Linear Hamiltonian systems

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Abstract

We study linear Hamiltonian systems using bilinear and quadratic differential forms. Such a representation-free approach allows to use the same concepts and techniques to deal with systems isolated from their environment and with systems subject to external influences, and allows to study systems described by higher-order differential equations, thus dispensing with the usual point of view in classical mechanics of considering first and second-order differential equations only.

1 Introduction

This communication aims to give a unified and general treatment of linear Hamiltonian systems using the formalism of quadratic differential forms introduced in [11]. We consider both autonomous systems, i.e. systems without external influences, and non-autonomous ones, in which external inputs are present, and we deal with both cases using the same techniques and the same concepts. We conduct our investigation in a representation-free way, thus dispensing with the usual point of view in mechanics and in physics of concentrating on first order representations in the (generalized) coordinates and the (generalized) momenta. Such representation-free approach allows to describe systems of different nature using the same formalism, independent of the domain of application, and it is especially relevant in view of the potential application of our techniques in the description of (possibly infinite-dimensional) non-mechanical systems, for example those arising in the theory of fields.

Instead of *postulating* the existence of a function (the Lagrangian, or the Hamiltonian) on the basis of physical considerations (conservation of energy, etc.) and deducing from it the equations of motion, we proceed by assuming that a set of linear differential equations with constant coefficients describing the system is given, and we *deduce* the Hamiltonian nature of the system from such equations, by proving the existence of certain bilinear functionals of the variables of the system and of their derivatives that satisfy a "skew-symmetry" and a

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"nondegeneracy" property. Proceeding directly from the equations of motion allows to study Hamiltonianity also for complex systems (for example those resulting from the interconnection of many simple subsystems), for which the identification of functionals representing the "conserved quantities" is not immediate.

In our approach to Hamiltonian systems, the concept of internal force also arises naturally from the equations describing the system: in this paper we show that generalized internal forces can be defined which depend on potential functions involving higher-order derivatives and not only first order ones at most, as happens with "velocity-dependent potentials" in classical mechanics.

The communication is organized as follows: in section 2 we define Hamiltonianity for autonomous systems. In section 3 we consider the notion of internal forces, which we propose to see as latent variables arising naturally from the equations describing an autonomous Hamiltonian system. The relationship between internal forces and external variables in an autonomous Hamiltonian system forms the basis for the definition of controllable Hamiltonian system proposed in section 4. In section 5 we discuss our results and outline some directions for future research. In this paper we assume that the reader has a solid background knowledge of behavioral system theory and of quadratic differential forms; we refer to [8] and [11] for a thorough exposition of such subjects.

A few words on notation. In this paper we denote the sets of real numbers respectively with $\mathbb{R}$. The space of n dimensional real vectors is denoted by $\mathbb{R}^n$, and the space of $m \times n$ real matrices by $\mathbb{R}^{m\times n}$. Whenever one of the two dimensions is not specified, a bullet $\bullet$ is used; so that for example, $\mathbb{R}^{\bullet\times n}$ denotes the set of complex matrices with n columns and an unspecified number of rows.

Given two column vectors x and y, we denote with $\operatorname{col}(x, y)$ the vector obtained by stacking x over y; a similar convention holds for the stacking of matrices with the same number of columns. If $A \in \mathbb{R}^{m\times n}$, then $A^T \in \mathbb{R}^{n\times m}$ denotes its transpose. Π_x denotes the projection map on the variable x: $\Pi_x(\operatorname{col}(x, y)) = x$.

The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the ring of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta, \eta]$. The space of all $n \times m$ polynomial matrices in the indeterminate ξ is denoted by $\mathbb{R}^{n\times m}[\xi]$, and that consisting of all $n \times m$ polynomial matrices in the indeterminates ζ and η by $\mathbb{R}^{n\times m}[\zeta, \eta]$. Given a matrix $R \in \mathbb{R}^{n\times m}[\xi]$, we define $R^{\sim}(\xi) := R(-\xi)^T \in \mathbb{R}^{m\times n}[\xi]$.

We denote with $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^q$, and with $\mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$ the subset of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ consisting of compact support functions.

2 Autonomous Hamiltonian systems

In this section we study Hamiltonianity for linear autonomous systems, i.e. systems with no inputs, on which no external influence is exerted. We define Hamiltonianity as a property arising from the interplay of a skew-symmetric bilinear form and the behavior. Then we prove the main result of this section, Theorem 2.1, in which a number of equivalent characterizations of Hamiltonianity are given.

