Non-symmetric Riccati theory and linear quadratic Nash games

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Abstract

The existence of a stabilizing solution to the non-symmetric algebraic Riccati equation is shown to be equivalent to the invertibility of a certain Toeplitz operator, whose symbol is the transfer matrix function of an exponentially dichotomic system. We also show that this theory has an important application to non-cooperative differential games.

1 Introduction

Ever since it emerged in the solution of the linear quadratic optimal control and filtering problems in the works of Kalman, the Riccati equation played a crucial role and has been a central topic in control theory. Popov's *positivity theory* [9] proved to be the natural framework for the linear quadratic topics, where the famous Kalman-Popov-Yakubovich Lemma may be seen as a result of the most theoretical relevance. The introduction of the two Riccati equation approach for the H^∞ control problem, increased substantially the range of control applications of the Riccati theory, extending the class of equations interesting from a practical point of view, with equations featuring *indefinite sign* free and quadratic terms. From this last perspective, much attention has been paid to the so-called *signature condition* (see [6]), which says that the standard algebraic Riccati equation has a stabilizing solution iff a certain signature condition holds on a selfadjoint rational transfer function, the so-called Popov function. Such condition is a natural way to extend Popov's positiveness condition to the game theoretic situation, which is synonymous to H^∞ control theory. Furthermore, the key result in time-domain states that the the Riccati equation has a stabilizing solution iff the Toeplitz operator associated with the L^2 input-output operator defined by the underlying Hamiltonian system has a bounded inverse. The basic idea of this work is to find a similar theory, that can be applied to non-cooperative differential games. These seem strongly related to coupled *non-symmetric* algebraic/differential Riccati equations of special form (see e.g. [2]). Moreover, it is the main task to find necessary and sufficient conditions for existence and uniqueness of equilibrium points. The only known iff-condition for open-loop linear quadratic Nash games is the invertibility of the so-called *decision operator* (see [8]), where additionally convexity assumptions are made.

This work has been motivated by the striking similarity between the above mentioned results from Nash games and from generalized Riccati theory, which are relating the existence of a stabilizing solution to the algebraic Riccati equation to the invertibility of some associated linear operator. The aim of the paper is to extend the generalized Riccati theory in [6] to *non-symmetric* algebraic Riccati equations, and, as an important consequence, to characterize unique Nash equilibria for linear quadratic open-loop Nash games on the infinite time horizon in the most general case. In fact this result is obtained as a simple application of the non-symmetric Riccati theory. We adopt a system theory viewpoint, following the methodology proposed in [6] and do not focus on the factorization approach as done for instance in [1].

The paper is organized as follows. Section 2. introduces some basic notions and gives several preliminary results. The next section contains the main results of the paper, stating that the existence of a stabilizing solution to the non-symmetric algebraic Riccati equation is equivalent to the invertibility of an associated Toeplitz operator. In Section 4. we show how these results can be applied to open-loop linear quadratic Nash games.

Subsequently the following notations will be adopted. By $\mathbb{R}$ and $\mathbb{R}^{m\times n}$ we denote respectively, the real axis and the set of $m \times n$ real matrices. Further, $RH^{\infty}_{m\times n}$ will stand for the set of $m \times n$ proper, real-rational matrices, without poles in the closed right part of the complex plane, while $RL^{\infty}_{m\times n}$ will stand for the set of $m\times n$ proper real-rational matrices without poles on the imaginary axis. If (A, B, C, D) is any state-space realization of a proper real-rational matrix $G(s)$, *i.e.* $G(s) = D + C(sI - A)^{-1}B$, then

$$G^*(s) := G^T(-s) = D^T + B^T(-sI - A^T)^{-1}C^T$$

denotes the *adjoint* of G.

