Approximate time-controllability versus time-controllability

Amol J. Sasane
Department of Mathematics
University of Twente
P.O. Box 217, 7500 AE Enschede
The Netherlands
A.J.Sasane@math.utwente.nl

Kanat M. Çamlıbel
Department of Mathematics
University of Groningen
P.O. Box 800, 9700 AV Groningen
The Netherlands
K.Camlibel@math.rug.nl

Abstract

This paper studies the notions of approximate time-controllability and (exact) time-controllability of behaviours. In the 1-D case, we show that these two notions are equivalent and in the n-D case we give an example of a behaviour which is approximately time-controllable, but not time-controllable. Finally, we discuss time-controllability of the heat equation. It turns out that it is time-controllable with respect to the so called Gevrey class of second order.

1 Introduction

Our purpose is to study the two notions of time-controllability (see [7]) of dynamical systems described by linear, constant coefficient ordinary and partial differential equations in the behavioural theory of Willems (see Polderman and Willems [5] for an elementary introduction to the subject). Although the property of (exact) time-controllability is desirable, very often one might be satisfied with an approximate, but reasonably good performance of the system in the future. This gives rise to the notion of approximate time-controllability. In this paper, we show that these two notions coincide in the case of 1-D behaviours. However, this is not the case for multidimensional behaviours in general. We illustrate this by means of an example which is approximately time-controllable, but not time-controllable. Moreover, this example shows that there is a large class of partial differential equations which constitute approximately, but not exactly time-controllable behaviours. Finally, the heat equation will be considered. We show that it is time-controllable with respect to Gevrey class of second order. An "algebraic" if and only if test on the polynomial matrix characterizing the time-controllability property of the corresponding behaviour remains an open problem.

The organization of the paper as follows. Section 2 is devoted to preliminaries. In Section 3 we discuss 1-D systems. This will be followed an example of 2-D behaviour which is approximately time-controllable but not time-controllable. In Section 5, time-controllability of the heat equation will be investigated. For the sake of easy reference, in the last section we have listed the definitions of terms that either do not have a unique universal meaning or are not well-known.

2 Preliminaries

This paper concerns *dynamical systems* $\Sigma = \left(\mathbb{R}^{\mathtt{m}+1}, \mathbb{C}^{\mathtt{w}}, \mathfrak{B}\right)$, where $\mathbb{C}^{\mathtt{w}}$ is called the *signal space* and $\mathfrak{B} \subset \mathcal{C}^{\infty}\left(\mathbb{R}^{\mathtt{m}+1}, \mathbb{C}^{\mathtt{w}}\right)$ is called the *behaviour* of the system Σ (see for example Pillai and Shankar [4]). The behaviour $\mathfrak{B}$ of a dynamical system is said to be *time-controllable* if for any w_1 and w_2

in $\mathfrak{B}$, there exist a $w \in \mathfrak{B}$ and a $\tau \geq 0$ such that

$$w(\bullet, t) = \begin{cases} w_1(\bullet, t) & \text{for all } t \leq 0 \\ w_2(\bullet, t-\tau) & \text{for all } t \geq \tau \end{cases}.$$

w is then said to *concatenate* w_1 and w_2. The behaviour $\mathfrak{B}$ of a dynamical system is said to be *approximately time-controllable* if for any w_1 and w_2 in $\mathfrak{B}$, and any given neighborhood N (in the topology of the topological vector space[1] $\mathcal{C}^\infty((0,\infty), \mathbb{C}^{\mathtt{w}})$ of the future of w_2, there exist a $w \in \mathfrak{B}$ and a $\tau \geq 0$ such that $w(\bullet, t) = w_1(\bullet, t)$ for all $t \leq 0$ and $w(\bullet, t-\tau)|_{t>0} \in N$.