In order to state our definition of autonomous Hamiltonian system, we recall the concept of nondegeneracy and skew-symmetry of a bilinear differential form. Let $\Phi \in \mathbb{R}^{q\times q}[\zeta,\eta]$ and $\mathfrak{B} \in \mathfrak{L}^q$. Then L_Ψ is *skew-symmetric* if $L_\Psi(w_1,w_2) = -L_\Psi(w_2,w_1)$ for all w_1, w_2. The concept of nondegeneracy is introduced as follows: observe that the BLDF L_Φ induces a bilinear form on the vector space $\mathfrak{B}$ by assigning to $(v,w) \in \mathfrak{B}\times\mathfrak{B}$ the real number $L_\Phi(v,w)(0)$. We denote such bilinear form by $L_{\Phi|\mathfrak{B}}$; observe that the rank and the nondegeneracy of such induced bilinear form are well-defined. In particular, $L_{\Phi|\mathfrak{B}}$ is *nondegenerate* if for all $w \in \mathfrak{B}$ we have $L_\Phi(\mathfrak{B},w)(0) = 0 \Leftrightarrow w = 0$.

The definition of autonomous Hamiltonian system is as follows.

Definition 2.1. *Let $\mathfrak{B} \in \mathcal{L}^q$ be autonomous. $\mathfrak{B}$ is Hamiltonian if there exists a bilinear differential form L_Ψ, such that*

(i) $\frac{d}{dt}L_\Psi(w_1,w_2) = 0$ *for all w_1, $w_2 \in \mathfrak{B}$;*

(ii) L_Ψ *is skew-symmetric;*

(iii) $L_{\Psi|\mathfrak{B}}$ *is a nondegenerate bilinear form.*

In Definition 2.1 no assumption on the number q of external variables of $\mathfrak{B}$ is made, and consequently also systems with an odd number of external variables can qualify for Hamiltonianity. This point of view is in contrast with the usual definition of autonomous Hamiltonian system, in which a symplectic structure on the space of the external variables (and consequently, an even number of such variables) is assumed (see for example [1], [2]). The authors believe that in order to investigate linear, finite-dimensional Hamiltonian systems, Definition 2.1 is a more natural starting point than the classical one. In order to support our claim, we consider the following example.

Example 2.1. Consider a spring-mass system without friction, with the position of the mass described by the equation

$$m\frac{d^2w}{dt^2} + kw = 0$$

Define $\Psi(\zeta,\eta) = m(\zeta-\eta)$ and consider the BLDF induced by such two-variable polynomial. Such BLDF is a skew-symmetric bilinear function of the state of $\mathfrak{B}$, namely $(w,\dot{w})$, which is nondegenerate and constant along its trajectories. Indeed, the coefficient matrix (see [11]) of $\Psi(\zeta,\eta)$ is

$$\tilde{\Psi} = \begin{pmatrix} 0 & -m & 0 & \cdots \\ m & 0 & 0 & \cdots \\ 0 & 0 & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

and has rank 2, equal to the McMillan degree of the system; this is sufficient to conclude that $L_{\Psi|\mathfrak{B}}$ is nondegenerate (see [9]). Moreover, for every w_1, w_2 satisfying the equations of the system $L_\Psi(w_1, w_2) = m\dot{w}_1 w_2 - m\dot{w}_2 w_1$ and consequently $\frac{d}{dt} L_\Psi(w_1, w_2) = 0$. It follows that this spring-mass system with only one external variable is Hamiltonian according to Definition 2.1. It is difficult to understand why in order to study the Hamiltonianity of such system, where the position of the mass is the only external variable, one should first transform the natural second-order differential equation description, coming up with a first order representation in which the position *and* the momentum are the external variables; and then study the symplectic structure of the resulting state-space system.

The previous example illustrates one situation in which a representation-free definition of Hamiltonianity appears to be more natural than the classical one. The argument for a general definition of Hamiltonianity, independent of the particular representation adopted for a system, becomes even stronger when considering the very frequent occurrence in applications, of dynamical systems described by sets of higher order differential equations, for example because of the elimination of auxiliary variables.

The following theorem consists of a series of conditions on $\mathfrak{B}$ and on its representations, equivalent to Hamiltonianity; we omit the proof, which will be given elsewhere.