A real-rational matrix $\Xi \in RH^{\infty}_{m\times m}$ is said to be a *unit* (in $RH^{\infty}_{m\times m}$) if it is invertible and its inverse Ξ^{-1} belongs also to $RH^{\infty}_{m\times m}$. We say that $G \in RL^{\infty}_{m\times m}$ is *anti-analytic factorizable* if there exist two units $\Psi_1, \Psi_2 \in RH^{\infty}_{m\times m}$ such that $G = \Psi_2^*\Psi_1$ holds.

2 Non-symmetric Riccati equations

We first introduce the notion of $\mathcal{P}$-system, which is the key object of our subsequent development. Several mathematical objects are associated with the $\mathcal{P}$-system, such as the Popov function or the non-symmetric algebraic Riccati equation.

Definition 2.1. *Let $A_1 \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m}$, $A_2 \in \mathbb{R}^{l\times l}$, $B_2^T \in \mathbb{R}^{m\times l}$, $Q \in \mathbb{R}^{l\times n}$, $L_1 \in \mathbb{R}^{l\times m}$, $L_2^T \in \mathbb{R}^{m\times n}$ and $R \in \mathbb{R}^{m\times m}$. The continuous-time system defined by*

$$\Sigma : \quad \begin{aligned} \dot{x} &= A_1 x + B_1 u, \quad x(0) = \xi && \text{(2.1a)} \\ \dot{\lambda} &= -Qx - A_2^T \lambda - L_1 u && \text{(2.1b)} \\ \nu &= L_2^T x + B_2^T \lambda + Ru && \text{(2.1c)} \end{aligned}$$

will be called a $\mathcal{P}$-system. *Here $x(t) \in \mathbb{R}^n$ is the state, $\lambda(t) \in \mathbb{R}^l$ is the "dual" state, $u(t), \nu(t) \in \mathbb{R}^m$ are respectively, the input and the output. We shall frequently use the abbreviation*

$$\Sigma = (A_1, B_1; A_2^T, B_2^T; Q, L_1, L_2^T, R), \tag{2.2}$$

which outlines explicitly the coefficients of the system (2.1a-2.1c). Its associated transfer matrix function is given by

$$\Pi_\Sigma(s) := R + B_2^T(-sI - A_2^T)^{-1} L_1 + L_2^T(sI - A_1)^{-1} B_1 + B_2^T(-sI - A_2^T)^{-1} Q (sI - A_1)^{-1} B_1 \tag{2.3}$$

and will be termed as the (non-selfadjoint) Popov function *associated with the $\mathcal{P}$-system Σ.*

Consider a $\mathcal{P}$-system given by (2.2). For any $X \in \mathbb{R}^{l\times n}$ associate with Σ the matrix

$$D_\Sigma(X) := \begin{pmatrix} A_2^T X + X A_1 + Q & X B_1 + L_1 \\ B_2^T X + L_2^T & R \end{pmatrix}. \tag{2.4}$$

Definition 2.2. *The system of equations*

$$\begin{pmatrix} A_2^T X_1 + X_1 A_1 + Q & X_1 B_1 + L_1 \\ B_2^T X_1 + L_2^T & R \end{pmatrix} \begin{pmatrix} I \\ F_1 \end{pmatrix} = 0 \tag{2.5}$$

in the unknowns X_1, $F_1 \in \mathbb{R}^{m\times n}$ will be called the right *non-symmetric algebraic Riccati system associated with Σ, NARS(Σ). A solution (X_1, F_1) to (2.5) is called a right* stabilizing *solution, if $A_1 + B_1 F_1$ is stable.*

Similarly, the left *non-symmetric algebraic Riccati system associated with Σ is defined as*

$$\begin{pmatrix} I & F_2^T \end{pmatrix} \begin{pmatrix} A_2^T X_2 + X_2 A_1 + Q & X_2 B_1 + L_1 \\ B_2^T X_2 + L_2^T & R \end{pmatrix} = 0. \tag{2.6}$$

If $A_2 + B_2 F_2$ is stable for a pair $(X_2, F_2) \in \mathbb{R}^{l\times n} \times \mathbb{R}^{m\times l}$ satisfying (2.6), then (X_2, F_2) is said to be a left stabilizing *solution.*