It is easy to see that if a behaviour is time-controllable, then it is approximately time-controllable. We remark that the above definitions of time-controllability and approximate time-controllability could be given more generally for a subspace $\mathcal{W}$ of all distributional solutions satisfying the given set of PDEs. One then insists that for two trajectories in this subspace $\mathcal{W}$, the concatenating trajectory also lies in $\mathcal{W}$, and in the case of approximate time-controllability with respect to $\mathcal{W}$, one uses the natural topology of $\mathcal{W}|_{t>0}$. For example for certain applications, it might be natural to consider approximate time-controllability with respect to the space $\mathcal{W} = \mathcal{C}\left(\mathbb{R}, L_2\left(\mathbb{R}^{\mathtt{m}}\right)^{\mathtt{w}}\right)$. However, for the sake of simplicity, we will restrict ourselves to the definitions with respect to $\mathcal{C}^\infty$ in this paper.

Let us denote the polynomial ring $\mathbb{C}[\eta_1, \dots, \eta_{\mathtt{m}}, \xi]$ by $\mathcal{A}$. Consider the polynomial matrix $R = [r_{\mathtt{kl}}]_{\mathtt{g}\times\mathtt{w}} \in \mathcal{A}^{\mathtt{g}\times\mathtt{w}}$, with each entry in $\mathcal{A}$. The polynomial matrix R gives rise to a map $D_R : \mathcal{C}^\infty\left(\mathbb{R}^{\mathtt{m}+1}, \mathbb{C}^{\mathtt{w}}\right) \to \mathcal{C}^\infty\left(\mathbb{R}^{\mathtt{m}+1}, \mathbb{C}^{\mathtt{g}}\right)$, which is given by:

$$D_R \begin{bmatrix} w_1 \\ \vdots \\ w_{\mathtt{w}} \end{bmatrix} = \begin{bmatrix} \sum_{\mathtt{k}=1}^{\mathtt{w}} r_{1\mathtt{k}}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{\mathtt{m}}}, \frac{\partial}{\partial t}\right) w_{\mathtt{k}} \\ \vdots \\ \sum_{\mathtt{k}=1}^{\mathtt{w}} r_{\mathtt{gk}}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{\mathtt{m}}}, \frac{\partial}{\partial t}\right) w_{\mathtt{k}} \end{bmatrix}.$$

Such maps will be called *differential maps* in the sequel. Given a behaviour, say $\mathfrak{B}$, corresponding to some kernel representation given by a polynomial matrix R_*, define

$$\langle R \rangle_{\mathfrak{B}} = \left\{ r = \begin{bmatrix} r_1 & \dots & r_{\mathtt{w}} \end{bmatrix} \in \mathcal{A}^{\mathtt{w}} \mid D_r(w) = 0 \text{ for all } w \in \mathfrak{B} \right\}.$$

It was shown in Oberst [3] that given any $\langle R \rangle_{\mathfrak{B}} = \langle R_* \rangle$, where $\langle R_* \rangle$ denotes the $\mathcal{A}$-submodule of $\mathcal{A}^{\mathtt{w}}$ generated by the rows of the polynomial matrix R_*.

Let $R \in \mathbb{C}[\eta_1, \dots, \eta_{\mathtt{m}}, \xi]^{\mathtt{g}\times\mathtt{w}}$ and $\mathfrak{B}$ be the corresponding behaviour. Let us consider the following four statements

$A1$. The $\mathbb{C}(\eta_1, \dots, \eta_{\mathtt{m}})[\xi]$-module $\mathbb{C}(\eta_1, \dots, \eta_{\mathtt{m}})[\xi]^{\mathtt{w}} / \mathbb{C}(\eta_1, \dots, \eta_{\mathtt{m}})[\xi]^{\mathtt{g}} R$ is torsion free.

$A2$. $\neg\, [\exists\, \chi \in \mathcal{A}^{\mathtt{w}} \setminus \langle R \rangle$ and $\exists\, (0 \neq)\, p \in \mathcal{A}$ such that $p \cdot \chi \in \langle R \rangle$, and $\deg(p) = \deg(\jmath(p))]$, where $\jmath$ is the homomorphism $p(\xi, \eta_1, \dots, \eta_{\mathtt{m}}) \mapsto p(\xi, 0, \dots, 0) : \mathbb{C}[\xi, \eta_1, \dots, \eta_{\mathtt{m}}] \to \mathbb{C}[\xi]$.

$B1$. $\mathfrak{B}$ is time-controllable.

$B2$. $\mathfrak{B}$ is approximately time-controllable.