Theorem 2.1. *Let $\mathfrak{B} \in \mathcal{L}^q$ be autonomous. Then the following conditions are equivalent:*

1. *$\mathfrak{B}$ is Hamiltonian;*

2. *There exists a kernel representation $\mathfrak{B} = ker(R(\frac{d}{dt}))$ induced by a nonsingular $R \in \mathbb{R}^{q \times q}[\xi]$, such that any invariant polynomial of R is either even, or it is odd; moreover, the odd invariant polynomials can be divided in pairs, so that the odd polynomials of each pair have the root zero with the same multiplicity;*

3. *For any $R \in \mathbb{R}^{\bullet \times q}[\xi]$ such that $\mathfrak{B} = \ker(R(\frac{d}{dt}))$, we have rank$(R) = q$ and any invariant polynomial of R is either even, or it is odd; moreover, the odd invariant polynomials can be divided in pairs, so that the odd polynomials of each pair have the root zero with the same multiplicity;*

4. *There exists a minimal state space representation $\dot{x} = Ax$, $w = Cx$ of $\mathfrak{B}$ and a nonsingular skew-symmetric matrix $K \in \mathbb{R}^{n(\mathfrak{B}) \times n(\mathfrak{B})}$ such that $A^T K + KA = 0$;*

5. *For any minimal state representation $\dot{x} = Ax$, $w = Cx$ of $\mathfrak{B}$ there exists a nonsingular skew-symmetric matrix $K \in \mathbb{R}^{n(\mathfrak{B}) \times n(\mathfrak{B})}$ such that $A^T K + KA = 0$.*

Observe that from Theorem 2.1 it follows that any Hamiltonian system has an even McMillan degree. Another consequence of Theorem 2.1 is that the state matrix A of any minimal state-space representation of an autonomous Hamiltonian system is similar to $-A^T$, with the similarity induced by a skew-symmetric matrix. Such matrices are called *Hamiltonian*, and a thorough investigation of their properties is given in [5] (see also [3] and [4]).

We conclude this section with an example of autonomous Hamiltonian system.

Example 2.2. Consider two masses m_1 and m_2 attached to springs with constants k_1 and k_2. The first mass is interconnected with the second one via the first spring, and the second mass is connected to a "wall" with the second spring. Considering w_1 and w_2 as external variables, we can write down the equation of the system as

$$\left(\begin{pmatrix} m_1 \frac{d^2}{dt^2} & 0 \\ 0 & m_2 \frac{d^2}{dt^2} \end{pmatrix} + \begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{pmatrix} \right) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$$

By eliminating w_2 from the equations, we take the position w of the first mass as external variable. The resulting equation is

$$r(\frac{d^2}{dt^2})w = m_1 m_2 \frac{d^4}{dt^4} w + (k_1 m_1 + k_2 m_1 + k_1 m_2) \frac{d^2}{dt^2} w + k_1 k_2 w = 0$$

Such equation describes an Hamiltonian system, since $r(\xi)$ is even, as it can be seen by applying statement 2 of Theorem 2.1.

3 Internal forces

Autonomous systems have no external influence exerted on them by the environment: in mechanics, they correspond to isolated systems on which external forces do not act. When considering systems of particles, there are forces (for example, gravitational ones) acting on some particles of the system which are due to all other particles of the system; in classical mechanics they are called *internal* forces, because they arise from some potential which is usually considered to be a function of the configuration variables and of their velocities.

In this section we investigate how internal forces can be accommodated into the framework we are developing for the analysis of Hamiltonian systems. To begin with, we investigate special latent-variable representations of Hamiltonian systems. In order to keep the exposition simple, in the rest of this section we consider Hamiltonian systems with only even invariant polynomials; the general case will be investigated elsewhere. The first result we present concerns the existence of second-order latent variable representations for higher-order Hamiltonian systems; keeping up with the traditional notation, we will denote somewhat ambiguously the latent variable of such representations with the symbol "q".