Remark 2.1. *(***Duality***) The transpose of the left NARS(Σ) (2.6) is, in fact, the right non-symmetric algebraic Riccati system associated with the* dual *of Σ, that is, $\Sigma_* = (A_2, B_2; A_1^T, -B_1^T; -Q^T, -L_2, L_1^T, R^T)$, in the unknowns X_2^T, F_2. Notice also that $\Pi_{\Sigma_*}(s) = \Pi_\Sigma^*(s)$, i.e. the Popov function associated with Σ_* is the adjoint of the Popov function associated with Σ.*

Proposition 2.1. *If (X_1, F_1) and (X_2, F_2) are respectively, a right and a left stabilizing solution to the NARS(Σ), then $X_1 = X_2$.*

Proposition 2.2. *Let the $\mathcal{P}$-system $\Sigma = (A_1, B_1; A_2^T, B_2^T; Q, L_1, L_2^T, R)$ be given. Then one has:*

1. *If (X, F_1) is a solution to the right NARS(Σ) (2.5) and (X, F_2) is simultaneously a solution to the left NARS(Σ) (2.6), then the following factorization holds:*

$$\Pi_\Sigma(s) = S_{F_2}^*(s)\, R\, S_{F_1}(s), \tag{2.7}$$

   where $S_{F_i}(s) = I - F_i(sI - A_i)^{-1}B_i$, $i = 1, 2$.

2. *Assume that R is invertible, and that A_1 and A_2 are both stable. Let (X, F_1) and (X, F_2) be respectively, a right and a left* stabilizing *solution to the NARS(Σ) (see Proposition 2.1). Then S_{F_1} and S_{F_2} are both units in $RH^\infty_{m\times m}$ (because $A_i + B_iF_i$ are stable, $i = 1, 2$), and (2.7) can be rewritten as*

$$\Pi_\Sigma(s) = \Psi_2^*(s)\Psi_1(s). \tag{2.8}$$

   Here $\Psi_i := V_iS_{F_i}$, $i = 1, 2$, with $R = V_2^TV_1$ (V_1, V_2 invertible), are also units in $RH^\infty_{m\times m}$. In other words, the Popov function Π_Σ is anti-analytic factorizable.

Definition 2.3. *Consider the $\mathcal{P}$-system $\Sigma = (A_1, B_1; A_2^T, B_2^T; Q, L_1, L_2^T, R)$ and assume that R is invertible. From the second line in the matrix equality (2.5), one obtains*

$$F_1 = -R^{-1}(B_2^TX + L_2^T). \tag{2.9}$$

By substituting now (2.9) into the first line of (2.5), one obtains the **non-symmetric algebraic Riccati equation** *associated with Σ, NARE(Σ),*

$$A_2^TX + XA_1 - (XB_1 + L_1)R^{-1}(B_2^TX + L_2^T) + Q = 0 \tag{2.10}$$

A solution X to (2.10) is said to be a right stabilizing *solution if $A_1 + B_1F_1$ is stable for F_1 given by (2.9).*

Remark 2.2.

1. *Assume that R in (2.6) is nonsingular. Clearly, X is a right stabilizing solution to the NARE(Σ) (2.10) iff (X, F_1) is a right stabilizing solution to the NARS(Σ) (2.5), with F_1 given by (2.9).*

2. *The NARE(Σ) can be alternatively obtained by eliminating F_2^T from the second line in the matrix equality (2.6),*

$$F_2^T = -(XB_1 + L_1)R^{-1}. \tag{2.11}$$

If $A_2 + B_2F_2$ is stable for F_2 given by (2.11), then X is called a left stabilizing *solution to the NARE(Σ). Equivalently, X^T is a right stabilizing solution of the transposed Riccati equation*

$$A_1^T X^T + X^T A_2 - (L_2 + X^T B_2) R^{-T} (L_1^T + B_1^T X^T) + Q^T = 0 \tag{2.12}$$

3. *If X_1 and X_2 are, respectively, a right and a left stabilizing solution of (2.10), then, according to Proposition 2.1 $X_1 = X_2 =: X$. In this case, X will be called simply a* stabilizing solution *to the NARE(Σ).*

