On a class of time-varying behaviors

Kanat Çamlıbel
Madhu N. Belur
Department of Mathematics, University of Groningen,
PO Box 800, 9700 AV, Groningen, The Netherlands

Amol J. Sasane
Department of Mathematics, University of Twente,
PO Box 217, 7500 AE, Enschede, The Netherlands

Jan C. Willems
ESAT/SISTA-COSIC, University of Leuven,
Kasteelpark Arenberg 10, B-3001 Leuven-Heverlee, Belgium

Abstract

We study a class of time-varying systems that we encounter when we look into decomposition of behaviors. This class is the set of behaviors that are themselves of polynomials in time, with coefficients as time-invariant behaviors. Operators that have such behaviors as their kernels are studied. It turns out that autonomous behaviors allow kernel representations of this kind. We are led to the study of skew polynomial ring as the underlying ring for such operators.

1 Introduction

In the behavioral approach, we look at a system as the set of all trajectories that it allows (for an introduction see for instance [5, 4]). When these trajectories are functions of one variable, we almost always consider them as functions of 'time'. Many such systems we encounter are governed by laws that themselves do *not* depend on time. This results in time-invariant systems. However, in this paper we study a class of systems/behaviors that are linear but not necessarily time-invariant. More precisely, we are interested in behaviors that are polynomials in the time variable with coefficients as time-invariant behaviors.

Our motivation for studying these systems stems from a 'decomposition' problem. Sometimes, decomposing a system into subsystems having certain properties helps in understanding its behavior. As a well-known example, we can think of Kalman decomposition in state space theory. The idea is to decompose the system into controllable and uncontrollable parts. By doing so, one can check, for instance, the stabilizability of the system. A very similar decomposition is possible also in behavioral theory: given any behavior, one can decompose it into the controllable part (by definition the largest controllable subbehavior) and an autonomous part. Here, what we mean by decomposition is that the direct sum of the two subbehaviors is the behavior itself. Knowing that the above decomposition is possible, we might take one step further and investigate under what conditions on a given behavior $\mathfrak{B}$ and a subbehavior $\mathfrak{B}' \subseteq \mathfrak{B}$ there exists another subbehavior $\mathfrak{B}'' \subseteq \mathfrak{B}$ such that $\mathfrak{B} = \mathfrak{B}' \oplus \mathfrak{B}''$. Not surprisingly, controllability plays a key role and it can be shown that $\mathfrak{B}''$ always exists whenever $\mathfrak{B}'$ is controllable. Up to now, we used the word 'behavior' as a synonym for linear time-invariant differential system. However, it is sometimes necessary to consider *time-varying* behaviors. For instance, in case $\mathfrak{B}$ and $\mathfrak{B}'$ are both autonomous and time-invariant, there always exists a behavior $\mathfrak{B}''$ such that $\mathfrak{B} = \mathfrak{B}' \oplus \mathfrak{B}''$, however, $\mathfrak{B}''$ is not time-invariant in general.

Although the above decomposition problem has been the main motivation, we do not restrict ourselves to just this issue. Some peripheral questions, such as kernel representations for this class of behaviors, are also explored. Related work on time-varying systems (though not explicitly in the behavioral framework) has appeared before, for example in [3] and in the references therein.

The paper is organized as follows. The rest of this section describes the notation we use. Section 2 is an exposition on polynomials of behaviors. The behavioral decomposition problem will be addressed in section 3. This will be followed by conclusions in section 4. The proofs of the results follow in the appendix.

1.1 Notation

We devote this subsection to main notational conventions used in the paper. The notations that are used 'locally' are defined just before their first appearance.

Sets. We denote the set of real numbers by $\mathbb{R}$ and the set of complex numbers by $\mathbb{C}$. As usual, $\mathbb{R}^n$ ($\mathbb{C}^n$) denotes the set of n-tuples of real (complex) numbers. The set of $n \times m$ matrices with entries in $\mathbb{R}(\mathbb{C})$ will be denoted by $\mathbb{R}^{n\times m}$ ($\mathbb{C}^{n\times m}$). When the specification of a dimension is not necessary, we use $\bullet$, i.e., we use $\mathbb{R}^{\bullet\times w}$ to denote the set of matrices with w columns.

Functions. We will mostly be interested in infinitely often differentiable functions. All such functions from $\mathbb{R}$ to a set Ω will be denoted by $\mathfrak{C}^\infty(\mathbb{R}, \Omega)$. The notation f^n will denote the product $f \cdot f \cdot \ldots \cdot f$ where the function $f \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ appears n times. By convention, f^0 is the mapping $t \mapsto 1$. With a (slight) abuse of notation, we write α instead of αf^0 for a scalar α. The notation $f|_\Delta$ stands for the restriction of the function f to the set Δ.

Operators. The kernel of an operator T is denoted by $\ker T$. Two special operators will be used often. For a real number s, we define the (time-)shift operator $\sigma_s : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w)$ by $(\sigma_s w)(t) := w(t + s)$ for all $t \in \mathbb{R}$ whereas the differentiation operator $D : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w)$ sends a function to its derivative.

Polynomials. For any set $\mathcal{U}$, $\mathcal{U}[\xi]$ denotes the set of polynomials in ξ with $\mathcal{U}$-coefficients. Sometimes, we consider polynomials in two (commuting) variables. Such polynomials in η and ξ with coefficients as real-valued $n \times m$ matrices will be denoted by $\mathbb{R}^{n\times m}[\eta, \xi]$.

Miscellaneous. Often, we add two subsets, say $\mathcal{V}$ and $\mathcal{W}$, of a given set, say $\mathcal{U}$ meaning

$\mathcal{V} + \mathcal{W} := \{v + w \mid v \in \mathcal{V} \text{ and } w \in \mathcal{W}\}$ where the addition is the usual addition in $\mathcal{U}$. If $\mathcal{V} \cap \mathcal{W} = \{0\}$ the sum $\mathcal{V} + \mathcal{W}$ is usually called the direct sum of $\mathcal{V}$ and $\mathcal{W}$ and denoted by $\mathcal{V} \oplus \mathcal{W}$. In a similar fashion, $v\mathcal{W}$ denotes the set $\{vw \mid w \in \mathcal{W}\}$ where $v \in \mathcal{V}$ and $\mathcal{V}\mathcal{W}$ denotes the set $\{vw \mid v \in \mathcal{V} \text{ and } w \in \mathcal{W}\}$. We use $\operatorname{span}_{\mathbb{R}} \mathcal{V}$ stands for all the finite linear combinations with real-valued coefficients of the elements of the set $\mathcal{V}$. Finally, for easy readability we use col that stacks up its arguments into a column.

2 Polynomials of behaviors

First, we briefly review some elementary facts from behavioral theory: a SYSTEM Σ is defined by a pair $(\mathfrak{U}, \mathfrak{B})$ where $\mathfrak{U}$ is the UNIVERSUM and $\mathfrak{B} \subseteq \mathfrak{U}$ is the BEHAVIOR. For reasons of brevity, we will consider only the case $\mathfrak{U} = \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathrm{w}})$ in this paper. Once $\mathfrak{U}$ is fixed, there is an obvious correspondence between systems and behaviors. Keeping this correspondence in mind, we will use the terms 'behavior' and 'system' synonymously.
We say that a behavior $\mathfrak{B}$ is

- LINEAR if it is a subspace of the vector space $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathrm{w}})$ over the field $\mathbb{R}$.
- TIME-INVARIANT if $\sigma_s(\mathfrak{B}) \subseteq \mathfrak{B}$ for all real numbers s.
- TIME-VARYING if it is not necessarily time-invariant.
- AUTONOMOUS if for $w_1, w_2 \in \mathfrak{B}$, $w_1|_{(-\infty,s)} = w_2|_{(-\infty,s)}$ for some $s \in \mathbb{R}$ implies $w_1 = w_2$.