[1]The topological vector space $\mathcal{C}^\infty(\Omega, \mathbb{C}^{\mathtt{w}})$, where Ω is an open subset of $\mathbb{R}^{\mathtt{m}}$, is a Fréchet space: it is locally convex, metrizable, Hausdorff and complete (see for instance, Trèves [8]).

It was shown in Sasane and Cotroneo [6] and Sasane et al. [7] that the following implications hold:

$$\begin{array}{ccc} A1 & \Rightarrow & B1 \\ \Downarrow \not\Uparrow & & \Downarrow \\ A2 & \Leftarrow & B2 \end{array}$$

In this article we will show that $B2 \not\Rightarrow B1$. The validity of $B1 \Rightarrow A1$ remains an open question.

3 The 1-D case

The following theorem states that in the 1-D case, the notions of (exact) time-controllability and approximate time-controllability are equivalent.

Theorem 3.1. *Let $R \in \mathbb{C}[\xi]^{\mathtt{g}\times\mathtt{w}}$, and let the corresponding behaviour be $\mathfrak{B}$. Then the following are equivalent:*

1. *$\mathfrak{B}$ is approximately time-controllable*
2. *$\mathfrak{B}$ is time-controllable*
3. *there exists a $\mathtt{r}_0 \in \mathbb{N} \cup \{0\}$ such that for all $\lambda \in \mathbb{C}$, rank$(R(\lambda)) = \mathtt{r}_0$*
4. *the $\mathbb{C}[\xi]$-module $\mathbb{C}[\xi]^{\mathtt{w}}/\langle R\rangle$ is torsion free.*

Proof We will prove that $4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 4$.

$4 \Rightarrow 3$. Let us assume that 3 does not hold. From Theorem B.1.4 (page 404, Polderman and Willems [5]), it follows that there exist unimodular matrices U and V such that $R = U\Sigma V$, where

$$\Sigma = \begin{bmatrix} \text{diag}(d_1, d_2, \cdots, d_{\mathtt{r}}) & 0 \\ 0 & 0 \end{bmatrix}_{\mathtt{g}\times\mathtt{w}}$$

and the $d_{\mathtt{k}}$'s are polynomials such that $d_{\mathtt{k}}$ divides $d_{\mathtt{k}+1}$ for all $\mathtt{k} \in \{1, \dots, \mathtt{r}-1\}$. Thus any vector r in $\langle R\rangle$ is of the form

$$r = uU\Sigma V = \tilde{u}\Sigma V = \tilde{u}_1 d_1 v_1 + \cdots + \tilde{u}_{\mathtt{r}} d_{\mathtt{r}} v_{\mathtt{r}}, \tag{3.1}$$

where $u \in \mathbb{C}[\xi]^{\mathtt{g}}$, $\tilde{u} = uU = \begin{bmatrix} \tilde{u}_1 & \tilde{u}_2 & \dots & \tilde{u}_{\mathtt{g}} \end{bmatrix}$ and $V = \text{col}(v_1, v_2, \dots, v_{\mathtt{w}})$. Now let $\chi := v_{\mathtt{r}}$. Furthermore, since 3 does not hold, it follows that $d_{\mathtt{r}}$ is not constant. Clearly $\chi \notin \langle R\rangle$: indeed otherwise we would obtain $\tilde{u}_1 d_1 v_1 + \cdots + \tilde{u}_{\mathtt{r}} d_{\mathtt{r}} v_{\mathtt{r}} = v_{\mathtt{r}}$, and so $\tilde{u}_{\mathtt{r}} d_{\mathtt{r}} = 1$, which contradicts the fact that $d_{\mathtt{r}}$ is not a constant. Moreover, from (3.1), it is easy to see that $d_{\mathtt{r}} \cdot \chi \in \langle R\rangle$. Hence $\chi + \langle R\rangle$ is a nonzero torsion element in the $\mathbb{C}[\xi]$-module $\mathbb{C}[\xi]^{\mathtt{w}}/\langle R\rangle$, and so it is not torsion free.

$3 \Rightarrow 2$. This follows from Theorem 5.2.5, page 154 of Polderman and Willems [5].

$2 \Rightarrow 1$. This is evident from the definitions.