Proposition 3.1. *Let $\mathfrak{B}$ be an autonomous behavior with q external variables. Let $R(\frac{d}{dt})w = 0$ be a minimal kernel representation of $\mathfrak{B}$, with q even invariant polynomials λ_i, $i = 1, \ldots, q$. Define $n := \frac{\sum_{i=1}^{q} \deg(\lambda_i)}{2}$.*

Then $\mathfrak{B}$ is Hamiltonian if and only if there exist $C_i \in \mathbb{R}^{q \times n}$, $i = 1, 2$ and symmetric matrices $M = M^T, K = K^T \in \mathbb{R}^{n \times n}$ with M nonsingular, such that

$$\begin{aligned} M \frac{d^2}{dt^2} q + K q &= 0 \\ C_1 q + C_2 \frac{d}{dt} q &= w \\ q &= Q(\frac{d}{dt}) w \end{aligned} \tag{3.1}$$

is an hybrid representation $\mathfrak{B}_f$ of $\mathfrak{B}$ with latent variable $q \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^n)$.

In classical mechanics, the variable q consists of the position of the masses, and the first block of equations (3.1) is obtained by writing down Newton's second law for each of the masses. Proposition 3.1 generalizes such procedure, showing that an Hamiltonian behavior described in kernel form by a matrix with only even polynomials, can be *always* interpreted as a *mechanical* system, with q a "generalized position" and $\dot{q}$ a "generalized velocity", M the mass matrix and K the matrix of elastic constants. In the following result we pursue such analogy one step further, relating the second-order representation (3.1) to the existence of an "energy" and a "Lagrangian" function. In order to state such result, we need to introduce two concepts, that of stationarity of a trajectory with respect to a QDF, and that of trimness of a latent variable.

In order to define stationarity, consider two arbitrary time instants t_1 and t_2, with $-\infty < t_1 \leq t_2 < \infty$. Let $w \in \mathfrak{C}^\infty$, and take a compact support variation $\epsilon\delta$ of w, with $\epsilon \in \mathbb{R}$, satisfying the following property: $\frac{d^k}{dt^k}(w + \epsilon\delta)(t_i) = \frac{d^k}{dt^k}(w)(t_i)$, $k \in \mathbb{N} \cup \{0\}$, $i = 1, 2$. We call w *stationary* w.r.t. a QDF Q_Φ if for all variations δ satisfying the above properties, and for all $t_1 \geq t_2$, the cost degradation

$$J_w(\epsilon\delta, t_1, t_2) := \int_{t_1}^{t_2} Q_\Phi(w + \epsilon\delta) - Q_\Phi(w)dt$$

satisfies $\lim_{\epsilon \to 0} \frac{J_w(\epsilon\delta, t_1, t_2)}{\epsilon} = 0$ in other words, if $J_w(\epsilon\delta, t_1, t_2)$ is of order ϵ^2.

The concept of *trimness* of a variable is as follows. Let $\mathfrak{B}$ be a behavior described in hybrid form, with an r-dimensional latent variable q and manifest variable w; we call the latent variable q *trim* if for every $v \in \mathbb{R}^r$ there exists a trajectory $(q, w) \in \mathfrak{B}$, such that $q(0) = v$.

Proposition 3.2. *Let $\mathfrak{B}$ be a system with McMillan degree $n(\mathfrak{B}) = 2r$, represented in hybrid form by the equations (3.1) with M, K $r \times r$ symmetric matrices with M nonsingular. Assume that the latent variable q is trim. Then the following statements are equivalent:*

1. *q satisfies $M\ddot{q} + Kq = 0$;*

2. *q is stationary with respect to the QDF $Q_L(q) := \dot{q}^T M \dot{q} - q^T K q$;*

3. *$\frac{d}{dt}(\dot{q}^T M \dot{q} + q^T K q) = 0$.*

Assume that any of the above statements hold, and let M' and K' be two symmetric $r \times r$ matrices with M' nonsingular. Then q is stationary w.r.t. $Q_{L'}(q) := \dot{q}^T M' \dot{q} - q^T K' q$ if and only if

$$\begin{aligned} M'(M^{-1}K) &= (M^{-1}K)^T M' \\ K' &= M'(M^{-1}K) \end{aligned} \tag{3.2}$$

If we interpret q as a "position" and $\dot{q}$ as a "velocity", then $\dot{q}^T M\dot{q}$ and $q^T K q$ can be interpreted respectively as a "kinetic energy" and a "potential energy". From this point of view, $\dot{q}^T M\dot{q} + q^T K q$ is the "total energy" of the system, while Q_L can be interpreted as the "Lagrangian" of the system.

The following example provides an illustration of the generalizations of classical mechanics concepts introduced until now.