In this paper, we consider only the linear behaviors and most of the time we skip the word 'linear'. The most typical example of a linear time-invariant behavior can be given as the kernel of a differential operator $R(D)$ where $R(\xi) \in \mathbb{R}^{\bullet \times \bullet}[\xi]$. It is also well-known that not every linear time-invariant behavior can be represented in this way. Consider, for instance, the behavior which consists of finite linear combinations (on $\mathbb{R}$) of the functions $\{t \mapsto e^{t(t-a)} \mid a \in \mathbb{R}\}$. It is a subspace of $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R})$ and hence linear. One can check that it is also time-invariant. However, it is not the kernel of any operator $R(D)$ with $R(\xi) \in \mathbb{R}[\xi]$. We say that $\mathfrak{B}$ is LINEAR TIME-INVARIANT DIFFERENTIAL if it is the kernel of such an operator $R(D)$. In this case, we say $R(D)$ is a KERNEL REPRESENTATION of $\mathfrak{B}$. The class of linear time-invariant differential behaviors of the universum $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathrm{w}})$ is denoted by $\mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}$.

Consider the polynomials in the indeterminate ζ with $\mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}$-coefficients, i.e., $\mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}[\zeta]$. Let $\mathfrak{B}(\zeta) \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}[\zeta]$. Defiine the function $\tau : \mathbb{R} \to \mathbb{R}$ as $\tau(t) := t$ for all $t \in \mathbb{R}$. Clearly, $\mathfrak{B}(\tau)$ is a subspace of the universum $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathrm{w}})$ and hence a behavior. It can be trivially verified that it is not necessarily time-invariant. Such behaviors will be the main object of study in this paper. We denote the set of such behaviors by $\mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}[\tau]$ and we write $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}[\tau]$ meaning that there exists $\mathfrak{B}'(\zeta) \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}[\zeta]$ such that $\mathfrak{B} = \mathfrak{B}'(\tau)$. In the sequel, we will investigate certain properties, such as time-invariance and autonomy, of these behaviors.

2.1 Time-invariance

The following theorem establishes necessary and sufficient conditions for time-invariance of $\mathfrak{L}_D[\tau]$-behaviors.

Theorem 2.1. *Let $\mathfrak{B}_i \in \mathfrak{L}_D^w$ for each i. The behavior $\mathfrak{B} = \mathfrak{B}_0 + \tau\mathfrak{B}_1 + \cdots + \tau^k\mathfrak{B}_k \in \mathfrak{L}_D^w[\tau]$ is time-invariant if and only if $\tau^m\mathfrak{B}_n \subseteq \mathfrak{B}$ for all $m = 0, 1, \ldots, n-1$ and $n = 0, 1, \ldots, k$.*

A curious class of behaviors are 'static' behaviors. We call a behavior $\mathfrak{B}$ STATIC if for all $w \in \mathfrak{B}$, $\sigma_s(w) = w$ for any $s \in \mathbb{R}$. We denote the set of all static behaviors by $\mathfrak{S}^w$ indicating that the underlying universum is $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w)$. It is not difficult to see that $\mathfrak{S}^w \subset \mathfrak{L}_D^w$. The following corollary is an application of theorem 2.1 to $\mathfrak{S}^w[\tau]$.

Corollary 2.1. *Let $\mathfrak{B}_i \in \mathfrak{S}^w$ for each i. The behavior $\mathfrak{B}_0 + \tau\mathfrak{B}_1 + \cdots + \tau^k\mathfrak{B}_k \in \mathfrak{S}^w[\tau]$ is time-invariant if and only if $\mathfrak{B}_k \subseteq \mathfrak{B}_{k-1} \subseteq \cdots \subseteq \mathfrak{B}_0$.*

2.2 Autonomy

Define $\mathfrak{A}^w$ as the set of all (possibly time-varying) autonomous behaviors with universum $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w)$. We define the vector space $\mathfrak{e}^\lambda := \text{span}_\mathbb{R}\{t \mapsto e^{\sigma t}\cos \omega t, t \mapsto e^{\sigma t}\sin \omega t\}$ where $\lambda = \sigma + i\,\omega \ \in \mathbb{C}$ and $\sigma, \omega \in \mathbb{R}$. Note that $\mathfrak{e}^\lambda$ is just $\text{span}_\mathbb{R}\{t \mapsto e^{\lambda t}\}$ in case λ is real.
The following proposition is just a restatement of [4, theorem 3.2.16].

Proposition 2.1. *For every autonomous behavior $\mathfrak{B} \in \mathfrak{A}^w \cap \mathfrak{L}_D^w$ there exist a finite set $\Lambda \subset \mathbb{C}$ and for each $\lambda \in \Lambda$ a unique time-invariant behavior $\mathfrak{B}_\lambda \in \mathfrak{S}^w[\tau] \cap \mathfrak{L}_D^w$ such that*

$$\mathfrak{B} = \bigoplus_{\lambda \in \Lambda} \mathfrak{B}_\lambda \mathfrak{e}^\lambda.$$

Moreover, Λ is unique up to the conjugation of every element.

As a step towards establishing an analogue of this result for autonomous behaviors of the class $\mathfrak{L}_D[\tau]$, we present the following theorem.

Theorem 2.2. *Let $\mathfrak{B}_i \in \mathfrak{L}_D^w$ for each i. The behavior $\mathfrak{B}_0 + \tau\mathfrak{B}_1 + \cdots + \tau^k\mathfrak{B}_k \in \mathfrak{L}_D^w[\tau]$ is autonomous if and only if each $\mathfrak{B}_i$ is autonomous. Stated differently, $\mathfrak{A}^w \cap \mathfrak{L}_D^w[\tau]$ coincides with $\mathfrak{A}^w[\tau] \cap \mathfrak{L}_D^w[\tau]$.*

Now, we are in a position to state an analogue of proposition 2.1.

Lemma 2.1. *For every autonomous behavior $\mathfrak{B} \in \mathfrak{A}^w \cap \mathfrak{L}_D^w[\tau]$ there exist a finite set $\Lambda \subset \mathbb{C}$ and for each $\lambda \in \Lambda$ a unique behavior $\mathfrak{B}_\lambda \in \mathfrak{S}^w[\tau]$ such that*

$$\mathfrak{B} = \bigoplus_{\lambda \in \Lambda} \mathfrak{B}_\lambda \mathfrak{e}^\lambda.$$

Moreover, Λ is unique up to the conjugation of every element.

