem is to find $H \in \mathcal{H}_\infty(\mathcal{U}, \mathcal{Y})$ such that

$$\|G - H\|_\infty \leq \alpha.$$

Now we put an extra constraint on H that $H(0) \in \mathcal{T}(\mathbb{R}^{q\times p})$. A constructive algorithm is given to this constrained Nehari extension problem in [10]. Here, we will present an explicit solution. To this end, let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be an anticausal realization for G, that is,

$$G(\lambda) = D + C(\lambda I - A)^{-1}B.$$

Let P and Q be the solutions of the following Lyapunov equations

$$P = APA^* + BB^* \text{ and } Q = A^*QA + C^*C.$$

Note that $\|G\|_H$ is equal to the spectral norm of QP. Choose $N_1 \in \mathcal{T}(\mathbb{R}^{p\times p})$ and $N_2 \in \mathcal{T}(\mathbb{R}^{q\times q})$ such that

$$\begin{aligned} N_1 N_1^* &= I + B^*(\alpha^2 I - QP)^{-1}QB \\ N_2^* N_2 &= I + C(\alpha^2 I - PQ)^{-1}PC^*. \end{aligned}$$

Let A_0 be the stable matrix defined by [9]

$$A_0 = (\alpha^2 I - A^*QAP)^{-1}A^*(\alpha^2 I - QP).$$

From Theorem VI. 8.1 in [9], the set of all solutions H is given by

$$H(\lambda) = \mathcal{F}_l(\Phi(\lambda), R(\lambda)) = \Phi_{11}(\lambda) + \Phi_{12}(\lambda)R(\lambda)(I - \Phi_{22}(\lambda)R(\lambda))^{-1}\Phi_{21}(\lambda) \tag{4.2}$$

where

$$\begin{aligned} \Phi(\lambda) &= \begin{bmatrix} \Phi_{11}(\lambda) & \Phi_{12}(\lambda) \\ \Phi_{21}(\lambda) & \Phi_{22}(\lambda) \end{bmatrix} \\ &= \left[\begin{array}{c|cc} A_0 & (\alpha^2 I - A^*QAP)^{-1}A^*QB & (\alpha^2 I - QP)^{-1}C^*N_2^{-1} \\ \hline CPA_0 & CP(\alpha^2 I - A^*QAP)^{-1}A^*QB & -N_2^{-1} \\ -\alpha N_1^{-1}B^* & \alpha N_1^{-1} & 0 \end{array}\right] \end{aligned}$$

and $R(\lambda) \in \mathcal{H}_\infty(\mathcal{U}, \mathcal{Y})$ with $\|R(\lambda)\|_\infty \leq 1$.

Lemma 4.1. The constrained Nehari extension problem is solvable if and only if there exists a matrix R_0 such that $\|R_0\| \leq 1$ and $\frac{1}{\alpha}N_1\Phi_{11}(0)N_2 - R_0 \in \mathcal{T}(\mathbb{R}^{q\times p})$. Furthermore, if such R_0 exists, then one solution is given by $H(\lambda) = \mathcal{F}_l(\Phi(\lambda), R_0)$.

Proof. It follows from (4.2) that

$$\begin{aligned} H(0) &= \Phi_{11}(0) + \Phi_{12}(0)R(0)\Phi_{21}(0) \\ &= \Phi_{11}(0) - \alpha N_2^{-1}R(0)N_1^{-1}. \end{aligned} \tag{4.3}$$

Pre- and postmultiply (4.3) by $\frac{1}{\alpha}N_2$ and N_1 respectively, to get

$$\frac{1}{\alpha}N_2 H(0) N_1 = \frac{1}{\alpha}N_2 \Phi_{11}(0) N_1 - R(0).$$

Therefore $H(0) \in \mathcal{T}(\mathbb{R}^{q\times p})$ if and only if $\frac{1}{\alpha}N_2 H(0) N_1 - R(0) \in \mathcal{T}(\mathbb{R}^{q\times p})$ since $N_1 \in \mathcal{T}(\mathbb{R}^{p\times p})$ and $N_2 \in \mathcal{T}(\mathbb{R}^{q\times q})$. If such $R(0)$, denoted as R_0, exists, we can choose $R(\lambda) \equiv R_0$ and get one solution $H(\lambda) = \mathcal{F}_l(\Phi(\lambda), R_0)$. □

The problem of finding R_0 such that $\|R_0\|_\infty \le 1$ and $\frac{1}{\alpha}N_1\Phi_{11}(0)N_2 - R_0 \in \mathcal{T}(\mathbb{R}^{p\times q})$ is called conractive matrix completion. Various methods exist in the literature [10, 22]. Actually, we can find the unique central solution of R_0 following the method in [22].

