Pole Placement Under Output Feedback: A Simplification of the Problem

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Abstract

For a given system defined by the matrix triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, which polynomials are characteristic polynomials of $\mathbf{A} + \mathbf{BKC}$ as $\mathbf{K}$ varies? When is this set of polynomials dense in the space of all monic polynomials of degree n? We show that a polynomial occurs as a characteristic polynomial of $\mathbf{A} + \mathbf{BKC}$ for some matrix $\mathbf{K}$ if and only if it occurs as a characteristic polynomial of $\mathbf{A}'+\mathbf{B}'\mathbf{K}'\mathbf{C}'$ for some matrix $\mathbf{K}'$, where $(\mathbf{A}', \mathbf{B}', \mathbf{C}')$ is related to $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ via an equivalence relation. Regarding the question of the density of the set of characteristic polynomials, our approach allows previously known necessary (and generically sufficient) conditions to be rewritten in terms of the following two conditions on the matrices $\mathbf{B}$ and $\mathbf{C}$: (rank $\mathbf{B}$)(rank $\mathbf{C}$) $\geq n$, and $\mathbf{CB}$ is not the zero matrix.

1 Introduction

Each matrix triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, with $\mathbf{A}$ in $k^{n\times n}$, $\mathbf{B}$ in $k^{n\times m}$, and $\mathbf{C}$ in $k^{v\times n}$, k a field, defines a time-invariant linear system with output:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned}$$

Output feedback refers to application of an input of the form $\mathbf{u} = \mathbf{K}\mathbf{y} + \mathbf{u}'$, with $\mathbf{K}$ in $k^{m\times v}$, thereby replacing the triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ with the triple $(\mathbf{A} + \mathbf{BKC}, \mathbf{B}, \mathbf{C})$.

For pole placement, we wish to identify the characteristic polynomials of the matrices $\mathbf{A} + \mathbf{BKC}$ as $\mathbf{K}$ varies. In Section 2, we show that this set of characteristic polynomials remains unchanged under many transformations of the triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. That is, a polynomial $p(x)$ is the characteristic polynomial of $\mathbf{A} + \mathbf{BKC}$ for some $m \times v$ matrix $\mathbf{K}$ if and only if it is the characteristic polynomial of $\mathbf{A}'+\mathbf{B}'\mathbf{K}\mathbf{C}'$ for some matrix $\mathbf{K}$, where the triples $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and $(\mathbf{A}', \mathbf{B}', \mathbf{C}')$ have certain properties in common. In Section 3, we relate these results to the work of Helten, Rosenthal and Wang in [1]. The results in this paper are based on ideas developed in M. Schilmoeller's dissertation [2], which focuses on equivalence of linear systems under output feedback.

2 A Simplification of the Problem

We begin by noting that the characteristic polynomial of $\mathbf{A}+\mathbf{BKC}$ remains unchanged under state space change of basis. Hence, without loss of generality, we may replace a given matrix triple $(\mathbf{A},\mathbf{B},\mathbf{C})$ with $(\mathbf{TAT}^{-1},\mathbf{TB},\mathbf{CT^{-1}})$ where the transformation $\mathbf{T}$ represents a change of basis. We choose an ordered basis $(x_1,\dots,x_n)$ for k^n as follows:

$$\begin{array}{rl} \{x_1,\dots,x_{p+q}\} & \text{spans } \ker\mathbf{C} \\ \{x_{p+1},\dots,x_{p+q}\} & \text{spans } \ker\mathbf{C}\cap\mathrm{im}\mathbf{B} \\ \{x_{p+1},\dots,x_{p+q+r}\} & \text{spans } \mathrm{im}\mathbf{B} \end{array}$$

where $q=\dim(\ker\mathbf{C}\cap\mathrm{im}\mathbf{B})$, $r=\mathrm{rank}\mathbf{B}-q$, and $p=n-\mathrm{rank}\mathbf{C}-q$. Let $s=n-(p+q+r)$.