3 Main results

We begin by introducing some concepts from the theory of linear operators. Denote by $L^{2,m}$ the space of m-dimensional square-integrable real-valued functions on $\mathbb{R}$. Let $L_-^{2,m}$ and $L_+^{2,m}$ be the closed subspaces of $L^{2,m}$, containing functions with support in $(-\infty, 0]$ and $[0, \infty)$, respectively. Furthermore, we define $P_\pm^r$ as the orthogonal projection of $L^{2,r}$ onto $L_\pm^{2,r}$. Let $\mathcal{G}$ be a bounded linear operator from $L^{2,m}$ to $L^{2,p}$. Let $\mathcal{G}^*$ stand for the adjoint of $\mathcal{G}$ and define the (causal) *Toeplitz operator* associated with $\mathcal{G}$ as $\mathbb{T}_\mathcal{G} := P_+^p \mathcal{G} P_+^m$. It is not difficult to check that $\mathbb{T}_\mathcal{G}^* = \mathbb{T}_{\mathcal{G}^*}$.

Let us briefly recall several known facts concerning L^2-evolutions of linear systems, applied in particular to $\mathcal{P}$-systems defined by (2.1a-2.1c). Henceforth we assume that both A_1 and A_2 are *stable*. For any $\xi \in \mathbb{R}^n$ and $u \in L_+^{2,m}$ the solution of the differential equation (2.1a) can be written as:

$$x = \Phi_1 \xi + \mathcal{L}_1 u,$$

where

$$\begin{aligned} \Phi_1 : \mathbb{R}^n \to L_+^{2,n} \qquad (\Phi_1 \xi)(t) &:= e^{A_1 t} \xi, \quad t \geq 0; \\ \mathcal{L}_1 : L_+^{2,m} \to L_+^{2,n} \qquad (\mathcal{L}_1 u)(t) &:= \int_0^t e^{A_1 (t-\tau)} B_1 u(\tau) d\tau, \quad t \geq 0, \end{aligned} \tag{3.13}$$

are both linear bounded operators. Similarly, one can define Φ_2 and $\mathcal{L}_2$ for the linear differential system $\dot{\tilde{x}} = A_2 \tilde{x} + B_2 \tilde{u}$. Under these circumstances, the adjoints of Φ_1, $\mathcal{L}_1$ and Φ_2, $\mathcal{L}_2$ are all well-defined bounded operators, and we give exemplarily those with index 2:

$$\Phi_2^* : L_+^{2,l} \to \mathbb{R}^l \qquad \Phi_2^* \lambda = \int_0^\infty e^{A_2^T t} \lambda(t) dt \;; \tag{3.14}$$

$$\mathcal{L}_2^* : L_+^{2,l} \to L_+^{2,m} \qquad (\mathcal{L}_2^* \lambda)(t) = \int_t^\infty B_2^T e^{-A_2^T (t-\tau)} \lambda(\tau) d\tau, \quad t \geq 0. \tag{3.15}$$

The next lemma establishes a connection between the above introduced operators and the L^2-evolutions of the system (2.1a-2.1c).

Lemma 3.1. *For any $\xi \in \mathbb{R}^n$ and $u \in L_+^{2,m}$ there exists a unique $L_+^{2,n} \times L_+^{2,l}$ state trajectory (x, λ) of the system (2.1a-2.1c) such that $x(0) = \xi$. This trajectory will be denoted by $(x^{\xi,u}, \lambda^{\xi,u})$. Moreover, the corresponding output (2.1c) is given by*

$$\nu^{\xi,u} = \mathcal{R}_\Sigma u + \mathcal{S}_{12,\Sigma}^* \xi, \tag{3.16}$$

where

$$\mathcal{R}_\Sigma := R + L_2^T \mathcal{L}_1 + \mathcal{L}_2^* L_1 + \mathcal{L}_2^* Q \mathcal{L}_1 \tag{3.17}$$

and $\mathcal{S}_{12,\Sigma}^* := (\mathcal{L}_2^* Q + L_2^T)\Phi_1$. Clearly, $\nu^{\xi,u}$ belongs to $L_+^{2,m}$.

