# Synthesis of strictly dissipative systems and the strictly suboptimal state space $\mathcal{H}_\infty$ control problem

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**Abstract**

In this short paper we study the problem of existence of a controlled behavior that is *strictly* dissipative with respect to a quadratic supply rate. The relation between strictness and the rank of a suitable qudratic differential form that couples the dissipativity properties of the hidden behavior and the orthogonal complement of the plant behavior is analyzed.

## 1 Introduction and notation

Recently, in [19] it was shown that, given a plant and a supply rate, the problem of designing a controller such that the interconnection is a dissipative system is equivalent to the problem of finding a behavior which satisfies the following three properties: (1) it is wedged in between the plant's hidden behavior and manifest behavior, (2) it is dissipative, and (3) its input cardinality is equal to the positive signature of the supply rate. In [19] necessary and sufficient conditions for the existence of such behavior were obtained. One of these conditions is a *coupling condition*, which requires that a certain quadratic differential form (called the *coupling QDF)*, coupling the dissipativity properties of the hidden behavior and manifest behavior, is non-negative. In this short paper we study the problem of how the coupling condition should be modified if, instead of a dissipative system behavior, we want to find a *strictly* dissipative behavior. We will show that in this case the coupling QDF should, in addition to being non-negative, have rank equal to the sum of the McMillan degrees of the hidden behavior and the manifest behavior.

The paper is structured as follows. In the rest of this section we introduce notation and review the most important behavioral definitions. The next section, section 2, contains the key notions concerning quadratic differential forms with an emphasis on their rank. We also prove a theorem about the rank of a QDF. This prepares the background for the subsequent section 3 which contains the main result of this paper. In section 4 we apply the main result to the classical strictly suboptimal state space $\mathcal{H}_\infty$ control problem. Finally, in section 5 we give some concluding remarks.

The notation that we use is standard. We use $\mathbb{R}$ to denote the field of real numbers and $\mathbb{C}$ to denote the complex plane. $\mathbb{R}^{\mathtt{n}}$ and $\mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_2}$ are the obvious extensions to vectors and matrices of the specified dimensions. We use $\mathbb{R}^{\bullet\times \mathtt{n}_2}$ when the context does not call for a specification of the row dimension (but just the column dimension) of the concerned matrix. We typically use the superscript $^{\mathtt{w}}$ (for example, $\mathbb{R}^{\mathtt{w}}$) when a generic element $w$ has $\mathtt{w}$ components. The ring of polynomials in the indeterminate $\xi$ with coefficients in $\mathbb{R}$ is denoted by $\mathbb{R}[\xi]$. $\mathbb{R}[\zeta,\eta]$ is the corresponding ring in two (commutative) indeterminates, and we use $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ and $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ to denote the sets of matrices with entries from the above rings, etc. The space of infinitely often differentiable functions with domain $\mathbb{R}$ and co-domain $\mathbb{R}^n$ is denoted by $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^n)$, and its subspace of compactly supported elements by $\mathfrak{D}(\mathbb{R},\mathbb{R}^n)$. The operator 'col' stacks its arguments into a column and is used for improving readability of matrix equations within text. We use $\text{rowdim}(M)$ to indicate the row dimension of a matrix $M$ and just $\dim(M)$ if $M$ is a vector or a square matrix.

A linear time-invariant differential system (or a behavior) is a subset $\mathfrak{B}\subseteq\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ such that, for some polynomial matrix $R\in\mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, we have $\mathfrak{B}=\{w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})\mid R(\frac{d}{dt})w=0\}$. We use $Ł^{\mathtt{w}}$ to denote the set of such behaviors. Here a behavior has been specified as the kernel of a differential operator. Hence we speak of this as a kernel representation of $\mathfrak{B}$. More generally, we might encounter a behavior as follows: for $R,\ M\ \in\mathbb{R}^{\bullet\times\bullet}[\xi]$,

$$\mathfrak{B}=\{w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})\mid\exists\ \ell\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}})\text{ such that }R(\frac{d}{dt})w=M(\frac{d}{dt})\ell\}\ .$$

It is a consequence of the *elimination theorem* that the set defined above is indeed a behavior in the sense we defined. A representation like the one above is called a *latent variable representation* (with $\ell$ as the latent variable here). The *full behavior* $\mathfrak{B}_{\text{full}}\in Ł^{\mathtt{w}+\mathtt{l}}$ is the set of all $(w,\ell)$ that satisfy the equation above.