$1 \Rightarrow 4$. Suppose that 4 does not hold and the behaviour is approximately time-controllable. Then there exists an element $\chi \in \mathbb{C}[\xi]^{\mathtt{w}} \setminus \langle R\rangle$ and a $0 \neq p \in \mathbb{C}[\xi]$ such that $p \cdot \chi \in \langle R\rangle$. As χ is not in $\langle R\rangle$, it follows that, it does not kill every element in $\mathfrak{B}$. Let $w_0 \in \mathfrak{B}$ be a trajectory such that

$D_\chi w_0 \neq 0$. Without loss of generality, we may assume that $(D_\chi w_0)\,|_{t>0} \neq 0$ (otherwise w_0 can be shifted to achieve this). Since the topology of $\mathcal{C}^\infty(\Omega, \mathbb{C})$ is Hausdorff, it follows that there exists a neighborhood N in $\mathcal{C}^\infty((0,\infty), \mathbb{C})$ of $(D_\chi w_0)\,|_{t>0}$ that does not contain 0. Since the map $D_\chi : \mathcal{C}^\infty((0,\infty), \mathbb{C})^{\mathtt{w}} \rightarrow \mathcal{C}^\infty((0,\infty), \mathbb{C})$ is continuous, it follows that there exists a neighborhood N_1 in $\mathcal{C}^\infty((0,\infty), \mathbb{C})^{\mathtt{w}}$ of $w_0|_{t>0}$ such that $w_1 \in N_1$ implies that $D_\chi w_1 \in N$. Consequently $(D_\chi w_1)\,|_{t>0} \neq 0$. Let $\tau \geq 0$ and let $w \in \mathfrak{B}$ be such that $w(t) = 0$ for all $t \leq 0$, and $w(\bullet + \tau) \in N_1$. Defining $u = D_\chi w$, we have $u|_{t>0} \neq 0$. Since $p \cdot \chi \in \langle R \rangle$, and $w \in \mathfrak{B}$, it follows that $D_p u = 0$. But since $w(t) = 0$ for all $t \in (0, \infty)$, it follows from the definition of u that $u(t) = 0$ for all $t \in (0, \infty)$. Hence it follows that u must be zero, but this a contradiction since $u \neq 0$. This completes the proof. ■

4 A behaviour that is approximately time-controllable, but not time-controllable

We quote the following (Theorem 12.7.8 on page 142 of Hörmander [1] and Theorem 8.6.8 on page 312 of Hörmander [2], respectively):

Theorem 4.1. *Let p be an irreducible polynomial with principal part $p_{\mathtt{N}}$ which is hyperbolic with respect to n. Let K be a convex compact subset of the plane $\langle x, n \rangle = 0$ and x_0 a point in this plane outside K. If $w \in \mathcal{C}^{\mathtt{N}}(H)$ where $H = \{x \mid \langle x, n \rangle \geq 0\}$, is a solution of the equation $D_p w = 0$ with Cauchy data vanishing outside K, and if the support of w is contained in $\{x_0\} + \Gamma^\circ(p_{\mathtt{N}}, n)$ except for a bounded set, it then follows that $w = 0$ identically in H.*

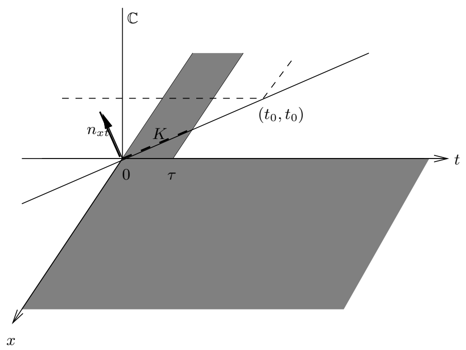
Theorem 4.2. *Let X_1 and X_2 be open convex sets in $\mathbb{R}^{\mathtt{n}}$ such that $X_1 \subset X_2$, and let $p \in \mathbb{C}[\zeta_1, \dots, \zeta_{\mathtt{n}}]$. Then the following conditions are equivalent:*

1. *Every $T \in \mathcal{D}'(X_2)$ such that $D_p T = 0$ in X_2 and vanishing in X_1 must also vanish in X_2.*
2. *Every hyperplane characteristic with respect to p and intersects X_2 also intersects X_1.*

Using the above theorems, we will show that exact and approximate time-controllability are not equivalent for a 2-D system.