5 Robust Stabilization of MP Systems

Now we return to the robust stabilization problem stated in section II: Given a nominal LTI G resulted from the lifting of G_{mp}, find the optimal robust stability margin b_{opt} in (3.1) and a suboptimal controller K with $\hat{K}(0) \in \mathcal{T}(\mathbb{R}^{p\times q})$ such that $b_{G,K} \ge \gamma$ for $\gamma < b_{opt}$. To solve this problem, we need some notation. Assume that G has a stabilizable and detectable state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with $D \in \mathcal{T}(\mathbb{R}^{q\times p})$. Let X and Y be the stabilizing solutions of Riccati equations

$$\begin{aligned} X &= A^*XA + C^*C - (A^*XB + C^*D)(B^*XB + I + D^*D)^{-1}(B^*XA + D^*C) && (5.4)\\ Y &= AYA^* + BB^* - (AYC^* + BD^*)(CYC^* + I + DD^*)^{-1}(CYA^* + DB^*). && (5.5) \end{aligned}$$

Denote

$$\begin{aligned} F &= -(B^*XB + I + D^*D)^{-1}(B^*XA + D^*C) && (5.6)\\ L &= -(AYC^* + BD^*)(CYC^* + I + DD^*)^{-1}. && (5.7) \end{aligned}$$

Here $(A+BF)$ and $(A+LC)$ are stable since X and Y are stablilizing solutions. The following equation [3, 12] gives a relationship between $A+BF$, $A+LC$, X and Y , which will be used later.

$$(A+LC)(I+YX) = (I+YX)(A+BF). \tag{5.8}$$

Using Cholesky factorization, we can get constant matrix $S \in \mathcal{T}(\mathbb{R}^{q\times q})$ satisfying [5],

$$SS^* = CYC^* + I + DD^*. \tag{5.9}$$

Denote

$$\alpha = (1-\gamma^2)^{\frac{1}{2}}. \tag{5.10}$$

and

$$W = \alpha^2 I + (\alpha^2 - 1)YX. \tag{5.11}$$

Let $N_1 \in \mathcal{T}(\mathbb{R}^{q\times q})$ be a constant matrix satisfying

$$N_1 N_1^* = I + S^{-1}C(I+YX)W^{-1}YC^*S^{*-1}. \tag{5.12}$$

Choose matrix $N_2 = \begin{bmatrix} N_{2,11} & N_{2,12} \\ N_{2,21} & N_{2,22} \end{bmatrix}$ with $N_{2,11} \in \mathcal{T}(\mathbb{R}^{p\times p})$, $N_{2,12} \in \mathcal{T}(\mathbb{R}^{p\times q})$, $N_{2,21} \in \mathcal{T}(\mathbb{R}^{q\times p})$ and $N_{2,22} \in \mathcal{T}(\mathbb{R}^{q\times q})$ satisfying

$$N_2^* N_2 = \begin{bmatrix} I + B^* X(I+YX)W^{-1}B & -B^* X(I+YX)W^{-1}L \\ -L^* X(I+YX)W^{-1}B & I + L^* X(I+YX)W^{-1}L \end{bmatrix}. \tag{5.13}$$

We know that there are normalized left coprime factorizations $G = \tilde{M}^{-1}\tilde{N}$ with $\tilde{M}(0) \in \mathcal{T}(\mathbb{R}^{q\times q})$ and $\tilde{N}(0) \in \mathcal{T}(\mathbb{R}^{q\times p})$. One particular realization of such factorization is as follows

$$\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & B + LD & L \\ \hline S^{-1}C & S^{-1}D & S^{-1} \end{array}\right]. \tag{5.14}$$

Now we are ready to present the main results of this paper.

Theorem 5.1. Given a lifted LTI plant $\hat{G}(\lambda) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with $D \in \mathcal{T}(\mathbb{R}^{q\times p})$, let X and Y be the stabilizing solutions of Riccati equations (5.4) and (5.5), and let F, L, S be defined as in (5.6)-(5.9). Then the optimal robust stabilization margin is

$$\begin{aligned} b_{opt} &= \sup_{K, \hat{K}(0) \in \mathcal{T}(\mathbb{R}^{p\times q})} b_{G,K} \\ &= \left\{ 1 - \max_r \left\| \begin{bmatrix} \Pi_{\mathcal{U}_r} & 0 & 0 \\ 0 & \Pi_{\mathcal{Y}_r} & 0 \\ 0 & 0 & I \end{bmatrix} \Gamma \begin{bmatrix} I - \Pi_{\mathcal{Y}_r} & 0 \\ 0 & I \end{bmatrix} \right\|^2 \right\}^{\frac{1}{2}}, \end{aligned} \tag{5.15}$$