Remark 3.1. *Since A_1 and A_2 are both stable, it follows that $\mathbf{A} := \begin{pmatrix} A_1 & 0 \\ -Q & -A_2^T \end{pmatrix}$ has no eigenvalues on the $j\omega$-axis. Hence $\mathbf{A}$ defines an exponentially dichotomic evolution. According to Theorem 1.1 in [3], for each $u \in L^{2,m}$, the system (2.1a-2.1b) has a unique solution (x^u, λ^u) in $L^{2,n} \times L^{2,l}$. In this case, one can also associate with the $\mathcal{P}$–system an L^2 input-output operator $\mathcal{R}_{\Sigma,e} : L^{2,m} \to L^{2,m}$, $\nu = \mathcal{R}_{\Sigma,e}\, u$, which is given by*

$$\mathcal{R}_{\Sigma,e} = R + L_2^T \mathcal{L}_{1,e} + \mathcal{L}_{2,e}^* L_1 + \mathcal{L}_{2,e}^* Q \mathcal{L}_{1,e}.$$

Here $(\mathcal{L}_{i,e} u)(t) := \int_{-\infty}^{t} e^{A_i(t-\tau)} B_i u(\tau) d\tau$ for any $u \in L^{2,m}$ and $i = 1, 2$. One can also check that $\mathcal{L}_i = \mathbb{T}_{\mathcal{L}_{i,e}}$ and consequently $\mathcal{R}_{\Sigma,e}$ is the (causal) Toeplitz operator *associated with $\mathcal{R}_{\Sigma,e}$. For more details on these aspects, see [6].*

Now we can state the key result of this section.

Theorem 3.1. *Let Σ be a $\mathcal{P}$-system where A_1 and A_2 are stable. If the Toeplitz operator $\mathcal{R}_\Sigma$ has a bounded inverse then R is nonsingular and the NARE(Σ) has a right stabilizing solution.*

The proof follows the main steps in the proof of Theorem 4.1.1 in [6], involving several lemmata.
Comparing Theorem 3.1 with Theorem 4.1.1 in [6], let us emphasize that the uniqueness of the stabilizing solution is not obtained in the non-symmetric case. Moreover, we state at this stage only a sufficient condition for the existence of the stabilizing solution. So far, the proofs involved the feedback matrix F_1 only. Now, we will also employ more of the structure, in order to use information provided by the feedback matrix F_2. This is done by considering the dual of the $\mathcal{P}$-system Σ.

Corollary 3.1. *Let the $\mathcal{P}$-system Σ be given such that A_1 and A_2 are stable. If the Toeplitz operator $\mathcal{R}_\Sigma$ has a bounded inverse then R is nonsingular and the NARE(Σ) has a stabilizing solution X.*

Proof. By Theorem 3.1 a right stabilizing solution X_1 of the NARE(Σ) exists. In view of points 2. and 3. in Remark 2.2, it suffices to show that the transposed Riccati equation (2.12) has a right stabilizing solution X_2^T. For, let us notice first that the adjoint of $\mathcal{R}_{\Sigma,e}$ is precisely the L^2 input-output operator defined by the dual $\mathcal{P}$-system of Σ, denoted by Σ_* (see Remark 2.1). Accordingly, $\mathcal{R}_{\Sigma,e}^* = \mathcal{R}_{\Sigma_*,e}$ and

$$\mathcal{R}_{\Sigma_*} = \mathbb{T}_{\mathcal{R}_{\Sigma_*,e}} = \mathbb{T}_{\mathcal{R}_{\Sigma,e}^*} = \mathbb{T}_{\mathcal{R}_{\Sigma,e}}^* = \mathcal{R}_{\Sigma}^*.$$

Since $\mathcal{R}_\Sigma$ is invertible, it follows from the above equality that the causal Toeplitz operator associated with the dual system Σ_*, that is, $\mathcal{R}_{\Sigma_*}$, is invertible as well. Invoking now Theorem 3.1 for $\mathcal{R}_{\Sigma_*}$ and the $\mathcal{P}$-system Σ_*, one deduces that the transposed Riccati equation (2.12) has a right stabilizing solution X_2^T and the proof is finished. □

The main result of the paper is given below.