Theorem 4.3. *The behaviour corresponding to $p(\eta, \xi) = 1 + \eta\xi$ is not time-controllable.*

Proof Let the corresponding behaviour be denoted by $\mathfrak{B}$. Since the plane with normal $n_x = (1, 0)$



is characteristic with respect to the differential operator D_p, it follows from Theorem 8.6.7 (page 310, Hörmander [2]) that there exists a trajectory, say w_2, such that $w_2(x,t) = 0$ if $x < 0$ and $\text{supp}(w_2) = \{(x,t) \in \mathbb{R}^2 \mid x \geq 0\}$.

Let $w_1 = 0$. Clearly $w_1 \in \mathfrak{B}$. If $\mathfrak{B}$ is time-controllable, then there exists a trajectory that concatenates w_1 and w_2: that is, there exists a $\tau \geq 0$ and a w such that

$$w(\bullet, t) = \begin{cases} 0 & t \leq 0 \\ w_2(\bullet, t-\tau) & t \geq \tau \end{cases}.$$

Clearly $w \neq 0$, since it matches the nonzero future of w_2.

We now show that p is irreducible. It is nonconstant. Moreover, if $p = p_1 p_2$ for some p_1 and p_2 in $\mathbb{C}[\eta, \xi]$, then it follows that $p_1 = a_0 + a_1\xi$ and $p_2 = b_0 + b_1\xi$, for some $a_0,\ a_1,\ b_0,\ b_1\ \in \mathbb{C}[\eta]$. Thus $a_0 b_0 = 1$ and so it follows that they are constants and furthermore, from $a_1 b_1 = 0$, it follows that one of them must be zero. Hence it follows that either p_1 or p_2 must be a constant.

$p_{\mathtt{N}} = \eta\xi$ is hyperbolic with respect to $n_{xt} = (-1, -1)$. Indeed, $(-1)\cdot(-1) \neq 0$ and with $\theta_0 = 0$, we have $p_{\mathtt{N}}\left((x,t) + i\theta n_{xt}\right) = (x - i\theta)(t - i\theta) \neq 0$ for all $(x,t) \in \mathbb{R}^2$ and $\theta < \theta_0$.

We will now find $\Gamma^\circ(p_{\mathtt{N}}, n_{xt})$. We know that $(x,t) \in \Gamma(p_{\mathtt{N}}, n_{xt})$ iff $p_{\mathtt{N}}\left((x,t) + \theta n_{xt}\right) = (x-\theta)(t-\theta) = 0$ implies $\theta < 0$. But $(x-\theta)(t-\theta) = 0$ iff $\theta = x$ or $\theta = t$. Hence $(x,t) \in \Gamma(p_{\mathtt{N}}, n_{xt})$ iff $x < 0$ and $t < 0$. Hence the dual cone of $\Gamma(p_{\mathtt{N}}, n_{xt}) = \{(x,t) \in \mathbb{R}^2 \mid x < 0 \text{ and } t < 0\}$ is $\Gamma^\circ(p_{\mathtt{N}}, n_{xt}) = \{(x,t) \in \mathbb{R}^2 \mid \forall (x_1, t_1) \in \Gamma(p_{\mathtt{N}}, n_{xt}),\ \langle (x,t), (x_1,t_1)\rangle \geq 0\} = \{(x,t) \in \mathbb{R}^2 \mid x \leq 0 \text{ and } t \leq 0\}$. Let $K = \{(x,t) \in \mathbb{R}^2 \mid 0 \leq t \leq \tau\} \cap \{(x,t) \in \mathbb{R}^2 \mid \langle (x,t), n_{xt}\rangle = 0\}$. Then K is a convex compact subset of the plane $\{(x,t) \in \mathbb{R}^2 \mid \langle (x,t), n_{xt}\rangle = 0\}$. Let (t_0, t_0) be any point with $t_0 > \tau$ (then this point belongs to the plane and is outside K) and denote the half plane $\{(x,t) \in \mathbb{R}^2 \mid \langle (x,t), n_{xt}\rangle \geq 0\}$ by H. It is then clear that $w \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{C})$ $(\subset \mathcal{C}^2(H, \mathbb{C}))$ is a solution of the equation $D_p w = 0$ with Cauchy data vanishing outside K, and the support of w is contained in $\{(t_0, t_0)\} + \Gamma^\circ(p_{\mathtt{N}}, n_{xt})$ except for the bounded set $\{(x,t) \in \mathbb{R}^2 \mid t \leq x \leq t_0 \text{ and } 0 \leq t \leq \tau\} \cap H$.