Under this change of basis, the matrices $\mathbf{B}$ and $\mathbf{C}$ have the following form:

$$\mathbf{B}=\left[\begin{array}{c} 0 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \\ 0 \end{array}\right]\begin{array}{l} \}p \\ \}q \\ \}r \\ \}s \end{array} \tag{2.1}$$

and

$$\mathbf{C}=[\quad \overbrace{0}^{p} \quad \overbrace{0}^{q} \quad \overbrace{\mathbf{C}_3}^{r} \quad \overbrace{\mathbf{C}_4}^{s} \quad] \tag{2.2}$$

where $\mathbf{B}$ has rank $q+r$ and $\mathbf{C}$ has rank $r+s$. For the remainder of the paper, we assume that $\mathbf{B}$ and $\mathbf{C}$ are in this form, which we call (p,q,r) *block forms.*

With the matrices $\mathbf{B}$ and $\mathbf{C}$ in (p,q,r) block form, the matrix $\mathbf{BKC}$ has the form

$$\begin{array}{c|cccc|} & \overbrace{\quad}^{p} & \overbrace{\quad}^{q} & \overbrace{\quad}^{r} & \overbrace{\quad}^{s} \\ p\{ & 0 & 0 & 0 & 0 \\ q\{ & 0 & 0 & * & * \\ r\{ & 0 & 0 & * & * \\ s\{ & 0 & 0 & 0 & 0 \end{array} \tag{2.3}$$

Proposition 2.1. *Let* $\mathbf{B}$ *and* $\mathbf{C}$ *be in* (p,q,r)*-block form and let* $\mathbf{S}$ *be an* $n\times n$ *matrix of the form (2.3). Then there is an* $m\times v$ *matrix* $\mathbf{K}$ *such that* $\mathbf{S}=\mathbf{BKC}$.

Proof. Since $\mathbf{S}$ has the form (2.3),

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{S}_{23} & \mathbf{S}_{24} \\ 0 & 0 & \mathbf{S}_{33} & \mathbf{S}_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $\mathbf{K}'$ be the $m \times v$ matrix given by

$$\mathbf{K}' = \begin{bmatrix} 0 & \mathbf{S}_{23} & \mathbf{S}_{24} \\ 0 & \mathbf{S}_{33} & \mathbf{S}_{34} \\ 0 & 0 & 0 \end{bmatrix}$$

Using a matrix $\mathbf{G} \in k^{m \times m}$ to column reduce $\mathbf{B}$ and a matrix $\mathbf{H} \in k^{v \times v}$ to row reduce $\mathbf{C}$, we have:

$$\mathbf{BG} = \begin{bmatrix} 0_{p \times q} & 0 & 0 \\ I_{q \times q} & 0 & 0 \\ 0 & I_{r \times r} & 0 \\ 0 & 0 & 0_{s \times (m-q-r)} \end{bmatrix}$$

and

$$\mathbf{HC} = \begin{bmatrix} 0_{(v-r-s) \times p} & 0 & 0 & 0 \\ 0 & 0_{r \times q} & I_{r \times r} & 0 \\ 0 & 0 & 0 & I_{s \times s} \end{bmatrix}$$

For $\mathbf{K} = \mathbf{GK}'\mathbf{H}$, we obtain:

$$\mathbf{BKC} = (\mathbf{BG})\mathbf{K}'(\mathbf{HC})$$

$$= \begin{bmatrix} 0_{p \times q} & 0 & 0 \\ I_{q \times q} & 0 & 0 \\ 0 & I_{r \times r} & 0 \\ 0 & 0 & 0_{s \times (m-q-r)} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{S}_{23} & \mathbf{S}_{24} \\ 0 & \mathbf{S}_{33} & \mathbf{S}_{34} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0_{(v-r-s) \times p} & 0 & 0 & 0 \\ 0 & 0_{r \times q} & I_{r \times r} & 0 \\ 0 & 0 & 0 & I_{s \times s} \end{bmatrix}$$