where

$$\mathcal{U}_r = \mathcal{U}(0) \oplus \cdots \oplus \mathcal{U}(r) \tag{5.16}$$
$$\mathcal{Y}_r = \mathcal{Y}(0) \oplus \cdots \oplus \mathcal{Y}(r) \tag{5.17}$$
$$\Gamma = \begin{bmatrix} -D^* S^{*-1} & -(B^* + D^* L^*)(X^{-1} + Y)^{-\frac{1}{2}} \\ S^{*-1} & L^*(X^{-1}+Y)^{-\frac{1}{2}} \\ Y^{\frac{1}{2}} C^* S^{*-1} & Y^{\frac{1}{2}}(A + LC)^*(X^{-1}+Y)^{-\frac{1}{2}} \end{bmatrix}. \tag{5.18}$$

Theorem 1 tells us the optimal robust stability margin. The next theorem provides us a suboptimal controller.

Theorem 5.2. Given a lifted LTI plant $\hat{G} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with $D \in \mathcal{T}(\mathbb{R}^{q\times p})$ and $\gamma < b_{opt}$. Let X, Y, F, L, S, W, N_1 and N_2 be defined as in (5.4)-(5.13). Then a suboptimal controller K exists if and only if there exists a constant matrix $R_0 \in \mathcal{M}(\mathbb{R}^{(p+q)\times q})$ with $\|R_0\|_\infty \le 1$ such that $E_1 \in \mathcal{T}(\mathbb{R}^{p\times q})$ and $E_2 \in \mathcal{T}(\mathbb{R}^{q\times q})$, where

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \alpha^{-1}(N_2^* N_2)^{-1} \begin{bmatrix} -D^* \\ I \end{bmatrix} S^{*-1} N_1 - N_2^{-1} R_0. \tag{5.19}$$

Furthermore, if such R_0 is found, a suboptimal controller K is given by

$$K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & H \end{array}\right] \tag{5.20}$$

where

$$
\begin{aligned}
A_K &= A + LC + (B + LD)(F - HC - HDF)W^{-1} && (5.21)\\
B_K &= BH + LDH - L && (5.22)\\
C_K &= (F - HC - HDF)W^{-1} && (5.23)\\
H &= E_1 E_2^{-1}. && (5.24)
\end{aligned}
$$

This controller can be written in the following general observer form

$$
\begin{aligned}
\hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L[C\hat{x}(k) + Du(k) - y(k)] && (5.25)\\
u(k) &= F_K\hat{x}(k) + \bar{H}[C\hat{x}(k) + Du(k) - y(k)] && (5.26)
\end{aligned}
$$

where $\bar{H} = -(I - HD)^{-1}H$ and $F_K = (I - HD)^{-1}(C_K + HC)$.

Remark 5.1. If there is no causality constraint, we can simply take $R_0 = 0$. Assume the plant G is strictly proper, it can be shown that

$$H = B^*X(W + BB^*X)^{-1}L.$$

Then the controller is given by

$$
\begin{aligned}
\hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L[C\hat{x}(k) - y(k)] && (5.27)\\
u(k) &= [(F - HC)W^{-1} + HC]\hat{x}(k) - H[C\hat{x}(k) - y(k)]. && (5.28)
\end{aligned}
$$

This is exactly the same as the controller of (2.5)-(2.6) in [15].

Remark 5.2. The problem to design a strictly proper suboptimal controller for an LTI strictly proper plant studied in [15] is a special case of Theorem 2. Actually, if there exists R_0 such that $E_1 = 0$, then the suboptimal controller is given by

$$
\begin{aligned}
\hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L[C\hat{x}(k) + Du(k) - y(k)] && (5.29)\\
u(k) &= FW^{-1}\hat{x}(k). && (5.30)
\end{aligned}
$$

The above controller is the same as Theorem 5 of [15].

Remark 5.3. The extra burden to design a robust controller for an MP system is to solve a contractive matrix completion problem. A unique central solution can be obtained following the method in [22]. In this way, we can get a unique central controller.

To prove the theorems, we need the following lemma.

Lemma 5.1. Given a lifted LTI plant G with $\hat{G}(0) \in \mathcal{T}(\mathbb{R}^{q\times p})$, a controller K with $\hat{K}(0) \in \mathcal{T}(\mathbb{R}^{p\times q})$ satisfies $b_{G,K} > \gamma$ if and only if K has an coprime factorization: $K = UV^{-1}$ for some $U, V \in \mathcal{RH}_\infty$ with $\hat{U}(0) \in \mathcal{T}(\mathbb{R}^{p\times q})$ and $\hat{V}(0) \in \mathcal{T}(\mathbb{R}^{q\times q})$ satisfying

$$\left\| \begin{bmatrix} -\tilde{N}^{\sim} \\ \tilde{M}^{\sim} \end{bmatrix} + \begin{bmatrix} U \\ V \end{bmatrix} \right\|_\infty \leq \alpha \tag{5.31}$$

where α is given in (5.10).