2.3 Kernel representations

Our next aim is to study certain types of kernel representations for $\mathfrak{L}_D[\tau]$-type behaviors. To do this, we begin by briefly recalling the notion of the skew polynomial ring. For a detailed treatment, we refer to [2, 1]. Let $\mathfrak{R}$ be a ring. An additive homomorphism $\delta : \mathfrak{R} \to \mathfrak{R}$ is said to be a Derivation of the Ring $\mathfrak{R}$ if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathfrak{R}$. The set of all polynomials in δ with $\mathfrak{R}$-coefficients forms a ring (called Skew Polynomial Ring in δ over $\mathfrak{R}$) with respect to the operations of addition of polynomials and multiplication induced by the relation $\delta a = a\delta + \delta(a)$ for all $a \in \mathfrak{R}$.
Let ξ be a derivation of the ring $\mathbb{R}[\eta]$ with

$$\xi(\eta) = 1 \tag{2.1a}$$

and let $\mathbb{R}_{\text{ORE}}[\eta, \xi]$ [1] denote the skew polynomial ring in ξ over $\mathbb{R}[\eta]$. Note that

$$\xi\eta - \eta\xi = 1 \tag{2.1b}$$

by definition of the derivation. Every element $R(\eta, \xi) \in \mathbb{R}^{q\times w}_{\text{ORE}}[\eta, \xi]$ induces a mapping from $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w)$ to $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ as follows:

$$R(\tau, D)w = R_0(D)w + \tau R_1(D)w + \cdots + \tau^k R_k(D)w$$

where $R(\eta, \xi) = R_0(\xi)+\eta R_1(\xi)+\cdots+\eta^k R_k(\xi)$ with $R_n(\xi) = R^0_n+\xi R^1_n+\cdots+\xi^{\ell_n} R^{\ell_n}_n \in \mathbb{R}^{q\times w}[\xi]$ for each n and $R_n(D)w = R^0_n w+R^1_n Dw+\cdots+R^{\ell_n}_n D^{\ell_n} w$. Note that (2.1a) and (2.1b) become

$$D(\tau) = 1, \tag{2.2a}$$

$$(D\tau - \tau D)w = w \text{ for all } w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet). \tag{2.2b}$$

In particular, equation (2.2b) yields

$$R(\tau, D)(\tau w) = [R'(\tau, D) + \tau R(\tau, D)](w) \text{ for all } w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w) \tag{2.3}$$

where $R(\eta, \xi) \in \mathbb{R}^{q\times w}_{\text{ORE}}[\eta, \xi]$ and $R'(\eta, \xi)$ denotes the partial derivative of $R(\eta, \xi)$ with respect to ξ.
We consider the kernel representations induced by the above mentioned operators. Let $\mathfrak{L}^w_{\tau,D}$ denote the set $\{\ker R(\tau, D) \mid R(\eta, \xi) \in \mathbb{R}^{\bullet\times w}_{\text{ORE}}[\eta, \xi]\}$. By observing that $\mathfrak{L}_{\tau,D} \ni \ker[1 + (1 + \tau)D] = \{c(1 + \tau) \mid c \in \mathbb{R}\} \notin \mathfrak{L}_D[\tau]$, we can conclude that $\mathfrak{L}^w_{\tau,D} \nsubseteq \mathfrak{L}^w_D[\tau]$. However, the time-invariant behaviors that are contained in $\mathfrak{L}^w_D[\tau]$ or $\mathfrak{L}^w_{\tau,D}$ should be of the type $\mathfrak{L}^w_D$ as stated next.

Lemma 2.2. *The following statements hold.*

1. *If* $\mathfrak{B} \in \mathfrak{L}^w_D[\tau]$ *is time-invariant then* $\mathfrak{B} \in \mathfrak{L}^w_D$.

[1] Oystein Ore (1899–1968), the Norwegian mathematician who initiated the study of such rings.

2. *If* $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{\tau,\mathrm{D}}$ *is time-invariant then* $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}$.

Later, we will show that $\mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}[\tau] \nsubseteq \mathfrak{L}^{\mathrm{w}}_{\tau,\mathrm{D}}$. We first study their intersection. Consider a (time-varying) behavior $\mathfrak{B}$. If there exists a subbehavior $\mathfrak{B}' \subseteq \mathfrak{B}$ such that $\mathfrak{B}'$ is time-invariant and $\mathfrak{B}'' \subseteq \mathfrak{B}'$ for all time-invariant $\mathfrak{B}'' \subseteq \mathfrak{B}$ then we call $\mathfrak{B}'$ THE LARGEST TIME-INVARIANT BEHAVIOR CONTAINED IN $\mathfrak{B}$ and denote it by LTIB($\mathfrak{B}$). It is easy to see that the largest time-invariant behavior is unique if it exists. In fact, for any $\mathfrak{B}$ existence of LTIB($\mathfrak{B}$) can be shown, for instance, by invoking Zorn's [2] lemma.
Our next result is on the LTIB of an $\mathfrak{L}_{\tau,\mathrm{D}}$-behavior.

Theorem 2.3. *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{\tau,\mathrm{D}}$. *Also let* $R(\eta,\xi) \in \mathbb{R}^{q\times w}_{\mathrm{ORE}}[\eta,\xi]$ *with* $R(\eta,\xi) = R_0(\xi) + \eta R_1(\xi) + \cdots + \eta^k R_k(\xi)$ *where* $R_i(\xi) \in \mathbb{R}^{q\times w}[\xi]$ *for each* i, *be such that* $\ker R(\tau, D) = \mathfrak{B}$. *The* LTIB($\mathfrak{B}$) *exists and equals* $\bigcap_{i=0}^{k} \ker R_i(D)$.

An application of this theorem is the characterization of behaviors in the intersection of $\mathfrak{L}^{\mathrm{w}}_{\tau,\mathrm{D}}$ and $\mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}[\tau]$.

Theorem 2.4. *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{\tau,\mathrm{D}} \cap \mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}[\tau]$. *Also let* $R(\eta,\xi) \in \mathbb{R}^{q\times w}_{\mathrm{ORE}}[\eta,\xi]$ *with* $R(\eta,\xi) = R_0(\xi) + \eta R_1(\xi) + \cdots + \eta^k R_k(\xi)$ *induce a kernel representation for* $\mathfrak{B}$. *Define*

$$\mathcal{R}_0(\xi) := \mathrm{col}(R_0(\xi), R_1(\xi), \ldots, R_k(\xi))$$

and

$$\mathcal{R}_{n+1}(\xi) := \mathrm{col}(\mathcal{R}'_n(\xi), 0_{q\times w}) + \mathrm{col}(0_{q\times w}, \mathcal{R}_n(\xi)) \text{ for } n = 0, 1, \ldots$$

where $\mathcal{R}'(\xi)$ *denotes the derivative of* $\mathcal{R}(\xi)$ *with respect to* ξ. *Then, there exists an* ℓ *such that* $\mathfrak{B} = \ker \mathcal{R}_0(D) + \tau \ker \mathcal{R}_1(D) + \cdots + \tau^\ell \ker \mathcal{R}_\ell(D)$ *and* $\ker \mathcal{R}_n(D) = \ker \mathcal{R}_l(D)$ *for all* $n > \ell$.

Consider the behavior $\mathfrak{B} = \tau\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \in \mathfrak{L}_{\mathrm{D}}[\tau]$. Suppose that $\mathfrak{B} \in \mathfrak{L}_{\tau,\mathrm{D}}$ and $R(\eta,\xi) = R_0(\xi) + \eta R_1(\xi) + \cdots + \eta^k R_k(\xi) \in \mathbb{R}^{\bullet\times 1}_{\mathrm{ORE}}[\eta,\xi]$ with $R_k(\xi) \not\equiv 0$ induces a kernel representation for $\mathfrak{B}$. Theorem 2.4 implies that $\ker R_k(D) \supseteq \ker \mathcal{R}_1(D) = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$. But, this means that $R_k(\xi) = 0$. The contradiction that we have just reached proves $\mathfrak{B} \notin \mathfrak{L}_{\tau,\mathrm{D}}$, and hence $\mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}[\tau] \nsubseteq \mathfrak{L}^{\mathrm{w}}_{\tau,\mathrm{D}}$.
Now, we are in a position to state that at least autonomous $\mathfrak{L}_{\mathrm{D}}[\tau]$-behaviors admit kernel representations, i.e., $(\mathfrak{A}^{\mathrm{w}} \cap \mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}[\tau]) \subset \mathfrak{L}^{\mathrm{w}}_{\tau,\mathrm{D}}$. Since this is a theorem that provides much of the our motivation for studying this class of time-varying behaviors, we refer to this theorem as the main theorem.