Thus the assumptions of Theorem 4.1 apply, and it follows that $w = 0$ in H. Finally, we apply Theorem 4.2, with $X_2 = \mathbb{R}^2$, $X_1 = \{(x,t) \in \mathbb{R}^2 \mid \langle (x,t), n_{xt}\rangle > 0\}$ to conclude that $w = 0$, which is a contradiction. ■

From the above proof, it is clear that one can construct a host of other polynomials (for instance, polynomials in $\eta\xi$ that are irreducible) such that the corresponding behaviour is approximately time-controllable, but not time-controllable.

5 The heat equation and open questions

In this section we study the case of the heat equation $[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}]w = 0$. We have not been able to show that the behaviour is time-controllable, but we will show that in a large subspace $\mathcal{W}$ of the behaviour, trajectories can be concatenated in $\mathcal{W}$, that is, the behaviour is time-controllable "with respect to $\mathcal{W}$". Before defining the set $\mathcal{W}$, we recall the definition of the (small) Gevrey class of order 2, denoted by $\gamma^{(2)}(\mathbb{R})$: $\gamma^{(2)}(\mathbb{R})$ is the set of all $\varphi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ such that for every compact set K and every $\epsilon > 0$ there exists a constant C_ϵ such that for every $\mathtt{k} \in \mathbb{N}$, $|\varphi^{(\mathtt{k})}(t)| \leq C_\epsilon \epsilon^{\mathtt{k}}(\mathtt{k}!)^2$ for all $t \in K$. $\mathcal{W}$ is the set of all $w \in \mathfrak{B}$ such that $w(0, \bullet) \in \gamma^{(2)}(\mathbb{R})$. Since the heat operator is $(1,2)$-hypoelliptic (see Theorem 7.9, page 446, Trèves [8]), it follows that for a fixed t, any solution w is an analytic function in x. Let w_1 and w_2 belong to $\mathcal{W}$. Since w_1 is an analytic function in x for a fixed t, we obtain

$$\begin{aligned} w_1(x,t) &= \sum_{\mathtt{k}=0}^{\infty} \frac{\partial^{\mathtt{k}}}{\partial x^{\mathtt{k}}} w_1(0,t) \frac{x^{\mathtt{k}}}{\mathtt{k}!} = \sum_{\mathtt{k}=0}^{\infty} \frac{\partial^{2\mathtt{k}}}{\partial x^{2\mathtt{k}}} w_1(0,t) \frac{x^{2\mathtt{k}}}{(2\mathtt{k})!} + \sum_{\mathtt{k}=0}^{\infty} \frac{\partial^{2\mathtt{k}+1}}{\partial x^{2\mathtt{k}+1}} w_1(0,t) \frac{x^{2\mathtt{k}+1}}{(2\mathtt{k}+1)!} \\ &= \sum_{\mathtt{k}=0}^{\infty} \frac{\partial^{\mathtt{k}}}{\partial t^{\mathtt{k}}} w_1(0,t) \frac{x^{2\mathtt{k}}}{(2\mathtt{k})!} + \sum_{\mathtt{k}=0}^{\infty} \frac{\partial^{\mathtt{k}+1}}{\partial t^{\mathtt{k}} \partial x} w_1(0,t) \frac{x^{2\mathtt{k}+1}}{(2\mathtt{k}+1)!}, \end{aligned} \tag{5.2}$$

where we have used the relations $\frac{\partial^{2\mathtt{k}}}{\partial x^{2\mathtt{k}}} w_1(0,t) = \frac{\partial^{\mathtt{k}}}{\partial t^{\mathtt{k}}} w_1(0,t)$ and $\frac{\partial^{2\mathtt{k}+1}}{\partial x^{2\mathtt{k}+1}} w_1(0,t) = \frac{\partial^{\mathtt{k}+1}}{\partial t^{\mathtt{k}} \partial x} w_1(0,t)$. Similarly