Proof. Assume K is a stabilizing controller with $\hat{K}(0) \in \mathcal{T}(\mathbb{R}^{p\times q})$ satisfying $b_{G,K} > \gamma$. Then K has a coprime factorization $K = U_1V_1^{-1}$ with $\hat{U}_1(0) \in \mathcal{T}(\mathbb{R}^{p\times q})$ and $\hat{V}_1(0) \in \mathcal{T}(\mathbb{R}^{q\times q})$, an equivalent coprime factorization is (U, V) with

$$\begin{bmatrix} U \\ V \end{bmatrix} = -\gamma^{-2} \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} (\tilde{M}V_1 - \tilde{N}U_1)^{-1}$$

since $(\tilde{M}V_1 - \tilde{N}U_1)^{-1} \in \mathcal{H}_\infty$ by internal stability. From the fact that $\hat{U}_1(0) \in \mathcal{T}(\mathbb{R}^{p\times q})$, $\hat{V}_1(0) \in \mathcal{T}(\mathbb{R}^{q\times q})$, $\tilde{M}(0) \in \mathcal{T}(\mathbb{R}^{q\times q})$ and $\tilde{N}(0) \in \mathcal{T}(\mathbb{R}^{q\times p})$, we have $\hat{U}(0) \in \mathcal{T}(\mathbb{R}^{p\times q})$ and $\hat{V}(0) \in \mathcal{T}(\mathbb{R}^{q\times q})$. The remaining proof of necessity and proof of sufficiency follow the same discussion of Theorem 4.1 in [12]. □

Lemma 5.2. Let $G = \tilde{M}^{-1}\tilde{N}$ be a normalized left coprime factorization with $\tilde{M}(0) \in \mathcal{T}(\mathbb{R}^{q\times q})$ and $\tilde{N}(0) \in \mathcal{T}(\mathbb{R}^{q\times p})$. Then the optimal robust stability margin is

$$b_{opt} = \left\{1 - \inf_{\hat{U},\hat{V}} \| \begin{bmatrix} -\tilde{N}^{\sim} \\ \tilde{M}^{\sim} \end{bmatrix} + \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \|^2 \right\}^{\frac{1}{2}} \tag{5.32}$$

where $\hat{U}, \hat{V} \in \mathcal{RH}_\infty$ with $\hat{U}(0) \in \mathcal{T}(\mathbb{R}^{p\times q})$ and $\hat{V}(0) \in \mathcal{T}(\mathbb{R}^{q\times q})$.

The proof is similar to that of Theorem 4.2 in [12] since we get Lemma 5.1.

Proof of Theorem 1: We know that $G = \tilde{M}^{-1}\tilde{N}$ is a normalized left coprime factorization, where $\tilde{M}$ and $\tilde{N}$ are given by (5.14). It is then straightforward to check that

$$\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] := \left[\begin{array}{c|c} A^* + C^*L^* & C^*S^{*-1} \\ \hline -B^* - D^*L^* & -D^*S^{*-1} \\ L^* & S^* \end{array}\right] \tag{5.33}$$

is an anticausal realization of $\begin{bmatrix} -\tilde{N}^{\sim} \\ \tilde{M}^{\sim} \end{bmatrix}$, that is,

$$\begin{bmatrix} -\tilde{N}^{\sim} \\ \tilde{M}^{\sim} \end{bmatrix} = \bar{D} + \bar{C}(\lambda I - \bar{A})^{-1}\bar{B}.$$

Let P and Q be the solutions of the following Lyapunov equations

$$P = \bar{A}P\bar{A}^* + \bar{B}\bar{B}^* \tag{5.34}$$
$$Q = \bar{A}^*Q\bar{A} + \bar{C}^*\bar{C}. \tag{5.35}$$

It is shown in [10] that

$$\begin{aligned} &\inf_{\hat{U},\hat{V},\hat{U}(0)\in\mathcal{T}(\mathbb{R}^{p\times q}),\hat{V}(0)\in\mathcal{T}(\mathbb{R}^{p\times q})} \left\| \begin{bmatrix} -\tilde{N}^{\sim} \\ \tilde{M}^{\sim} \end{bmatrix} + \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right\|_\infty \\ = &\max_r \left\| \begin{bmatrix} \Pi_{\mathcal{U}_r} & 0 & 0 \\ 0 & \Pi_{\mathcal{Y}_r} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \bar{D} & \bar{C}P^{\frac{1}{2}} \\ Q^{\frac{1}{2}}\bar{B} & Q^{\frac{1}{2}}\bar{A}P^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} I - \Pi_{\mathcal{U}_r} & 0 \\ 0 & I \end{bmatrix} \right\|, \end{aligned}$$

where $\mathcal{U}_r$ and $\mathcal{Y}_r$ are defined as in (5.16-5.17). Note that P and Q be simple functions of X and Y defined in (5.4-5.5): $P = X(I + YX)^{-1}$ and $Q = Y$ [12]. The proof is then completed by Lemma 5.2. □