Theorem 2.5. *For every autonomous behavior* $\mathfrak{B} \in \mathfrak{A}^{\mathrm{w}} \cap \mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}[\tau]$, *there exists* $R(\eta,\xi) \in \mathbb{R}^{w\times w}_{\mathrm{ORE}}[\eta,\xi]$ *such that* $\mathfrak{B} = \ker R(\tau, D)$.

Here, $\mathfrak{B}_1 = \tau\mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$, $\mathfrak{B}_2 = \ker \begin{pmatrix} 0 & 1+\tau-\tau D \end{pmatrix}$, $\mathfrak{B}_3 = \{w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^2) \mid w_1 = c(1+\tau) \text{ with } c \in \mathbb{R}\}$, $\mathfrak{B}_4 = \{w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \mid w = c(1+\tau) \text{ with } c \in \mathbb{R}\}$, and $\mathfrak{B}_5 = \mathrm{span}\{e^{t(t-a)} \mid a \in \mathbb{R}\}$.

[2] Max Zorn (1906-1993), the German mathematician who was the first one to use a *maximal principle* in algebra[6].

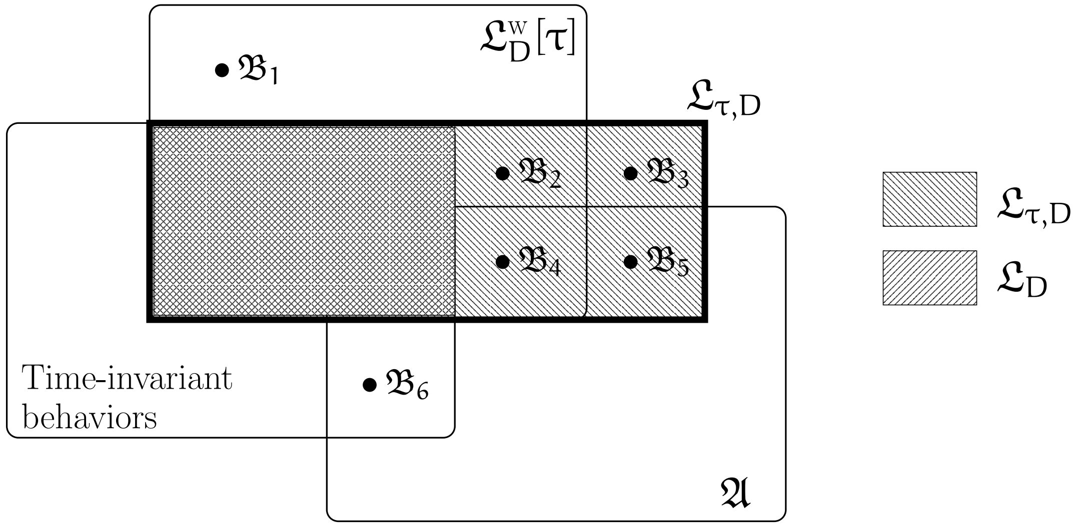

Figure 1: Venn diagram for some classes of behaviors

3 Decompositions of behaviors

In this section, we investigate decomposition of a behavior into a direct sum of two subbehaviors. The motivation for this may come from the need to capture a certain property (such as controllability, losslessness, or time-reversibility etc.) into one of the subbehaviors. The formulation of the decomposition problem can be stated as follows: given a time-invariant behavior $\mathfrak{B} \in \mathfrak{L}_D^w$ and a time-invariant subbehavior $\mathfrak{L}_D^w \ni \mathfrak{B}' \subseteq \mathfrak{B}$ find $\mathfrak{B}'' \in \mathfrak{L}_D^w$ such that $\mathfrak{B} = \mathfrak{B}' \oplus \mathfrak{B}''$.

It turns out that 'controllability' plays a key role here. We say a behavior $\mathfrak{B} \in \mathfrak{L}_D^w$ is CONTROLLABLE if for any w_1 and $w_2 \in \mathfrak{B}$ there exists a $w \in \mathfrak{B}$ such that $w|_{(-\infty,0]} = w_1|_{(-\infty,0]}$ and $w|_{[s,\infty)} = w_2|_{[s,\infty)}$ for some real number $s > 0$. The following well-known proposition solves the above problem for the controllability property.

Proposition 3.1. [4, theorem 5.2.14] *Let $\mathfrak{B} \in \mathfrak{L}_D^w$. There exist subbehaviors $\mathfrak{B}_c$ and $\mathfrak{B}_a$ of $\mathfrak{B}$ such that $\mathfrak{B} = \mathfrak{B}_c \oplus \mathfrak{B}_a$ where $\mathfrak{B}_c$ is controllable and $\mathfrak{B}_a$ is autonomous. Moreover, the controllable part $\mathfrak{B}_c$ is unique.*

The importance of controllability is already well-acknowledged in the study of open systems. It also comes into play in our context as stated below.

Theorem 3.1. *Let $\mathfrak{B}$, $\mathfrak{B}' \in \mathfrak{L}_D^w$ be such that $\mathfrak{B}' \subseteq \mathfrak{B}$ and let $\mathfrak{B}'$ be controllable. Then, there exists a $\mathfrak{B}'' \in \mathfrak{L}_D^w$ such that $\mathfrak{B} = \mathfrak{B}' \oplus \mathfrak{B}''$. Moreover, if $\mathfrak{B}$ is also controllable, then $\mathfrak{B}''$ is controllable.*

One can view proposition 3.1 as a special case of the above theorem. When the behavior $\mathfrak{B}$ does not contain any nontrivial controllable subbehaviors (in other words when it is autonomous), $\mathfrak{L}_D[\tau]$-behaviors have to be called in. In this case, the following theorem solves the decomposition problem in a slightly more general framework.

Theorem 3.2. *Let autonomous behaviors* $\mathfrak{B},\ \mathfrak{B}' \in \mathfrak{L}_{\mathrm{D}}^{\mathrm{w}}[\tau]$ *be such that* $\mathfrak{B}' \subseteq \mathfrak{B}$. *Then, there exists an autonomous behavior* $\mathfrak{B}'' \in \mathfrak{L}_{\mathrm{D}}^{\mathrm{w}}[\tau]$ *such that* $\mathfrak{B} = \mathfrak{B}' \oplus \mathfrak{B}''$.

Even if $\mathfrak{B}$ and $\mathfrak{B}'$ are both time-invariant there is no escape from $\mathfrak{L}_{\mathrm{D}}[\tau]$, in general. This makes worthwhile stating the following corollary separately.

Corollary 3.1. *Consider time-invariant autonomous behaviors* $\mathfrak{B},\ \mathfrak{B}' \in \mathfrak{L}_{\mathrm{D}}^{\mathrm{w}}$ *such that* $\mathfrak{B}' \subseteq \mathfrak{B}$. *Then, there exists an autonomous behavior* $\mathfrak{B}'' \in \mathfrak{L}_{\mathrm{D}}^{\mathrm{w}}[\tau]$ *such that* $\mathfrak{B} = \mathfrak{B}' \oplus \mathfrak{B}''$. *Moreover, there exists an* $\mathsf{R}(\eta, \xi) \in \mathbb{R}_{\mathrm{ORE}}^{\bullet\times\mathrm{w}}[\eta, \xi]$ *such that* $\mathfrak{B}'' = \ker \mathsf{R}(\tau, \mathsf{D})$.