Theorem 3.2. *Let a $\mathcal{P}$-system Σ be given such that A_1, A_2 are both stable. Then the following statements are equivalent:*

1. *The operator $\mathcal{R}_\Sigma$ has a bounded inverse.*
2. *R is nonsingular and the NARE(Σ) has a stabilizing solution X.*
3. *The non-symmetric Popov function Π_Σ is anti-analytic factorizable.*

Proof. The implication 1. ⇒ 2. is precisely Corollary 3.1. To conclude 2. ⇒ 3., we invoke point 2. in Proposition 2.2, which shows that (2.8) is an anti-analytic factorization of Π_Σ. In order to prove 3. ⇒ 1., let $\widehat{\mathcal{R}}_\Sigma = \mathbb{T}_{\Pi_\Sigma}$ be the *frequency-domain* Toeplitz operator with symbol the Popov function Π_Σ (see, for more details, [5]). If $\Pi_\Sigma = \Psi^*\Omega$ is an anti-analytic factorization of the Popov function, then $\widehat{\mathcal{R}}_\Sigma = \mathbb{T}_{\Pi_\Sigma} = \mathbb{T}_\Psi^*\mathbb{T}_\Omega$. Since Ψ, Ω are units in $RH^\infty_{m\times m}$, one has $\mathbb{T}_\Omega^{-1} = \mathbb{T}_{\Omega^{-1}}$ and $\mathbb{T}_{\Psi^*}^{-1} = \mathbb{T}_{\Psi^{-*}}$, and hence $\widehat{\mathcal{R}}_\Sigma$ must be invertible. As $\mathcal{R}_\Sigma$ and $\widehat{\mathcal{R}}_\Sigma$ are unitarily equivalent, the conclusion follows. □

4 Application to open-loop linear quadratic Nash games

The non-symmetric Riccati theory is now applied to a certain type of Nash games. We consider two player differential games on the infinite time horizon having linear dynamics of the form

$$\dot{x} = Ax + B^1u_1 + B^2u_2, \quad x(0) = \xi, \tag{4.18}$$

where the matrices $A \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times m_i}$ $(i = 1, 2)$ are constant and $n, m_1, m_2 \in \mathbb{N}$. The matrix A is assumed to be *stable.* The cost-functionals are of quadratic type

$$\begin{aligned} J_1(u_1, u_2) &= \int_0^\infty w_1^T(t)P_1w_1(t)dt, \\ J_2(u_1, u_2) &= \int_0^\infty w_2^T(t)P_2w_2(t)dt, \end{aligned} \tag{4.19}$$

where $w_i(t) := \begin{pmatrix} x(t) \\ u_i(t) \\ u_j(t) \end{pmatrix}$ $(i = 1, 2; j := 3 - i)$ and where $x(t)$ satisfies (4.18) for $u_1(t)$ and $u_2(t)$ and ξ. The weighting matrix P_i is defined by

$$P_i := \begin{pmatrix} Q_i & L^i & M^i \\ L^{i,T} & R_{ii} & N^i \\ M^{i,T} & N^{i,T} & R_{ij} \end{pmatrix} = P_i^T,$$

with $Q_i = Q_i^T \in \mathbb{R}^{n \times n}, L^i \in \mathbb{R}^{n \times m_i}, M^i \in \mathbb{R}^{n \times m_j}, R_{ii} = R_{ii}^T \in \mathbb{R}^{m_i \times m_i}, N^i \in \mathbb{R}^{m_i \times m_j}, R_{ij} = R_{ij}^T \in \mathbb{R}^{m_j \times m_j}$, and $P_i \in \mathbb{R}^{(n+m_i+m_j) \times (n+m_i+m_j)}$.
For $L^i = 0, M^i = 0, N^i = 0$ $(i = 1, 2)$ this is the purely quadratic cost case, which is mostly considered in the literature. As equilibrium concept we take the Nash concept:

Definition 4.1. *The pair (u_1^*, u_2^*) is called a Nash (equilibrium) strategy if*

$$\begin{aligned} J_1(u_1^*, u_2^*) &\leq J_1(u_1, u_2^*) \quad \textit{and} \\ J_2(u_1^*, u_2^*) &\leq J_2(u_1^*, u_2) \end{aligned}$$

for all admissible strategies u_1, u_2.

Since A is stable any pair (u_1, u_2) with $u_i \in L_+^{2,m_i}$ is admissible. Furthermore, open-loop information structure is assumed, which means that no measurements during the game are available and the players only know the initial state ξ.
In [8], a Hilbert space method is used in order to obtain a necessary and sufficient conditions for the existence of a unique Nash equilibrium. Unless it is stated therein for the finite time horizon the method also works in the infinite time horizon case. Furthermore, the results in [8] are derived under the convexity assumption, i.e. for $R_{ii} > 0$ and $Q_i \geq 0$ for $i = 1, 2$. The key object in [8] is the so-called decision operator. Define

$$\mathcal{B}_i : L_+^{2,m_i} \to L_+^{2,n}, \qquad (\mathcal{B}_i u)(t) \ := \ \int_0^t \mathrm{e}^{A(t-\tau)} B^i u(\tau) d\tau, \quad t \geq 0, i = 1, 2.$$

With that the *decision operator* $\mathcal{D} : L_+^{2,m_1} \otimes L_+^{2,m_2} \to L_+^{2,m_1} \otimes L_+^{2,m_2}$ is given as:

$$\mathcal{D} := \begin{pmatrix} \mathcal{B}_1^* Q_1 \mathcal{B}_1 + \mathcal{B}_1^* L^1 + L^{1,T} \mathcal{B}_1 + R_{11} & \mathcal{B}_1^* Q_1 \mathcal{B}_2 + L^{1,T} \mathcal{B}_2 + N^1 \\ \mathcal{B}_2^* Q_2 \mathcal{B}_1 + L^{2,T} \mathcal{B}_1 + N^2 & \mathcal{B}_2^* Q_2 \mathcal{B}_2 + \mathcal{B}_2^* L^2 + L^{2,T} \mathcal{B}_2 + R_{22} \end{pmatrix}. \tag{4.20}$$

Under the convexity condition a game has a unique Nash equilibrium iff the decision operator is invertible. In [8] such a game is called playable. Note, that situations are known in which the decision operator fails to be invertible and a unique Nash equilibrium for the game exists. The same consequence has the following extension of the notion of playability for the existence of unique Nash equilibria:

Definition 4.2. *An open-loop Nash game is called playable if and only if the following conditions both hold:*

1. The decision operator has a bounded inverse.

2. $\mathcal{B}_i^* Q_i \mathcal{B}_i + \mathcal{B}_i^* L^i + L^{i,T}\mathcal{B}_i + R_{ii} > 0$ *for* $i = 1, 2$.