Proof of Theorem 2: To get the controller, we need to solve the constrained suboptimal Nehari extension problem (5.31) stated in Lemma 4.1. For the anticausal realization of $\begin{bmatrix} -\tilde{N}^{\sim} \\ \tilde{M}^{\sim} \end{bmatrix}$ in (5.33), we know that the solution of Lyapunov equations (5.34-5.35) is $P = X(I+YX)^{-1}$ and $Q = Y$. Now it is easy to obtain that

$$\begin{aligned} N_1 N_1^* &= I + S^{-1}C(I+YX)W^{-1}YC^*S^{*-1} \\ N_2^* N_2 &= \begin{bmatrix} I + B^*X(I+YX)W^{-1}B & -B^*X(I+YX)W^{-1}L \\ -L^*X(I+YX)W^{-1}B & I + L^*X(I+YX)W^{-1}L \end{bmatrix} \\ A_0 &= (\alpha^2 I - \bar{A}^*Q\bar{A}P)^{-1}\bar{A}^*(\alpha^2 I - QP). \end{aligned}$$

By Lemma 4.1, the solutions of the suboptimal Nehari extension problem (5.31) are as follows

$$\begin{aligned} \begin{bmatrix} U(\lambda) \\ V(\lambda) \end{bmatrix} &= \mathcal{F}_l(\Phi, R_0) = \Phi_{11} + \Phi_{12}R_0(I - \Phi_{22}R_0)^{-1}\Phi_{21}^{-1} \\ &= \left[\Phi_{11}\Phi_{21}^{-1} + (\Phi_{21} - \Phi_{11}\Phi_{21}^{-1}\Phi_{22})R_0\right]\left[\Phi_{21}^{-1}(I - \Phi_{22}R_0)\right]^{-1} \end{aligned}$$

where

$$\begin{aligned} \Phi &:= \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \\ &= \left[\begin{array}{c|cc} A_0 & (\alpha^2 I - \bar{A}^*Q\bar{A}P)^{-1}\bar{A}^*Q\bar{B} & (\alpha^2 I - QP)^{-1}\bar{C}^*N_2^{-1} \\ \hline \bar{C}PA_0 & \bar{C}P(\alpha^2 I - \bar{A}^*Q\bar{A}P)^{-1}\bar{A}^*Q\bar{B} + \bar{D} & -N_2^{-1} \\ -\alpha N_1^{-1}\bar{B}^* & \alpha N_1^{-1} & 0 \end{array}\right] \end{aligned}$$

and $\|R_0\| \le 1$ satisfying $\frac{1}{\alpha}N_2\Phi_{11}(0)N_1 - R_0 \in \begin{bmatrix} \mathcal{T}(\mathbb{R}^{p\times q}) \\ \mathcal{T}(\mathbb{R}^{q\times q}) \end{bmatrix}$. Noting that $\Phi_{21}^{-1}(I - \Phi_{22}R_0)$ is a unit in $\mathcal{H}_\infty$, then from basic coprime factorization theory, the coprime factors of the controller are also given by

$$\begin{bmatrix} U_1(\lambda) \\ V_1(\lambda) \end{bmatrix} = \Phi_{11}\Phi_{21}^{-1} + (\Phi_{21} - \Phi_{11}\Phi_{21}^{-1}\Phi_{22})R_0. \tag{5.36}$$

We need the following claim which is proved in the appendix.

Claim 5.1. *Equation (5.36) can be written as*

$$\begin{bmatrix} U_1(\lambda) \\ V_1(\lambda) \end{bmatrix} = \left[\begin{array}{c|c} A + LC & (I+YX)W^{-1}[(B+LD)E_1 - LE_2] \\ \hline \begin{bmatrix} F \\ C + DF \end{bmatrix}(I+YX)^{-1} & \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \end{array}\right] \tag{5.37}$$

where $\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$ *is given in (5.19).*

After some algebra manipulations, the controller is given by

$$K = U_1V_1^{-1} = \left[\begin{array}{c|c} A_C & B_C \\ \hline C_C & H \end{array}\right] \tag{5.38}$$

where

$$\begin{aligned} A_C &= A + LC + (I+YX)W^{-1}(L - BH - LDH)(C+DF)(I+YX)^{-1} \\ B_C &= (I+YX)W^{-1}(BH + LDH - L) \\ C_C &= (F - HC - HDF)(I+YX)^{-1} \\ H &= E_1E_2^{-1}. \end{aligned}$$