4 Conclusions

As a prelude for the behavioral decomposition problem, we discussed a class of time-varying behaviors, more precisely, polynomials in the time variable of time-invariant differential behaviors. Of course, this class is interesting in its own right. Sometimes by digressing from the main course, we investigated some issues which are not directly related to our initial motivation. We addressed issues like time-invariance and autonomy. We also showed that such autonomous behaviors admit a certain type of kernel representation. By means of an example, it was illustrated that not every nonautonomous behavior can be described by such representations. The question under what conditions a behavior can be represented as such remains unresolved.

Furthermore, we solved the decomposition problem for controllable and for autonomous behaviors. It turned out that one has to consider time-varying systems for the autonomous case. Still, the problem needs to be solved in the most general setting. This will be a subject of future research.

A Appendix

Before we begin with the proofs, we collect some basic results into the following subsection.

A.1 Preliminaries

We denote the i-th derivative of $\mathsf{R}(\xi) \in \mathbb{R}^{\bullet\times\mathrm{w}}[\xi]$ with respect to ξ by $\mathsf{R}^{(i)}(\xi)$.

Lemma A.1. *Let* $\mathsf{P}(\xi) \in \mathbb{R}^{\bullet\times\mathrm{w}}[\xi]$ *and* $\mathsf{Q}(\eta, \xi) \in \mathbb{R}_{\mathrm{ORE}}^{\bullet\times\mathrm{w}}[\eta, \xi]$. *The following statements hold for all* $w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathrm{w}})$.

1. *For any integer* m,
$$\mathsf{P}(\mathsf{D})(\tau^{m} w) = \sum_{\ell=0}^{m} \binom{m}{\ell} \tau^{\ell} \mathsf{P}^{(m-\ell)}(\mathsf{D}) w.$$

2. *For any real number* s,
$$\mathsf{Q}(\tau, \mathsf{D})(\sigma_{s} w) = \mathsf{Q}(\tau - s, \mathsf{D}) w.$$

A.2 Proofs for section 2

This subsection of the appendix contains the proofs of the results in section 2 of this paper.

Proof of theorem 2.1

'if': Note that $\sigma_s(\mathfrak{B}' + \mathfrak{B}'') = \sigma_s\mathfrak{B}' + \sigma_s\mathfrak{B}''$ for all real numbers s and subspaces $\mathfrak{B}', \mathfrak{B}''$ of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w)$. Hence, we have $\sigma_s\mathfrak{B} = \sigma_s\mathfrak{B}_0 + \sigma_s(\tau\mathfrak{B}_1) + \cdots + \sigma_s(\tau^k\mathfrak{B}_k)$. By using linearity, one can show that $\sigma_s(\tau^n\mathfrak{B}_n) \subseteq \mathfrak{B}_n + \tau\mathfrak{B}_n + \cdots + \tau^n\mathfrak{B}_n$. It follows that $\sigma_s\mathfrak{B} \subseteq \mathfrak{B}$ for all real numbers s.

'only if': Note that the implication

$$w \in \mathfrak{B}' \Rightarrow Dw \in \mathfrak{B}' \tag{A.4}$$

holds for $\mathfrak{B}' \in \mathfrak{L}^w_D$. Since we already know that $\tau^n\mathfrak{B}_n \subseteq \mathfrak{B}$ for each $n = 0, 1, \ldots, k$, the following implication for $m = 0, 1, \ldots, n-1$

$$\tau^{m+1}\mathfrak{B}_n \subseteq \mathfrak{B} \Rightarrow \tau^m\mathfrak{B}_n \subseteq \mathfrak{B} \tag{A.5}$$

would complete the proof. Take any $w' \in \mathfrak{B}_n$. Clearly, $w := \tau^{m+1}w' \in \tau^{m+1}\mathfrak{B}_n$. If $\tau^{m+1}\mathfrak{B}_n \subseteq \mathfrak{B}$, we have $w \in \mathfrak{B}$. It follows from time-invariance of $\mathfrak{B}$ and the implication (A.4) that

$$Dw = (m+1)\tau^m w' + \tau^{m+1}Dw' \in \mathfrak{B}. \tag{A.6}$$

Note that $Dw' \in \mathfrak{B}_n$ due to the implication (A.4) again. This means that the second summand on the right hand side of equation (A.6) is an element of $\tau^{m+1}\mathfrak{B}_n$ and hence $\mathfrak{B}$. Consequently, $\tau^m w' \in \mathfrak{B}$. This proves the implication (A.5). □

Proof of corollary 2.1

Since all the concerned behaviors are static, the following statements are equivalent.

1. For all $n = 0, 1, \ldots, k$ and $m = 0, 1, \ldots, n-1$, $\tau^m\mathfrak{B}_n \subseteq \mathfrak{B}$.
2. For all $n = 0, 1, \ldots, k$ and $m = 0, 1, \ldots, n-1$, $\tau^m\mathfrak{B}_n \subseteq \tau^m\mathfrak{B}_m$.
3. $\mathfrak{B}_k \subseteq \mathfrak{B}_{k-1} \subseteq \cdots \subseteq \mathfrak{B}_0$. □

Proof of theorem 2.2

(if part:) Here, we note that if a behavior $\mathfrak{B}$ is autonomous, then every subbehavior $\mathfrak{B}' \subseteq \mathfrak{B}$ is also autonomous. Let $\mathfrak{L}^w_D[\tau] \ni \mathfrak{B} = \mathfrak{B}_0 + \tau\mathfrak{B}_1 + \cdots + \tau^k\mathfrak{B}_k$ with $\mathfrak{B}_i \in \mathfrak{L}^w_D$, then $\tau^i\mathfrak{B}_i \subseteq \mathfrak{B}$. Hence we have that $\tau^i\mathfrak{B}_i$ is autonomous for each i. This implies that $\mathfrak{B}_i$ is autonomous too.

(only if part:) An interesting fact about autonomous behaviors in $\mathfrak{L}^w_D[\tau]$ or in $\mathfrak{L}^w_D$ is that they are subspaces of the space of real analytic functions. Hence, finite sums of these behaviors are still subspaces of this space. Hence, for $\mathfrak{B} \in \mathfrak{L}^w_D[\tau]$, if $w \in \mathfrak{B}$ is such that $w|_I = 0$ for some open interval $I \subset \mathbb{R}$, then $w = 0$. Thus $\mathfrak{B}$ is autonomous. □