$$= \mathbf{S}$$

□

From Proposition 2.1, we see that the pole placement problem is reduced to identifying characteristic polynomials of matrices $\mathbf{A} + \mathbf{S}$, where $\mathbf{S}$ varies over the set $\mathcal{S}$ of matrices of the form (2.3). Furthermore, the set of characteristic polynomials is unchanged if $\mathbf{A}$ is replaced with $\mathbf{TAT}^{-1}$, for any invertible $n \times n$ matrix $\mathbf{T}$ that fixes the set $\mathcal{S}$. It is easy to verify that the invertible matrices which fix the set $\mathcal{S}$ under conjugation are those of the form

$$\mathbf{T} = \begin{array}{c} \begin{array}{cccc} \overbrace{\hphantom{\mathbf{T}_{11}}}^{p} & \overbrace{\hphantom{\mathbf{T}_{22}}}^{q} & \overbrace{\hphantom{\mathbf{T}_{33}}}^{r} & \overbrace{\hphantom{\mathbf{T}_{44}}}^{s} \end{array} \\ \left[\begin{array}{cccc} \mathbf{T}_{11} & 0 & 0 & \mathbf{T}_{14} \\ \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} & \mathbf{T}_{24} \\ 0 & 0 & \mathbf{T}_{33} & \mathbf{T}_{34} \\ 0 & 0 & 0 & \mathbf{T}_{44} \end{array} \right] \end{array} \begin{array}{l} \\ \}\, p \\ \}\, q \\ \}\, r \\ \}\, s \end{array}$$

We denote the set of invertible matrices of this form by $\mathcal{T}_{p,q,r}$.

Definition 2.1. *Let $p(x)$ be a monic polynomial of degree n. We say that $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ produces $p(x)$ if there is an $m \times v$ matrix $\mathbf{K}$ such that*

$$p(x) = \det(x\mathbf{I} - \mathbf{A} - \mathbf{BKC})$$

Let $\mathcal{P}_{\mathbf{A},\mathbf{B},\mathbf{C}}$ denote the set of polynomials that are produced by $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. That is,

$$\mathcal{P}_{\mathbf{A},\mathbf{B},\mathbf{C}} = \{\det(x\mathbf{I} - \mathbf{A} - \mathbf{BKC}) | \mathbf{K} \in k^{m \times v}\}$$

The following theorem is an immediate consequence of the remarks above.

Theorem 2.1. *Let the matrices $\mathbf{B} \in k^{n \times m}$ and $\mathbf{C} \in k^{v \times n}$ be in (p, q, r) block form. If the matrix triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ produces the polynomial $p(x)$, then $(\mathbf{A}, \mathbf{B}', \mathbf{C}')$ also produces $p(x)$ for all pairs $(\mathbf{B}', \mathbf{C}')$ in (p, q, r) block form. Furthermore, the triple $(\mathbf{TAT}^{-1}, \mathbf{B}, \mathbf{C})$ produces the same polynomials as $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ if $\mathbf{T} \in \mathcal{T}_{p,q,r}$. In other words, $\mathcal{P}_{\mathbf{A},\mathbf{B},\mathbf{C}} = \mathcal{P}_{\mathbf{TAT}^{-1},\mathbf{B}',\mathbf{C}'}$ if the pairs $(\mathbf{B}, \mathbf{C})$ and $(\mathbf{B}', \mathbf{C}')$ are both in (p, q, r) block form and $\mathbf{T} \in \mathcal{T}_{p,q,r}$.*

Using the convention that $\mathbf{B}$ and $\mathbf{C}$ are in (p, q, r) block form, we see that the set $\mathcal{P}_{\mathbf{A},\mathbf{B},\mathbf{C}}$ depends only on the $\mathcal{T}_{p,q,r}$-conjugacy class of $\mathbf{A}$ and on the invariants p, q, r.