The remaining thing is to write the controller K of (5.38) to a general observer form using the state-space transformation. Recall that $W = \alpha^2 I + (\alpha^2 - 1)YX$, then it follows from equation (5.8) that

$$W(I+YX)^{-1}(A+LC)(I+YX) = (A+LC)W + BF - LC.$$

So we have

$$\begin{aligned} & W(I+YX)^{-1}A_C(I+YX)W^{-1} \\ = \; & W(I+YX)^{-1}(A+LC)(I+YX)W^{-1} + (L - BH - LDH)(C+DF)W^{-1} \\ = \; & A_K, \end{aligned} \tag{5.39}$$

$$W(I+YX)^{-1}B_C = BH + LDH - L = B_K, \tag{5.40}$$

$$C_C(I+YX)W^{-1} = (F - HC - DF)W^{-1} = C_K. \tag{5.41}$$

Therefore, the controller is given by (5.20). It is then straightforward to check that the controller K can be written in the general observer form of (5.25-5.26) by taking that $\bar{H} = -(I - HD)^{-1}H$ and $F_K = (I - HD)^{-1}(C_K + HC)$. This completes the proof. □

6 Conclusion

In this paper, we present a state space solution to the robust stabilization problem of discrete-time periodic and multirate systems. First, we give a general setup of MP systems and show how the robust stabilization problem of multirate systems with ν-gap metric uncertainty can be converted to a constrained $\mathcal{H}_\infty$ optimal control problem. An explicit solution is presented to the Nehari extension problem with a structural constraint. The optimal robust stabilization margin is explicitly computed and an observer form suboptimal controller is presented. The computational burden is to solve two Riccati equations and an contractive matrix completion problem.

A Appendix

To prove Claim 5.1, we need two important equalities as follows:

$$B^*X(A+BF) + D^*(C+DF) = F \tag{A.42}$$

$$L^*X(I+YX)^{-1}(A+LC) + (SS^*)^{-1}C = (C+DF)(I+YX)^{-1}. \tag{A.43}$$

Proof. It is easy to check (A.42) directly. From equation (5.8) and the following form of the Riccati equation (5.4)

$$X = A^*X(A+BF) + C^*(C+DF),$$

we have

$$\begin{aligned} & L^*X(I+YX)^{-1}(A+LC) + (SS^*)^{-1}C \\ = \; & L^*X(A+BF)(I+YX)^{-1} + (SS^*)^{-1}C = (C+DF)(I+YX)^{-1}. \end{aligned}$$

The last equality comes from (A.42). □

Proof of Claim 5.1. First it is straightforward to check that

$$\begin{aligned} \Phi_{11}\Phi_{21}^{-1} &= \left[\begin{array}{c|c} \bar{A}^* & (\alpha^2 I - \bar{A}^* Q \bar{A} P)^{-1} \bar{A}^* Q \bar{B} \alpha^{-1} N_1 \\ \hline \bar{C} P \bar{A}^* + \bar{D}\bar{B}^* & \left[\bar{C}P(\alpha^2 I - \bar{A}^* Q \bar{A} P)^{-1} \bar{A}^* Q \bar{B} + \bar{D}\right] \alpha^{-1} N_1 \end{array}\right] \\ \Phi_{21} - \Phi_{11}\Phi_{21}^{-1}\Phi_{22} &= \left[\begin{array}{c|c} \bar{A}^* & (\alpha^2 I - QP)^{-1}\bar{C}^* N_2^{-1} \\ \hline \bar{C} P \bar{A}^* + \bar{D}\bar{B}^* & -N_2^{-1} \end{array}\right]. \end{aligned}$$

Then (5.36) can be written as

$$\left[\begin{array}{c} U_1 \\ V_1 \end{array}\right] = \left[\begin{array}{c|c} \bar{A}^* & (\alpha^2 I - \bar{A}^* Q \bar{A} P)^{-1} \bar{A}^* Q \bar{B} \alpha^{-1} N_1 + (\alpha^2 I - QP)^{-1} \bar{C}^* N_2^{-1} R_0 \\ \hline \bar{C} P \bar{A}^* + \bar{D}\bar{B}^* & \left[\bar{C}P(\alpha^2 I - \bar{A}^* Q \bar{A} P)^{-1} \bar{A}^* Q \bar{B} + \bar{D}\right] \alpha^{-1} N_1 - N_2^{-1} R_0 \end{array}\right]. \tag{A.44}$$