Proof of lemma 2.1

Let $\mathfrak{B} \in \mathfrak{A}^w \cap \mathfrak{L}^w_D[\tau]$, i.e., $\mathfrak{B} = \mathfrak{B}_0 + \tau\mathfrak{B}_1 + \cdots + \tau^k\mathfrak{B}_k$ with $\mathfrak{B}_i \in \mathfrak{L}^w_D$. Using theorem 2.2, we have that each $\mathfrak{B}_i$ is autonomous. Further, by proposition 2.1, each $\mathfrak{B}_i$ can be written as $\mathfrak{B}_i = \bigoplus_{\lambda \in \Lambda_i} \mathfrak{B}_\lambda \mathfrak{e}^\lambda$ with $\mathfrak{B}_\lambda \in \mathfrak{S}^w[\tau] \cap \mathfrak{L}^w_D$ and $\Lambda_i \subset \mathbb{C}$, a finite set. In addition, $\mathfrak{S}^w[\tau]$ is closed under addition and multiplication with τ, i.e., for $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{S}^w[\tau]$, $\mathfrak{B}_1 + \mathfrak{B}_2 \in \mathfrak{S}^w[\tau]$ and $\tau\mathfrak{S}^w[\tau] \subset \mathfrak{S}^w[\tau]$. Hence if the Λ_i are disjoint, then $\sum_{i=0}^k \tau^i \mathfrak{B}_i \in \mathfrak{S}^w[\tau]$. Suppose there are common elements in Λ_i. Let $\lambda \in \Lambda_{i_1} \cap \Lambda_{i_2}$ for some i_1, i_2 , then $(\tau^{i_1}\mathfrak{B}_{\lambda,i_1} + \tau^{i_2}\mathfrak{B}_{\lambda,i_2}) \in \mathfrak{S}^w[\tau]$, and hence for the case of some Λ_i not disjoint also, the theorem is proved. □

Proof of lemma 2.2

1: We will need the following lemma.

Lemma A.2. *Let* $\mathfrak{B} \in \mathfrak{L}^w_D$ *and* $\tilde{\mathfrak{B}}_k = \mathfrak{B} + \tau\mathfrak{B} + \cdots + \tau^k\mathfrak{B}$ *for* $k = 0, 1, \ldots$. *Then,* $\tilde{\mathfrak{B}}_k \in \mathfrak{L}^w_D$.

Proof: We have the following facts.

i. $\tilde{\mathfrak{B}}_0 \in \mathfrak{L}^w_D$.

ii. Suppose that $\tilde{\mathfrak{B}}_n \in \mathfrak{L}^w_D$, i.e., there exists $R(\xi) \in \mathbb{R}^{q \times w}[\xi]$ such that

$$\tilde{\mathfrak{B}}_n = \ker R(D). \tag{A.7}$$

iii. Since for $m \leq n$, $\tau^m\mathfrak{B} \subseteq \tilde{\mathfrak{B}}_n = \ker R(D)$, we have

$$R(D)(\tau^m w) = 0$$

for all $w \in \mathfrak{B}$ and $m = 0, 1, \ldots, n$. Together with lemma A.1, this results in

$$R^{(m)}(D)w = 0. \tag{A.8}$$

It follows from the same lemma that

$$R^{(n+1)}(D)w = R(D)(\tau^{n+1}w) \tag{A.9}$$

for all $w \in \mathfrak{B}$.

iv. Let $F(\xi) \in \mathbb{R}^{\bullet \times q}[\xi]$ be such that

$$\ker F(D) = R^{(n+1)}(D)\mathfrak{B}. \tag{A.10}$$

v. Let $w \in \tilde{\mathfrak{B}}_{n+1}$. Since $\tilde{\mathfrak{B}}_{n+1} = \tilde{\mathfrak{B}}_n + \tau^{n+1}\mathfrak{B}$, $w = w_1 + \tau^{n+1}w_2$ for some $w_1 \in \tilde{\mathfrak{B}}_n$ and $w_2 \in \mathfrak{B}$. Then,

$$\begin{aligned} F(D)R(D)w &= F(D)R(D)w_1 + F(D)R(D)(\tau^{n+1}w_2) \\ &= F(D)R^{(n+1)}(D)w_2 && \text{(by (A.7) and (A.9))} \\ &= 0 && \text{(by (A.10)).} \end{aligned}$$

Consequently, $\tilde{\mathfrak{B}}_{n+1} \subseteq \ker F(D)R(D)$.

vi. Let $w \in \ker F(D)R(D)$. This implies that $R(D)w \in \ker F(D)$. Hence, $R(D)w = R^{(n+1)}(D)w_1$ for some $w_1 \in \mathfrak{B}$ due to (A.10). By employing (A.9), one can get $R(D)w = R(D)(\tau^{n+1}w_1)$. Therefore, $w = \tau^{n+1}w_1 + w_2$ where $w_2 \in \ker R(D) = \tilde{\mathfrak{B}}_n$. Consequently, $w \in \tilde{\mathfrak{B}}_{n+1}$ and $\ker F(D)R(D) \subseteq \tilde{\mathfrak{B}}_{n+1}$.

The facts, i, ii, v, and vi constitute a proof by induction for the claim. □

Now, we turn to the proof of lemma 2.2 item 1. The following facts will lead us to the proof.

1. Define $\overline{\mathfrak{B}}_n := \mathfrak{B}_n + \tau\mathfrak{B}_n + \cdots + \tau^{n-1}\mathfrak{B}_n$ for $n = 1, 2, \ldots, k$ and $\overline{\mathfrak{B}} := \overline{\mathfrak{B}}_1 + \overline{\mathfrak{B}}_2 + \cdots + \overline{\mathfrak{B}}_k$. It follows from theorem 2.1 that $\overline{\mathfrak{B}} \subseteq \mathfrak{B}$ since $\mathfrak{B}$ is time-invariant.

2. Lemma A.2 implies that $\overline{\mathfrak{B}}_n \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}$ and hence $\overline{\mathfrak{B}} \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}$. Let $R(\xi) \in \mathbb{R}^{q\times w}[\xi]$ be such that $\ker R(D) = \overline{\mathfrak{B}}$. Since $\tau^m \mathfrak{B}_n \subseteq \overline{\mathfrak{B}}$ for all $m = 0, 1, \ldots, n-1$ and $n = 1, 2, \ldots, k$, from lemma A.1 we have for all $w \in \mathfrak{B}_n$

$$R^{(m)}(D)w = \begin{cases} 0 & \text{if } m = 0, 1, \ldots, n-1, \\ R(D)(\tau^n w) & \text{if } m = n. \end{cases} \tag{A.12}$$

3. Let $F(\xi) \in \mathbb{R}^{\bullet\times q}[\xi]$ be such that

$$\ker F(D) = R(D)\mathfrak{B}_0 + R^{(1)}(D)\mathfrak{B}_1 + \cdots + R^{(k)}(D)\mathfrak{B}_k. \tag{A.13}$$

4. Let $w \in \mathfrak{B}$. Clearly, $w = w_0 + \tau w_1 + \cdots + \tau^k w_k$ where $w_n \in \mathfrak{B}_n$. So, (A.12) gives $R(D)w = R(D)w_0 + R^{(1)}(D)w_1 + \cdots + R^{(k)}(D)w_k$. Hence, $F(D)R(D)w = 0$. Consequently, $w \in \ker F(D)R(D)$.

5. Let $w \in \ker F(D)R(D)$. This immediately means that $R(D)w \in \ker F(D)$. Then, we have

$$\begin{aligned} R(D)w &= R(D)w_0 + R^{(1)}(D)w_1 + \cdots + R^{(k)}(D)w_k \quad \text{(from (A.13))}, \\ &= R(D)(w_0 + \tau w_1 + \cdots + \tau^k w_k) \quad \text{(from (A.12))} \end{aligned}$$

for some $w_n \in \mathfrak{B}_n$. Therefore, $w = (w_0 + \tau w_1 + \cdots + \tau^k w_k) + \overline{w}$ where $\overline{w} \in \ker R(D) = \overline{\mathfrak{B}}$. This implies that $w \in \mathfrak{B}$ since $\overline{\mathfrak{B}} \subseteq \mathfrak{B}$ due to 1.