Example 3.1. Consider the system presented in Example 2.2. In order to simplify the notation, define $r_0 := k_1k_2$, $r_2 := k_1m_1 + k_2m_1 + k_1m_2$ and $r_4 := m_1m_2$. A second-order representation (3.1) is obtained with

$$M := \begin{pmatrix} r_2 & r_4 \\ r_4 & 0 \end{pmatrix} \text{ and } K := \begin{pmatrix} r_0 & 0 \\ 0 & -r_4 \end{pmatrix}$$

From Proposition 3.2 it follows that all energy/Lagrangian functions are induced by matrices M' and K' satisfying (3.2). Denote the entries of M with m_{ij}; then, choosing the values of $m_{11} := m_1 + m_2$, $m_{12} := \frac{m_1m_2}{k_1}$, and $m_{22} := \frac{m_1^2 m_2}{k_1^2}$, we obtain M' which satisfies the first equation in (3.2). Such choice yields a matrix K given by the second equation in (3.2), and consequently the QDF

$$\begin{aligned} Q_\Phi(w, \dot{w}, \ddot{w}, w^{(3)}) = \quad & \begin{pmatrix} w & \ddot{w} \end{pmatrix} \begin{pmatrix} k_2 & \frac{k_2m_1}{k_1} \\ \frac{k_2m_1}{k_1} & \frac{m_1^2(k_1+k_2)}{k_1^2} \end{pmatrix} \begin{pmatrix} w \\ \ddot{w} \end{pmatrix} \\ & + \begin{pmatrix} \dot{w} & w^{(3)} \end{pmatrix} \begin{pmatrix} m_1 + m_2 & \frac{m_1m_2}{k_1} \\ \frac{m_1m_2}{k_1} & \frac{m_1^2m_2}{k_1^2} \end{pmatrix} \begin{pmatrix} \dot{w} \\ w^{(3)} \end{pmatrix} \end{aligned}$$

Such QDF measures the physical total energy of the system. Indeed, from the second-order description given above in terms of the two variables w_1 and w_2 it follows that

$$\begin{aligned} E_{\text{kin}}(w_1, w_2) &= \frac{1}{2}(m_1\dot{w}_1{}^2 + m_2\dot{w}_2{}^2) \\ E_{\text{pot}}(w_1, w_2) &= \frac{1}{2}(k_1w_1^2 - 2k_1w_1w_2 + (k_1 + k_2)w_2^2) \end{aligned}$$

Using the fact the generalized positions $\mathrm{col}(w, w^{(2)})$ are related to the actual positions $\mathrm{col}(w_1, w_2)$ by a nonsingular linear map as

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & \frac{m_1}{k_1} \end{pmatrix} \begin{pmatrix} w \\ \ddot{w} \end{pmatrix}$$

it can be readily verified that the sum of such two quantities yields the same value as Q_Φ defined above.

We proceed to define the notion of generalized internal force. In classical mechanics it is customary to take the positions q as external variables; the vector of internal forces is then defined from the potential energy E_{pot} as $f := \frac{\partial}{\partial q} E_{\text{pot}}$; observe that in this way an internal force is paired with each position. Taking our moves from the generalization of position and potential energy given in Proposition 3.1 and Proposition 3.2, it is natural to call the i-th component of $-Kq$ the *generalized internal force acting on the i-th generalized position.* The following Theorem provides the main result of this section, and shows that in an Hamiltonian system an internal force can always be coupled with each external variable. In such manner a hybrid representation of autonomous Hamiltonian systems is obtained in which the internal forces and the external variables are present.

Theorem 3.1. *Let $\mathfrak{B}$ be an autonomous Hamiltonian behavior with q external variables, having only even invariant polynomials. Then there exists a q-dimensional auxiliary variable f, called the* internal force*, and polynomial matrices $R, M, F \in \mathbb{R}^{q\times q}[\xi]$ such that $\mathfrak{B}$ has the hybrid representation*

$$\begin{aligned} R(\frac{d}{dt})w &= M(\frac{d}{dt})f \\ F(\frac{d}{dt})w &= f \end{aligned} \tag{3.3}$$

Moreover, R is nonsingular, R and M are left-coprime, $R^{-1}M$ is a proper rational function satisfying $R^{-1}M = (R^{-1}M)^{\sim}$, and F satisfies $F = F^{\sim}$.