Note that

$$\bar{A}^* Q \bar{B} + \bar{C}^* \bar{D} = (A + LC) Y C^* S^{*-1} + BD^* S^{*-1} + LDD^* S^{*-1} + LS^{*-1} = 0.$$

So we have

$$(\alpha^2 I - \bar{A}^* Q \bar{A} P)^{-1} \bar{A}^* Q \bar{B} = -(\alpha^2 I - QP + \bar{C}^* \bar{C})^{-1} \bar{C}^* \bar{D} = -(\alpha^2 I - QP)^{-1} \bar{C}^* (N_2^* N_2)^{-1} \bar{D}. \tag{A.45}$$

The proof is then completed by direct computation as follows

$$\begin{aligned} & [\bar{C}P(\alpha^2 I - \bar{A}^* Q \bar{A} P)^{-1} \bar{A}^* Q \bar{B} + \bar{D}] \alpha^{-1} N_1 - N_2^{-1} R_0 \\ = \ & [-\bar{C}P(\alpha^2 I - QP)^{-1} \bar{C}^* (N_2^* N_2)^{-1} \bar{D} + \bar{D}] \alpha^{-1} N_1 - N_2^{-1} R_0 \\ = \ & (N_2^* N_2)^{-1} \bar{D} \alpha^{-1} N_1 - N_2^{-1} R_0 \\ = \ & \alpha^{-1} (N_2^* N_2)^{-1} \left[\begin{array}{c} -D^* \\ I \end{array}\right] S^{*-1} N_1 - N_2^{-1} R_0, \end{aligned} \tag{A.46}$$

$$\begin{aligned} & (\alpha^2 I - \bar{A}^* Q \bar{A} P)^{-1} \bar{A}^* Q \bar{B} \alpha^{-1} N_1 + (\alpha^2 I - QP)^{-1} \bar{C}^* N_2^{-1} R_0 \\ = \ & -(\alpha^2 I - QP)^{-1} \bar{C}^* \left[(N_2^* N_2)^{-1} \bar{D} \alpha^{-1} N_1 - N_2^{-1} R_0\right] \\ = \ & (I + YX) W^{-1} [BE_1 + LDE_1 - LE_2], \end{aligned} \tag{A.47}$$

and

$$\begin{aligned} & \bar{C} P \bar{A}^* + \bar{D} \bar{B}^* \\ = \ & -\left[\begin{array}{c} B^* + D^* L^* \\ -L^* \end{array}\right] X (I + YX)^{-1} (A + LC) + \left[\begin{array}{c} -D^* S^{*-1} \\ S^{*-1} \end{array}\right] S^{-1} C \\ = \ & \left[\begin{array}{c} F \\ C + DF \end{array}\right] (I + YX)^{-1}. \end{aligned} \tag{A.48}$$

Note that equations (A.42-A.43) are used to get (A.48).

References

[1] M. Araki and K. Yamamato. "Multivariable multirate sampled-data systems: state-space description, transfer characteristics, and Nyquist criterion". *IEEE Trans. Automat. Contr.*, 30:145–154, 1986.

[2] S. Bittanti and P. Colaneri. "Invariant representations of discrete-time periodic systems". *Automatica*, 36:1777–1793, 2000.

[3] R. Bucy. "The Riccati equation and its bounds". *J. Comput. Syst. Sci.*, 6:343–353, 1972.

[4] T. Chen and B. A. Francis. "Design of multirate filter banks by $\mathcal{H}_\infty$ optimization". *IEEE Trans. Signal Processing*, 43:2822–2830, 1995.

[5] T. Chen and L. Qiu. "$\mathcal{H}_\infty$ design of general multirate sampled-data control systems". *Automatica*, 30:1139–1152, 1994.

[6] T. Chen, L. Qiu, and E. Bai. "General mulrirate building blocks and their application in nonumiform filter banks". *IEEE Trans. on Circuits and Systems, Part II*, 45:948–958, 1998.

[7] A. Feuer and G. C. Goodwin. *Sampling in Digital Signal Processing and Control*. Birkhäuser, 1996.

[8] G. Floquet. "Sur Les equations differentielles linearies a cofficients periodicques". *Annales de l'Ecole Normale Superieur*, 12:47–89, 1883.

[9] C. Foias, A. E. Frazho, I. Gohberg, and M. A. Kaashoek. *Metric Constrained Interpolation, Commutant Lifting and Systems*. Birkhäuser, 1998.

[10] T. T. Georgiou and P. P. Khargonekar. "A constructive algorithm for sensitility optimization of periodic systems". *SIAM J. Control and Optimization*, 25:334–340, 1987.

[11] T. T. Georgiou and M. C. Smith. "Optimal robustness in the gap metric". *IEEE Trans. Automat. Contr.*, 35:673–687, 1990.