We introduce the following $\mathcal{P}$-systems. With the definitions $m := m_1 + m_2, l := 2n$ we choose and partition the matrices of Σ as follows.

$$A_1 := A \in \mathbb{R}^{n\times n} \qquad B_1 := \begin{pmatrix} B^1 & B^2 \end{pmatrix} \in \mathbb{R}^{n\times(m_1+m_2)};$$

$$A_2 := \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \in \mathbb{R}^{2n\times 2n} \qquad B_2 := \begin{pmatrix} B^1 & 0 \\ 0 & B^2 \end{pmatrix} \in \mathbb{R}^{2n\times(m_1+m_2)};$$

$$Q := \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \in \mathbb{R}^{2n\times n}, R := \begin{pmatrix} R_{11} & N^1 \\ N^2 & R_{22} \end{pmatrix} \in \mathbb{R}^{(m_1+m_2)\times(m_1+m_2)}$$

$$L_1 := \begin{pmatrix} L^1 & M^1 \\ M^2 & L^2 \end{pmatrix} \quad L_2^T = \begin{pmatrix} L^{1,T} \\ L^{2,T} \end{pmatrix}.$$

With these settings the $\mathcal{P}$–system Σ coincides with that derived by the application of variational principles to the problem of finding an equilibrium point (see Chapter 6.5.1 in [2]). Moreover, we need the subsystems Σ^1 and Σ^2, which are Popov triplets in the classical sense (cf. [6]):

$$\Sigma^i = (A, B^i, Q_i, L^i, R_{ii}), \qquad i = 1, 2.$$

By the theory of $\mathcal{P}$-systems and inspection we get:

Lemma 4.1. *For an open-loop Nash game and the* $\mathcal{P}$*–systems* Σ, Σ^1 *and* Σ^2 *the following identities are valid:*

1. $\mathcal{R}_\Sigma = \mathcal{D}$

2. $\mathcal{R}_{\Sigma^i} = \mathcal{B}_i^* Q_i \mathcal{B}_i + \mathcal{B}_i^* L^i + L^{i,T}\mathcal{B}_i + R_{ii}$ *for* $i = 1, 2$.

With the preceding Lemma and Theorem 3.1 we easily find a characterization for the playability of a Nash game on the infinite time horizon.

Theorem 4.1. *An open-loop Nash game is playable if and only if the following conditions are all satisfied:*

1. R *is invertible.*

2. The algebraic open-loop Nash Riccati equation

$$A_2^T X + X A_1 - (L_1 + X B_1) R^{-1} (L_2^T + B_2^T X) + Q = 0 \tag{4.21}$$

has a (unique) stabilizing solution X.

3. *The matrices* $R_{ii} > 0$ *for* $i = 1, 2$.

4. *The optimal control* symmetric *algebraic Riccati equations associated with* Σ^i

$$A^T X_i + X_i A - (X_i B^i + L^i) R_{ii}^{-1} (B^{i,T} X_i + L^{i,T}) + Q_i = 0,$$

have stabilizing solutions X_i, $i = 1, 2$.

Proof. We apply Theorem 3.1 to the first condition in Definition 4.2. Hence, we get that R is invertible and a short calculation shows that (4.21) has a stabilizing solution. The argument also works vice versa.
Conditions 3. and 4. follow by combining point 2. in Lemma 4.1 with the positivity theory in [6]. □

For the case that Theorem 4.1 guarantees the playability of the game the unique Nash equilibrium strategy (u_1^*, u_2^*) is explicitly given in **feedback form** as:

$$\begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} = -R^{-1} B_2^T \tilde{x}, \qquad i = 1, 2,$$

with $\tilde{x}$ being the solution of $\dot{\tilde{x}} = (A - B_1 R^{-1} B_2^T)\tilde{x}, \quad \tilde{x}(0) = \xi$. A weaker version of Theorem 4.1 was proved in [7].

5 Conclusion

The Riccati equation (4.21) is well known in the purely quadratic case, where it rewrites as:

$$0 = -\begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} - \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} A - \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} + \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{pmatrix} B^{1,T} R_{11}^{-1} B^1 & B^{2,T} R_{22}^{-1} B^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

In [4] it was shown under the convexity condition that right stabilizing solutions of this equation are related to Nash equilibria. Our approach clarifies the essential role of a stabilizing solution for the playability of Nash games. It should also be emphasized that the results rely by no means on the convexity condition as it is the case in the known literature. This enlarges the class of games, for which playability statements can be deduced.

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