$$w_2(x,t) = \sum_{\mathtt{k}=0}^{\infty} \frac{\partial^{\mathtt{k}}}{\partial t^{\mathtt{k}}} w_2(0,t) \frac{x^{2\mathtt{k}}}{(2\mathtt{k})!} + \sum_{\mathtt{k}=0}^{\infty} \frac{\partial^{\mathtt{k}+1}}{\partial t^{\mathtt{k}} \partial x} w_2(0,t) \frac{x^{2\mathtt{k}+1}}{(2\mathtt{k}+1)!}.$$

From the remark following Definition 12.7.3 (page 137 of Hörmander [1]), it follows that $\gamma^{(2)}(\mathbb{R})$ is an algebra and one can find cut-off functions there. Consequently, it is easy to see that given any $\tau > 0$, one can find functions f and g in $\gamma^{(2)}(\mathbb{R})$ such that

$$f(t) = \begin{cases} w_1(0,t) & t \le 0 \\ w_2(0,t-\tau) & t \ge \tau \end{cases}, \text{ and } g(t) = \begin{cases} \frac{\partial}{\partial x} w_1(0,t) & t \le 0 \\ \frac{\partial}{\partial x} w_2(0,t-\tau) & t \ge \tau \end{cases}.$$

Now let $w(x,t) = \sum_{\mathtt{k}=0}^{\infty} f^{(\mathtt{k})}(t) \frac{x^{2\mathtt{k}}}{(2\mathtt{k})!} + \sum_{\mathtt{k}=0}^{\infty} g^{(\mathtt{k})}(t) \frac{x^{2\mathtt{k}+1}}{(2\mathtt{k}+1)!}$. Since f and g belong to the class $\gamma^{(2)}(\mathbb{R})$, it is easy to see that the convergence is uniform on compact subsets of $\mathbb{R}^2$. Furthermore, we know that the class $\gamma^{(2)}(\mathbb{R})$ is closed under differentiation. Consequently, we have

$$\frac{\partial}{\partial t} w(x,t) = \sum_{\mathtt{k}=0}^{\infty} f^{(\mathtt{k}+1)}(t) \frac{x^{2\mathtt{k}}}{(2\mathtt{k})!} + \sum_{\mathtt{k}=0}^{\infty} g^{(\mathtt{k}+1)}(t) \frac{x^{2\mathtt{k}+1}}{(2\mathtt{k}+1)!} = \frac{\partial^2}{\partial x^2} w(x,t)$$

and so w satisfies the heat equation. Moreover, it is clear that w concatenates w_1 and w_2. Hence the behaviour is time-controllable with respect to $\mathcal{W}$.

An interesting question that now arises is the following: since we have only one variable w, how is the control implemented? (5.2) shows that a solution w in $\mathcal{W}$ is fixed once $w(0,\bullet)$ and $\frac{\partial}{\partial x} w(0,\bullet)$ are specified. Hence the control could be implemented by the two *point control* input functions acting at the point $x = 0$: $u_1(t) = w(0,t)$ and $u_2(t) = \frac{\partial}{\partial x} w(0,t)$ for all $t \in \mathbb{R}$. Another interesting problem is to construct an example of a trajectory in the behaviour that is not in the class $\mathcal{W}$. Also whether the behaviour of the heat equation is time-controllable or not is an open question. The answers to these questions would either strengthen or discard the conjecture that the behaviour corresponding to $p \in \mathbb{C}[\eta_1, \dots, \eta_{\mathtt{m}}, \xi]$ is time-controllable iff $p \in \mathbb{C}[\eta_1, \dots, \eta_{\mathtt{m}}]$, which would eventually help in settling the conjecture that $A1$ and $B1$ are equivalent.

6 Glossary of technical terms

Irreducible polynomial: A polynomial $p \in \mathbb{C}[\zeta_1, \dots, \zeta_n]$ is said to be irreducible if it is nonconstant and is not the product of two nonconstant polynomials in $\mathbb{C}[\zeta_1, \dots, \zeta_n]$.