3 Relationship to Known Results

In the paper [1], Helton, Rosenthal, and Wang consider the pole placement problem over the field of complex numbers $\mathbb{C}$. Identifying the set of degree n monic polynomials with the affine space $\mathbb{C}^n$, they give two conditions on the vector space $\{\mathbf{BKC} \mid \mathbf{K} \in \mathbb{C}^{m \times v}\}$ that are necessary for the set $\mathcal{P}_{\mathbf{A},\mathbf{B},\mathbf{C}}$ to be a dense subset of $\mathbb{C}^n$:

1. $\dim\{\mathbf{BKC} \mid \mathbf{K} \in \mathbb{C}^{m \times v}\} \geq n$, and
2. There exists a matrix $\mathbf{K} \in \mathbb{C}^{m \times v}$ such that $\mathbf{BKC}$ has nonzero trace.

From the previous section and using the convention that the matrices $\mathbf{B}$ and $\mathbf{C}$ are in (p, q, r) block form, we know that the vector space $\{\mathbf{BKC} \mid \mathbf{K} \in \mathbb{C}^{m \times v}\}$ depends only on the triple of non-negative integers (p, q, r).

Definition 3.1. *For a triple of non-negative integers* (p,q,r) *such that* $p+q+r \leq n$*, define*

$$\mathbf{M}_{p,q,r} = \{\mathbf{A} \in \mathbb{C}^{n\times n} \mid \mathcal{P}_{\mathbf{A},\mathbf{B},\mathbf{C}} \textit{ is dense in } \mathbb{C}^n \textit{ for some pair } (\mathbf{B},\mathbf{C}) \textit{ in } (p,q,r) \textit{ block form}\}$$

The main result of the paper [1] is that, for a given pair $(\mathbf{B},\mathbf{C})$ satisfying the two conditions above, the set $\mathbf{M}_{p,q,r}$ is a dense subset of $\mathbb{C}^{n\times n}$. Using the (p,q,r) block representation, we have a nice interpretation of the two conditions:

Theorem 3.1. *Let* $\mathbf{B}$ *be an* $n \times m$ *matrix over* $\mathbb{C}$ *and let* $\mathbf{C}$ *be an* $v \times n$ *matrix over* $\mathbb{C}$ *. Then*

1. $\dim\{\mathbf{BKC} \mid \mathbf{K} \in \mathbb{C}^{m\times v}\} \geq n$ *if and only if* $(\mathit{rank}\mathbf{B})(\mathit{rank}\mathbf{C}) \geq n$
2. *There exists a matrix* $\mathbf{K} \in \mathbb{C}^{m\times v}$ *such that* $\mathbf{BKC}$ *has nonzero trace if and only if im*$\mathbf{B}$ *does not lie in* $\ker \mathbf{C}$.

Proof. It suffices to consider matrix pairs $(\mathbf{B},\mathbf{C})$ of the form (2.1) and (2.2). Proposition 2.1 states that the vector space $\{\mathbf{BKC} \mid \mathbf{K} \in \mathbb{C}^{m\times v}\}$ consists of all $n \times n$ matrices of the form (2.3). Consequently, this vector space has dimension $(q+r)(r+s)$. Since $q+r$ is rank$\mathbf{B}$ and $r+s$ is rank$\mathbf{C}$, the conclusion (1) follows. By inspection, the set of matrices of the form (2.3) fails to contain a matrix with non-zero trace only when $r=0$. But $r=0$ if and only if the image of $\mathbf{B}$ lies entirely inside $\ker \mathbf{C}$. □

References

[1] W. Helton, J. Rosenthal, X. Wang, "Matrix Extensions and Eigenvalue Completions, The Generic Case," Trans. AMS 349 (1997), 3401-3408.

[2] M. Schilmoeller, "A Complete Invariant for Linear Dynamical Systems Under Output Feedback," dissertation approved June 13, 2000, published May 2001, UMI.