Theorem 3.1 shows that the external behavior of an autonomous Hamiltonian system is the projection on the external variables of the interconnection of two systems, one described by the first block of equations (3.3), and the other represented by the second block of equations (3.3). The number of variables in the interconnection is $2q$: there are as many internal forces as there are external variables w. Observe also that in the system described by the first block of (3.3), the internal forces are input variables, while w is an output; and moreover, that the transfer function $G_{f\to w}$ from f to w satisfies $G_{f\to w} = G^{\sim}_{f\to w}$. It is from such pairing of inputs and outputs, together with the symmetry property of the transfer function between the two, that we take our moves to define Hamiltonianity for the controllable case in section 4. Before considering such issue, we briefly return to Example 3.1 in order to compute one representation (3.3) for such system.

Example 3.2. In Example 3.1 we have computed the “potential energy” as

$$E_{\text{pot}}(w) = k_2 w^2 + 2\frac{k_2 m_1}{k_1} w\ddot{w} + \frac{m_1^2}{k_1^2}(k_1 + k_2)\ddot{w}^2$$

We define the internal force as $f := -(k_2 w + \frac{k_2 m_1}{k_1}\ddot{w})$. Choose $M(\xi) = 1$ and $R(\xi) = \frac{m_1 m_2}{k_1}\xi^4 + (m_1 + m_2)\xi^2$; we obtain the hybrid representation

$$\begin{aligned} \frac{m_1 m_2}{k_1} w^{(4)} + (m_1 + m_2) w^{(2)} &= f \\ -(k_2 w + \frac{m_1 k_2}{k_1} w^{(2)}) &= f \end{aligned}$$

of the external behavior. In order to obtain a physical interpretation of such internal force, observe that since $w + \frac{m_1}{k_1} w^{(2)} = w_2$ (see Example 3.1), then $f = -k_2 w_2$. Such expression describes the reaction force of the second spring, and it corresponds to the sum of the two internal forces $f_1 := -k_1 w_1 + k_1 w_2$ and $f_2 := k_1 w_1 - (k_1 + k_2) w_2$ that we would obtain considering the second-order representation of the system in the external variables w_1 and w_2, provided at the beginning of Example 2.2.

4 Controllable Hamiltonian systems

In the previous section we have shown that an autonomous (i.e. isolated) Hamiltonian system admits an hybrid representation in which each external variable is paired with an auxiliary variable (the "internal force"). In such hybrid representation, the transfer function from the internal forces to the external variables is invariant under transposition and replacing of the indeterminate ξ with $-\xi$. Observe that from the second block of equations (3.3) it follows that such internal forces depend on the external variables, analogously to what happens in classical mechanics, where they originate from a potential function of the configuration variables and their velocities. If we assume that the forces f in the first block of equations (3.3) can be chosen *freely*, then we arrive at the definition of a controllable Hamiltonian system (see also [10], where the same definition is given).

Definition 4.1. *Let $\mathfrak{B} \subseteq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{2m})$ be a controllable linear differential behavior with m input variables, and let $J_{2m} \in \mathbb{R}^{2m \times 2m}$ be the skew-symmetric matrix*

$$J_{2m} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

inducing the bilinear differential form $L_{J_{2m}}$ on $\mathfrak{B} \times \mathfrak{B}$. Then $\mathfrak{B}$ is Hamiltonian if for all trajectories w_1, $w_2 \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^{2m})$ it holds that

$$\int_{-\infty}^{+\infty} L_{J_{2m}}(w_1, w_2) dt = 0$$

In the following, whenever the dimensions of the matrix J_{2m} are evident from the context, we will suppress the subscript. Observe that in a controllable Hamiltonian system the number of input variables is the same as that of the outputs.

We illustrate such definition with an example.