Degree of a polynomial: Let $p \in \mathbb{C}[\zeta_1, \ldots, \zeta_{\mathtt{n}}]$ be of the form

$$p = \sum_{|(\alpha_1,\ldots,\alpha_{\mathtt{n}})| \leq \mathtt{N}} a_{(\alpha_1,\ldots,\alpha_{\mathtt{n}})}\, \zeta_1^{\alpha_1} \ldots \zeta_{\mathtt{n}}^{\alpha_{\mathtt{n}}},$$

with $a_{(\alpha_1,\ldots,\alpha_{\mathtt{n}})} \neq 0$ for some $(\alpha_1, \ldots, \alpha_{\mathtt{n}})$ with $|(\alpha_1, \ldots, \alpha_{\mathtt{n}})|\,(:= \alpha_1 + \cdots + \alpha_{\mathtt{n}}) = \mathtt{N}$. The degree of p, denoted by $\deg(p)$, is $\mathtt{N}$.

Principal part of a polynomial: With the same notation as above, the principal part of p (denoted by $p_{\mathtt{N}}$) is defined by

$$p_{\mathtt{N}} = \sum_{|(\alpha_1,\ldots,\alpha_{\mathtt{n}})| = \mathtt{N}} a_{(\alpha_1,\ldots,\alpha_{\mathtt{n}})}\, \zeta_1^{\alpha_1} \ldots \zeta_{\mathtt{n}}^{\alpha_{\mathtt{n}}}.$$

Characteristc hyperplane: The hyperplane with normal $n \in \mathbb{R}^{\mathtt{n}}$, that is, $\{x \in \mathbb{R}^{\mathtt{n}} \mid \langle x, n\rangle = 0\}$ is said to be characteristic with respect to p if $p_{\mathtt{N}}(n) = 0$.

Hyperbolic polynomial: A polynomial p is called hyperbolic with respect to a real vector n if $p_{\mathtt{N}}(n) \neq 0$ and there exists a number θ_0 such that $p(x + i\theta n) \neq 0$ if $x \in \mathbb{R}^{\mathtt{n}}$ and $\theta < \theta_0$.

Cauchy data: Given a $n \in \mathbb{R}^{\mathtt{n}}$, let H be the half-space $\{x \mid \langle x, n\rangle \geq 0\}$. Given a polynomial p, and a solution $w \in \mathbb{C}^N(H)$ of the equation $D_p w = 0$ the Cauchy data is the restriction of the function w to the set $\{x \in \mathbb{R}^n \mid \langle x, n\rangle = 0\}$.

The covex cone $\Gamma(p, n)$: Given a polynomial p and a $n \in \mathbb{R}^{\mathtt{n}}$,

$$\Gamma(p, n) := \{x \in \mathbb{R}^{\mathtt{n}} \mid p_{\mathtt{N}}(x + \theta n) = 0 \text{ implies } t < 0\}.$$

The dual cone: Given a cone Γ, the dual cone, denoted by Γ° is defined by

$$\Gamma^{\circ} = \{x \in \mathbb{R}^{\mathtt{n}} \mid \langle y, x\rangle \geq 0 \text{ for all } y \in \Gamma\}.$$

References

[1] L. Hörmander. *The Analysis of Linear Partial Differential Operators II.* Springer-Verlag, 1983.

[2] L. Hörmander. *The Analysis of Linear Partial Differential Operators I.* Springer-Verlag, 1983.

[3] U. Oberst. Multidimensional constant linear systems. *Acta Appl. Math.*, 20: 1-175, 1990.

[4] H.K. Pillai and S. Shankar. A Behavioural Approach to the Control of Distributed Systems. *SIAM Journal on Control and Optimization*, 37: 388-408, November, 1998.

[5] J.W. Polderman and J.C. Willems. *Introduction to Mathematical Systems Theory.* Springer-Verlag, 1998.

[6] A.J. Sasane and T. Cotroneo. Conditions for time-controllability of behaviours. *International Journal of Control*, 75: 61-67, 2002.

[7] A.J. Sasane and E.G.F. Thomas and J.C. Willems. Time-autonomy versus time-controllability. *Systems and Control Letters*, 45: 145-153, 2002.

[8] F. Trèves. *Linear Partial Differential Equations with Constant Coefficients*. Gordon and Breach, 1966.