It follows from 4 and 5 that $\mathfrak{B} = \ker F(D)R(D)$ and thus $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{D}}$.

2: Let $R(\eta, \xi) \in \mathbb{R}^{\bullet\times w}_{\mathrm{ORE}}[\eta, \xi]$ be such that $\mathfrak{B} = \ker R(\tau, D)$. Also let $R(\eta, \xi)$ be of the form $R_0(\xi) + \eta R_1(\xi) + \ldots + \eta^k R_k(\xi)$. It can be verified that $\ker \mathrm{col}(R_0(D), R_1(D), \ldots, R_k(D)) \subseteq \mathfrak{B}$. We claim that the time-invariance of $\mathfrak{B}$ yields, in fact, an equality in the last inclusion. To see this, take any $w \in \mathfrak{B}$. Since $\mathfrak{B}$ is time-invariant, we get $R(\tau, D)(\sigma_s w) = 0$ for all $w \in \mathfrak{B}$ and $s \in \mathbb{R}$. Lemma A.1 implies that

$$\begin{aligned} 0 = R(\tau, D)(\sigma_s w) &= R(\tau - s, D)w \\ &= R_0(D)w + (\tau - s)R_1(D)w + \ldots + (\tau - s)^k R_k(D)w \end{aligned} \tag{A.14}$$

for all $w \in \mathfrak{B}$ and $s \in \mathbb{R}$. Note that the left hand side is a polynomial in s. This means that (A.14) holds if and only if the coefficients of the monomials s^n are all zero for $n = 0, 1, 2, \ldots, k$. Therefore, $R_k(D)w = R_{k-1}(D)w = \cdots = R_0(D)w = 0$. In other words, $\mathfrak{B} \subseteq \bigcap_{i=0}^{k} \ker R_i(D) = \ker \operatorname{col}(R_0(D), R_1(D), \ldots, R_k(D))$. □

Proof of theorem 2.3

Clearly, $\tilde{\mathfrak{B}} := \bigcap_{i=0}^{k} \ker R_i(D)$ is time-invariant and contained in $\mathfrak{B}$. By the definition of LTIB($\mathfrak{B}$), we have already $\overline{\mathfrak{B}} := \text{LTIB}(\mathfrak{B}) \supseteq \tilde{\mathfrak{B}}$. So, it remains to show that $\overline{\mathfrak{B}} \subseteq \tilde{\mathfrak{B}}$. To see this, take any $w \in \overline{\mathfrak{B}}$. It follows from time-invariance that $\sigma_s w \in \overline{\mathfrak{B}}$ for all $s \in \mathbb{R}$. Since $\overline{\mathfrak{B}} \subseteq \mathfrak{B}$, we have even $\sigma_s w \in \mathfrak{B}$. Therefore, $R(\tau, D)(\sigma_s w) = 0$ for all $s \in \mathbb{R}$. Lemma A.1 implies that

$$\begin{aligned} 0 = R(\tau, D)(\sigma_s w) &= R(\tau - s, D)w \\ &= R_0(D)w + (\tau - s)R_1(D)w + \ldots + (\tau - s)^k R_k(D)w. \end{aligned} \tag{A.15}$$

Note that (A.15) holds if and only if the coefficients of each monomial s^n are zero. This results in $R_k(D)w = R_{k-1}(D)w = \cdots = R_0(D)w = 0$. Thus, $w \in \tilde{\mathfrak{B}}$. Consequently, $\overline{\mathfrak{B}} \subseteq \tilde{\mathfrak{B}}$. □

Proof of theorem 2.4

From theorem 2.3 we have $\ker(\mathcal{R}_0) = \text{LTIB}(\mathfrak{B})$. Now, we show that $\ker \mathcal{R}_i(D)$ is the largest time-invariant behavior (say $\mathfrak{B}_i$) such that $\tau^i \mathfrak{B}_i \subseteq \mathfrak{B}$. To see this for $i = 1$, we write

$$R(\tau, D)(\tau w) = R_0(D)\tau w + \tau R_1(D)\tau w + \cdots + \tau^k R_k(D)\tau w \ .$$

Now we use the identity $R(D)\tau = R'(D) + \tau R(D)$ where $R'(\xi)$ denotes the derivative of the polynomial matrix $R(\xi)$ with respect to ξ. The $\text{LTIB}(\ker(R(\tau, D)\tau))$ is precisely equal to $\ker(\mathcal{R}_1(D))$. Similarly, $\ker(\mathcal{R}_n(D))$ is the largest time invariant behavior $\mathfrak{B}_n$ such that $\tau^n \mathfrak{B}_n \subseteq \mathfrak{B}$. This recursion terminates when $\tau \ker(\mathcal{R}_{n+1}(D)) \subseteq \ker(\mathcal{R}_n(D))$ for some n. Now because $\mathfrak{B} \in \mathfrak{L}_D^w[\tau]$ also, there exists an $n < \infty$ for which the recursion terminates. □

A.3 Proof of theorem 2.5

The proof is split into intermediate steps which can be formulated into auxiliary results in themselves. Hence we state them as lemmas and prove them also.

Let $\mathfrak{B} = \mathfrak{B}_0 + \tau\mathfrak{B}_1 + \cdots + \tau^k \mathfrak{B}_k$ with $\mathfrak{B}_i \in \mathfrak{L}_D^w$ be autonomous. We know that $\mathfrak{B}$ is finite dimensional. For simplicity, we assume that each $\mathfrak{B}_i$ has only real exponents, i.e., if $R_i(\xi) \in \mathbb{R}^{w \times w}[\xi]$ induces a kernel representation for $\mathfrak{B}_i$, then $\det(R_i(\xi))$ has only real roots. The general case can be treated by following the proof of the case of real roots, except that it involves more complicated computations.

A.3.1 The scalar case

We first prove the theorem for $w = 1$. We need to introduce a few notations here. For polynomials $p(\eta), q(\eta) \in \mathbb{R}[\eta]$ and $\lambda, \mu \in \mathbb{R}$ we define the function $v_{p,\lambda} := t \to p(t)e^{\lambda t}$ and the operator $r_{q,\mu} \in \mathbb{R}_{\text{ORE}}[\eta, \xi]$ by $r_{q,\mu}(\tau, D) = [(\dot{q}(\tau) + \mu q(\tau)) - q(\tau)D]$. In what follows, we will sometimes skip the arguments η, τ, ξ, D if they are evident from the context. The following result is useful and is proved by straightforward computation.

Lemma A.3. *Given* $v_{p,\lambda}$ *and* $r_{q,\mu} \in \mathbb{R}_{\text{ORE}}[\eta, \xi]$ *the following hold:*

$$r_{q,\mu} v_{p,\lambda} = v_{\bar{p},\lambda} \tag{A.16}$$

where $\bar{p} = (p\dot{q} - \dot{p}q + (\mu - \lambda)pq)$. *Further,* $r_{q,\mu} v_{p,\lambda} = 0$ *if and only if* $\mu = \lambda$ and $p = \text{constant} \cdot q$.