Example 4.1. Take the same system considered in Example 2.2, but with an external force applied to the first mass. Choose as external variables the position q of the first mass and the external force f; then it is easy to see that the set of admissible trajectories $w = (q, f)$ is described by the equation

$$m_1 m_2 \frac{d^4 q}{dt^4} + (m_1 k_1 + m_1 k_2 + m_2 k_1) \frac{d^2 q}{dt^2} + k_1 k_2 q = m_2 \frac{d^2 f}{dt^2} + (k_1 + k_2) f$$

In order for such system to be controllable, the polynomials $d(\xi) := m_1m_2\xi^4 + (m_1k_1 + m_1k_2 + m_2k_1)\xi^2 + k_1k_2$ and $n(\xi) := m_2\xi^2 + k_1 + k_2$ must be coprime. In such case the system also admits an observable image representation induced by the polynomial matrix

$$M(\xi) = \begin{pmatrix} m_2\xi^2 + k_1 + k_2 \\ m_1m_2\xi^4 + (m_2k_1 + m_1k_1 + m_1k_2)\xi^2 + k_1k_2 \end{pmatrix} \tag{4.4}$$

We now show that such system is an Hamiltonian. Observe that for every compact support trajectories $w_i = M(\frac{d}{dt})\ell_i$ $i = 1, 2$ with M as in (4.4), it holds that $L_J(w_1, w_2) = L_\Phi(\ell_1, \ell_2)$, where $\Phi(\zeta, \eta) := M(\zeta)^T J_2 M(\eta)$, that is,

$$\begin{aligned} \Phi(\zeta,\eta) = \quad & (m_2\zeta^2 + k_1 + k_2)(m_1m_2\eta^4 + (m_2k_1 + m_1k_1 + m_1k_2)\eta^2 + k_1k_2) \\ & -(m_1m_2\zeta^4 + (m_2k_1 + m_1k_1 + m_1k_2)\zeta^2 + k_1k_2)(m_2\eta^2 + k_1 + k_2) \end{aligned}$$

In order to prove that $\int_{-\infty}^{+\infty} L_J(w_1, w_2)dt = 0$, observe that $\Phi(-\xi, \xi) = 0$. Conclude from Theorem 3.1 of [11] that there exists $\Psi \in \mathbb{R}[\zeta, \eta]$ such that $\Phi(\zeta, \eta) = (\zeta + \eta)\Psi(\zeta, \eta)$, equivalently, $\frac{d}{dt}L_\Psi = L_\Phi$. Now use the fact that the latent variable trajectories ℓ_i are also compact-support, in order to conclude that $\int_{-\infty}^{+\infty} L_\Phi(\ell_1, \ell_2)dt = L_\Psi(\ell_1, \ell_2)\big|_{-\infty}^{+\infty} = 0$.

The following result consists in a series of conditions on a behavior $\mathfrak{B}$ and on its representations, equivalent to Hamiltonianity; the proof will be given elsewhere.

Theorem 4.1. *Let $\mathfrak{B}$ be a controllable behavior with m input and m output variables. Then the following statements are equivalent:*

1. *$\mathfrak{B}$ is Hamiltonian;*

2. *$\mathfrak{B} \subseteq (J\mathfrak{B})^\perp$;*

3. *Every controllable subbehavior of $\mathfrak{B}$ is Hamiltonian with respect to J;*

4. *For every input/output partition (u, y) of the external variables of $\mathfrak{B}$ there exists a $m \times m$ signature matrix Σ such that the transfer function G associated with such i/o partition satisfies $G^\sim\Sigma = \Sigma G$;*

5. *For every input-output partition (u, y) of w, there exists a signature matrix Σ and a minimal input-state-output representation*

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

of $\mathfrak{B}$ such that

$$\begin{aligned} J_{2n}A &= -A^T J_{2n} \\ \Sigma D &= D^T \Sigma \\ B^T J_{2n} &= -\Sigma C \end{aligned}$$

The characterization of Hamiltonian transfer functions given in statement 4 of Theorem 4.1 is the same given in [3] in the context of input-state-output systems and in [4] in the polynomial model context.

It follows immediately from statement 5 of Theorem 4.1 that the McMillan degree $n(\mathfrak{B})$ of a controllable Hamiltonian behavior is even. Using statement 5 it is also easy to see that the projection on the output variable of the zero-input subbehavior $\mathfrak{B}_z := \{(u, y) \in \mathfrak{B} \mid u = 0\}$ of $\mathfrak{B}$ is an autonomous Hamiltonian system. Indeed, observe that $\Pi_y \mathfrak{B}_z$ is described by the minimal state-space equations $\dot{x} = Ax$, $y = Cx$ and that the matrix A is Hamiltonian; now apply statement 4 of Theorem 2.1 in order to conclude that $\Pi_y \mathfrak{B}_z$ is also Hamiltonian.

We conclude this section with two examples of controllable Hamiltonian systems.