[12] K. Glover and D. C. McFarlane. "Robust stabilization of normalized coprime factor plant descriptions with $\mathcal{H}_\infty$-bounded uncertainty". *IEEE Trans. Automat. Contr.*, 34:821–830, 1989.

[13] I. Gohberg, M. A. Kaashoek and L. Lerer. "Minimality and realization of discrete time-varying systems". *Operator Theory: Advances and Applications*, 33:261–296, 1992.

[14] G. C. Goodwin and A. Feuer. "Linear periodic control: a frequency domain viewpoint". *Systems and Control Letters*, 19:379–390, 1992.

[15] P. A. Iglesias. "The strictly proper discrete-time controller for the normalized left-coprime factorization robust stabilization problem". *IEEE Trans. Automat. Contr.*, 45:516–520, 2000.

[16] E. I. Jury and F. J. Mullin. "The analysis of sampled-data control systems with a periodically time-varying sampling rate". *IRE Trans. Automat. Contr.*, 24:15–21, 1959.

[17] R. E. Kalman and J. E. Bertram. "A unified approach to the theory of sampling systems". *J. Franklin Inst.*, 267:405–436, 1959.

[18] P. P. Khargonekar, K. Poolla, and A. Tannenbaum. "Robust control of linear time-invariant plants using periodic compensation". *IEEE Trans. Automat. Contr.*, 30:1088–1096, 1985.

[19] G. M. Kranc. "Input-output analysis of multirate feedback systems". *IRE Trans. Automat. Contr.*, 3:21–28, 1957.

[20] S. Lall and G. Dullerud. "An LMI solution to the robust synthesis problem for multi-rate sampled-data systems". *Automatica*, 37:1909–1922, 2001.

[21] R. A. Meyer and C. S. Burrus. "A unified analysis of multirate and periodically time-varying digital filters". *IEEE Trans. on Circuits and Systems*, 22:162–168, 1975.

[22] L. Qiu and T. Chen. "Contractive completion of block matrices and its application to $\mathcal{H}_\infty$ control of periodic systems". In P. L. I. Gohberg and P. N. Shivakumar, editors, *Applications of Operator Theory*, pages 2506–2511. Birkhäuser, 1996.

[23] L. Qiu and T. Chen. "Multirate sampled-data systems : all $\mathcal{H}_\infty$ suboptimal controllers and the minimum entropy controllers". *IEEE Trans. Automat. Contr.*, 44:537–550, 1999.

[24] L. Qiu and E. J. Davison. "Pointwise gap metric on transfer matrices". *IEEE Trans. Automat. Contr.*, 37:770–780, 1992.

[25] L. Qiu and E. J. Davison. "Stability under simultaneous uncertainty in plant and controller". *Systems & Control Letters*, 18:9–22, 1992.

[26] L. Qiu and K. Tan. "Direct state space solution of multirate sampled-data $\mathcal{H}_2$ optimal control". *Automatica*, 34:234–245, 1998.

[27] R. G. Shenoy, D. Burnside, and T. W. Parks. "Linear periodic systems and multirate filter design". *IEEE Trans. Signal Processing*, 42:2242–2256, 1994.

[28] L. Tong, G. Xu, and T. Kailath. "Blind identification and equalization based on second-order statistics: A time domain approach". *IEEE Trans. Signal Processing*, 42:2242–2256, 1994.

[29] P. Vaidyanathan. *Multirate Systems and Filter Banks*. Prentice-Hall, 1993.

[30] A. Varga and P. V. Dooren. "Computational methods for periodic systems - an overview". In S. Bittanti and P. Colaneri, editors, *Preprints of IFAC Workshop on Periodic Control Systems*, pages 171–176. 2001.

[31] G. Vinnicombe. "Frequency domain uncertainty and the graph topology". *IEEE Trans. Automat. Contr.*, 38:1371–1383, 1993.

[32] P. G. Voulgaris, M. A. Dahleh, and L. Valavani. "$\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimal controllers for periodic and multirate systems ". *Automatica*, 30:251–263, 1994.

[33] D. J. Walker. "Robust stabilizability of discrete-time systems with normalized stable factor perturbation". *Int. J. Contr.*, 52:441–455, 1990.

[34] S. Wan and B. Huang. "Gap metric on the subgraphs of systems and the robustness problem". *IEEE Trans. Automat. Contr.*, 45:1522–1526, 2000.

[35] B. Xie and V. L. Syrmos. "Robust stabilization for discrete-time periodic systems with an $\mathcal{H}_\infty$-bounded uncertainty". In *Proceedings of the 36th Conference on Decision and Control*, pages 2175–2180, San Diego, USA, 1997.