Now, we are in a position to complete the proof of the scalar case. Let $\mathfrak{B}$ be expressed as the span of functions $\{v_{p_i,\lambda_i}\}_{i=1}^{k}$ for a suitable choice of $p_i \in \mathbb{R}[\eta]$ and $\lambda_i \in \mathbb{R}$. By employing A.3, we find an $r_1 \in \mathbb{R}_{\text{ORE}}[\eta, \xi]$ such that $\ker r_1(\tau, D) = \text{span}\{v_{p_1,\lambda_1}\}$. Lemma A.3 also guarantees that $r_1(\tau, D)V = \text{span}\{v_{\bar{p}_i,\lambda_i}\}_{i=2}^{k}$ for some polynomials $\bar{p}_i \in \mathbb{R}[\eta]$. By repeating the above argument one finds $r_2, r_3, \ldots, r_k$ such that $\mathfrak{B}$ is the kernel of $r_k(\tau, D) r_{k-1}(\tau, D) \cdots r_1(\tau, D)$.

A.3.2 Multivariable case

Let $W = \{w_{ij}\}$ be a matrix whose columns form a basis for $\mathfrak{B}$. Since $\mathfrak{B} \in \mathfrak{L}_D^{w}[\tau]$, w_{ij} can be chosen such that any two elements in each column are linearly dependent over $\mathbb{R}$, i.e., $w_{ij} = t \to \alpha_{ij} t^{n_j} e^{\lambda_j t}$. Without loss of generality, we can assume that $(n_{j_1}, \lambda_{j_1}) = (n_{j_2}, \lambda_{j_2})$ implies that either α_{ij_1} or α_{ij_2} is zero. Otherwise, one can always achieve this by post-multiplying W by a real nonsingular matrix (which amounts to a basis transformation).

Suppose W has the following special form:

$$W = \begin{bmatrix} \tilde{w}_1^T & 0 & \cdots & 0 \\ 0 & \tilde{w}_2^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{w}_w^T \end{bmatrix}. \tag{A.17}$$

We can construct kernel representations for the span of elements of each $\tilde{w}_i$. This would yield a kernel representation for the corresponding behavior and hence for $\mathfrak{B}$.

We claim that any W can be brought to this form by elementary row and column operations on W. To show this we first need the following two lemmas.

Lemma A.4. *Let* $p, q \in \mathbb{R}[\eta]$ *with* $p \neq 0$. *Then there exists an* $s \in \mathbb{R}_{\text{ORE}}[\eta, \xi]$ *such that* $s(\tau, D) v_{p,\lambda} = v_{q,\lambda}$.

Proof: Define $s(\tau, D) := q(\tau) \frac{(-\lambda + D)^{\deg(p)}}{\alpha(\deg(p))!}$ where α is the coefficient of the highest term of p. Note that $(-\lambda + D)^{\deg(p)} p(\tau) = \alpha(\deg(p))!$. □

Lemma A.5. *Let* $\alpha_i, \beta_i, \lambda_i \in \mathbb{R}$ *with* $\alpha_i \neq 0$ *for* $i = 1, \cdots, \ell$ *, and* distinct *functions* $t^{n_i} e^{\lambda_i t}$ *be given. Then, there exists an* $r \in \mathbb{R}_{\text{ORE}}[\eta, \xi]$ *such that*

$$r(\tau, D)\alpha_i t^{n_i} e^{\lambda_i t} = \beta_i t^{n_i} e^{\lambda_i t} \tag{A.18}$$

for $i = 1, \ldots, \ell$.

Proof: The proof goes by induction on ℓ. For $\ell = 1$, $r := \beta_1/\alpha_1$ does the job. Suppose there exists an $r_{\ell-1}$ such that equation (A.18) holds for $i = 1, \ldots, \ell - 1$. Define $r_\ell := r_{\ell-1} + q r_{\text{add}}$ where $q, r_{\text{add}} \in \mathbb{R}_{\text{ORE}}[\eta, \xi]$ are chosen in the following way. We choose r_{add} such that ker $r_{\text{add}}(\tau, D) = \text{span}\{t^{n_i} e^{\lambda_i t}\}_{i=1}^{\ell-1}$ by invoking the scalar version of theorem 2.5. Using lemma A.4 we define q as the solution of the following equation

$$\alpha_\ell q(\tau) r_{\text{add}}(\tau, D) t^{n_\ell} e^{\lambda_\ell t} = \beta_\ell t^{n_\ell} e^{\lambda_\ell t} - \beta_\ell r_{\ell-1}(\tau, D) t^{n_\ell} e^{\lambda_\ell t}.$$

It can be verified that

$$r_\ell(\tau, D)(\alpha_i t^{n_i} e^{\lambda_i t}) = \beta_i t^{n_i} e^{\lambda_i t}$$

for $i = 1, \ldots, \ell$. □

We now turn to the proof of theorem 2.5. Let P_1 be a permutation matrix such that

$$WP_1 = \begin{bmatrix} \tilde{w}_{11} & \tilde{w}_{12} & \cdots & \tilde{w}_{1\ell_1} & 0 & 0 & \cdots & 0 \\ * & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & * & \cdots & * & * & * & \cdots & * \end{bmatrix}.$$

Lemma A.5 gives $r_2, r_3, \ldots, r_w \in \mathbb{R}_{\text{ORE}}[\eta, \xi]$ such that

$$\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -r_2 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -r_w & 0 & \cdots & 1 \end{bmatrix}}_{R_1} WP_1 = \begin{bmatrix} \tilde{w}_{11} & \tilde{w}_{12} & \cdots & \tilde{w}_{1\ell_1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & * & * & \cdots & * \end{bmatrix}.$$

By repeating this process, we find $R_1, R_2, \ldots, R_w \in \mathbb{R}^{w \times w}_{\text{ORE}}[\eta, \xi]$ and $P_1, P_2, \ldots, P_w \in \mathbb{R}^{w \times w}$ such that

$$R_w \cdots R_2 R_1 W P_1 P_2 \cdots P_w = \begin{bmatrix} \tilde{w}_1^T & 0 & \cdots & 0 \\ 0 & \tilde{w}_2^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{w}_w^T \end{bmatrix}.$$

If $\tilde{R} \in \mathbb{R}^{w \times w}_{\text{ORE}}[\eta, \xi]$ induces a kernel representation for the behavior spanned by the columns of the right hand side, then $R := \tilde{R} R_w \cdots R_2 R_1$ is a kernel representation for the behavior $\mathfrak{B}$. This completes the proof of theorem 2.5. □

A.4 Proofs for section 3

Proof of theorem 3.1

We first prove the case when $\mathfrak{B}$ is controllable. Since $\mathfrak{B}'$ is controllable, it admits an observable image representation, say $w = M_1(D)\ell$. Since $\mathfrak{B}' \subseteq \mathfrak{B}$, we can find an $M_2 \in \mathbb{R}^{w\times\bullet}[\xi]$ such that $\mathfrak{B} = \text{Im}[M_1(D)\ M_2(D)]$ and the representation is observable. By defining $\mathfrak{B}'' = \text{Im}[M_2(D)]$, we have that $\mathfrak{B} = \mathfrak{B}' \oplus \mathfrak{B}''$. Note that $\mathfrak{B}''$ is *not* unique

For the case when $\mathfrak{B}$ is not controllable, we first decompose $\mathfrak{B} = \mathfrak{B}_{\text{cont}} \oplus \mathfrak{B}_{\text{aut}}$ using proposition 3.1, and then it follows that $\mathfrak{B}' \subseteq \mathfrak{B}_{\text{cont}}$. We first find $\mathfrak{B}''' \in \mathfrak{L}^{w}_{D}$ such that $\mathfrak{B}' \oplus \mathfrak{B}''' = \mathfrak{B}_{\text{cont}}$ and then define $\mathfrak{B}'' := \mathfrak{B}''' \oplus \mathfrak{B}_{\text{aut}}$ for the required decomposition of $\mathfrak{B}$.

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