Example 4.2. Newton's second law defines a controllable Hamiltonian system

$$\mathfrak{B} = \{(F, q) \mid F = m\ddot{q}\}$$

as it is easy to verify using for example statement 4 of Theorem 4.1.

Example 4.3. Consider a parallel interconnection of a capacitor C with an inductance L subject to an external current I_e. Assume that we choose as external variables for such system the external current and the magnetic flux ϕ_L in the inductance; it is easy to verify that in such case the system equation is $(\frac{d^2}{dt^2} + \frac{1}{CL})\phi_L - \frac{1}{C} I_e = 0$. We show that this system is Hamiltonian. It can be verified that an observable image representation of the system is induced by the matrix

$$M(\xi) = \begin{pmatrix} 1 \\ C\xi^2 + \frac{1}{L} \end{pmatrix}$$

Consider that $M(\zeta)^T J_2 M(\eta) = (\zeta + \eta)\Psi(\zeta, \eta)$ with $\Psi(\zeta, \eta) := C(\zeta - \eta)$. Consequently such BLDF satisfies $\frac{d}{dt} L_\Psi = L_J$ on $\mathfrak{B}$, and therefore $\int L_{J_2}(w_1, w_2)\mathrm{d}t = 0$ for all w_1, $w_2 \in \mathfrak{B}$. In order to prove the Hamiltonianity of $\mathfrak{B}$, one can alternatively use the fact that the only admissible input/output partition of the variables is with I_e the input and ϕ_L the output, and that the transfer function $G_{I_e \to \phi_L}$ satisfies statement 4 of Theorem 4.1.

5 Conclusions

In this communication we have used the formalism of bilinear and quadratic differential forms in order to study Hamiltonian systems. We have provided a series of conditions equivalent to Hamiltonianity for autonomous systems (Theorem 2.1) and for controllable ones (Theorem 4.1). We have also proposed a definition of generalized total energy, generalized Lagrangian and generalized internal forces for systems described by higher-order differential equations (Theorem 3.1). For lack of space we have been unable to illustrate in this communication the application of the concept of Hamiltonianity presented above to LQ-control problems; this will be done elsewhere.

The approach followed in this paper is representation-free: the definitions and results are not dependent on the existence of a special representation of the system, such as a transfer function or a state-space representation. In view of the encouraging results of the application of quadratic differential forms in the context of infinite-dimensional systems (see [6],[7]), it can therefore be hoped that some of the results presented in this paper can be generalized also to systems described by linear constant-coefficient partial differential equations.

References

[1] Abraham, R. and Marsden, J.E., *Foundations of mechanics*, Benjamin/Cummings, NY, 1978.

[2] Arnold, V.I., *Mathematical methods of classical mechanics*, Springer-Verlag, Berlin, 1978

[3] Brockett, R.W. and Rahimi, A., "Lie algebras and linear differential equations", in *Ordinary Differential Equations*, (L. Weiss, Ed.), Academic Press, NY, 1982.

[4] Fuhrmann, P.A., "On Hamiltonian rational transfer functions", *Linear Algebra and its Applications*, vol. 63, pp. 1-93, 1984.

[5] Laub, A.J. and Meyer, K., "Canonical forms for symplectic and Hamiltonian matrices", *Celestial Mechanics*, vol. 9, pp. 213-238, 1974.

[6] Pillai, H.K. and Shankar, S., "A behavioral approach to control of distributed systems", *SIAM J. Control Opt.*, vol. 37, pp. 388-408, 1999.

[7] Pillai, H.K. and Willems, J.C., "Dissipative distributed systems", submitted to *SIAM J. Control Opt.*, 2000.

[8] Polderman, J.W. and Willems, J.C., *Introduction to mathematical system theory: a behavioral approach*, Springer-Verlag, Berlin, 1997.

[9] Rapisarda, P. and Trentelman, H.L., "Higher order linear differential Hamiltonian systems", Research Report, Dept. Mathematics University of Maastricht. E-mail the first author at `P.Rapisarda@math.unimaas.nl` for a copy.

[10] Schaft, A.J. van der, *System theoretic description of physical systems*, CWI Tract. no. 3, CWI, Amsterdam, 1984.

[11] Trentelman, H.L. and Willems, J.C., "On quadratic differential forms", *SIAM J. Control Opt.*, vol. 36, no. 5, pp. 1703-1749, 1998.