In this paper, we restrict ourselves to controllable behaviors. Roughly speaking, controllable behaviors are defined as behaviors in which for any two of its elements there exists a third element which coincides with the first one on the past and the second one on the future (for details, see [8]). $Ł^{\mathtt{w}}_{\text{cont}}$ (a subset of $Ł^{\mathtt{w}}$) denotes this set of controllable behaviors. Given a behavior $\mathfrak{B}\in Ł^{\mathtt{w}}$, it is possible to choose some components of $w$ as any function in $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$. The maximal number of such components that can be chosen arbitrarily is called the *input cardinality* of $\mathfrak{B}$ and is denoted as $\mathtt{m}(\mathfrak{B})$. We also need the notion of state for a behavior. We refer to [9] for a detailed exposition, with only a brief review here. A latent variable representation of $\mathfrak{B}\in Ł^{\mathtt{w}}$ is called a state representation if the latent variable (denoted here by $x$ ) has the *property of state*, i.e.: if $(w_1,x_1),(w_2,x_2)\in\mathfrak{B}_{\text{full}}$ are such that $x_1(0)=x_2(0)$ then $(w_1,x_1)\wedge(w_2,x_2)$, the concatenation (at $t=0$, here), belongs to the $\mathfrak{L}_1^{\text{loc}}$-closure of $\mathfrak{B}_{\text{full}}$. We call such an $x$ a state for $\mathfrak{B}$. A state map for a $\mathfrak{B}$ is a differential operator $X(\frac{d}{dt})$ (induced by $X\in\mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$) such that $X(\frac{d}{dt})w$ is a state for $\mathfrak{B}$. A state map $X\in\mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ is minimal if every other state map has at least as many rows as those of $X$, and this minimal number of state variables (called the *McMillan degree*) of $\mathfrak{B}$ is denoted by $\mathtt{n}(\mathfrak{B})$. The rows of such an $X$ are linearly independent over $\mathbb{R}$. A minimal state map has a property called *trimness*, i.e., for all $x_0\in\mathbb{R}^{\mathtt{x}}$, there exists a $(w,X(\frac{d}{dt})w)\in\mathfrak{B}_{\text{full}}$ such that $(X(\frac{d}{dt})w)(0)=x_0$ . Issues concerning existence and constructive algorithms about state maps have been dealt in [9].

## 2 Quadratic differential forms

This section contains a brief review of bilinear differential forms, quadratic differential forms and other necessary notions like the rank of a QDF, etc. A bilinear form (BF) on the vector spaces $(\mathbb{V}_1, \mathbb{V}_2)$ is a mapping $\ell : \mathbb{V}_1 \times \mathbb{V}_2 \to \mathbb{R}$ that is linear in each of its two arguments. Given such an $\ell$, its *rank* is the number of independent linear functionals $\ell(\cdot, v_2)$ where $v_2$ ranges over $\mathbb{V}_2$, or equivalently the number of independent linear functionals $\ell(v_1, \cdot)$ where $v_1$ ranges over $\mathbb{V}_1$. When $\mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}$, a BF $\ell$ on $(\mathbb{V}, \mathbb{V})$ is called symmetric if $\ell(v_1, v_2) = \ell(v_2, v_1)$. Also, when $\mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}$, we speak of the quadratic form (QF) induced by $\ell$ on $\mathbb{V}$, defined by $q(v) := \ell(v, v)$. The rank of a QF is the rank of the symmetric BF that induces it. A QF $q$ on $\mathbb{V}$ can be expressed as $q = \sum_{k=1}^{n_+} |f_k^+(v)|^2 - \sum_{k=1}^{n_-} |f_k^-(v)|^2$ with the $f_k^+$'s and $f_k^-$'s linear functionals on $\mathbb{V}$, if (and only if) $q$ has finite rank. We can choose $f_1^+, f_2^+, \ldots, f_{n_+}^+, f_1^-, f_2^-, \ldots, f_{n_-}^-$ linearly independent over $\mathbb{R}$. In this case $n_-$ and $n_+$ are individually minimal over all such decompositions of $q$ as a sum and difference of squares. We call the corresponding pair of integers $(n_-, n_+)$ the *signature* of $q$ and denote it as $\operatorname{sign}(q) = (\sigma_-(q), \sigma_+(q))$. The rank of $q$ equals $\sigma_-(q) + \sigma_+(q)$.

The QF on $\mathbb{R}^n$ induced by the matrix $S = S^T \in \mathbb{R}^{n \times n}$ is defined as $q_S(x) := x^T S x$. We shall also use $|x|_S^2$ to denote it, and when $S = I$ the subscript is often dropped. We denote the signature of $S$ by $\operatorname{sign}(S) = (\sigma_-(S), \sigma_+(S))$ where $\sigma_-(S)$ and $\sigma_+(S)$ are respectively the number of negative and positive eigenvalues of $S$. Further, $\operatorname{sign}(S) = \operatorname{sign}(q_S)$. We have $\sigma_-(q_S) = 0 \Leftrightarrow q_S(x) \geq 0$ for all $x \in \mathbb{R}^n$. We call such a $q_S$ non-negative. Also, the usual definition of positive definiteness (of matrices) gives us that $\sigma_+(q_S) = \mathtt{n} \Leftrightarrow q_S(x) > 0$ for all $x \neq 0$.

We now move over to the notions of BDF's and QDF's. Let $\Phi \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta]$ be written out as a finite sum $\Phi(\zeta, \eta) = \Sigma_{k,l \in \mathbb{Z}_+} \Phi_{k\ell} \zeta^k \eta^\ell$ with $\Phi_{k\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}$ – its coefficient matrices. Let $\mathfrak{B}_1 \in \text{Ł}^{\mathtt{w}_1}$ and $\mathfrak{B}_2 \in \text{Ł}^{\mathtt{w}_2}$. Then, $\Phi$ induces the map $L_\Phi : \mathfrak{B}_1 \times \mathfrak{B}_2 \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$, defined by

$$L_\Phi(w_1, w_2) := \sum_{k,\ell \in \mathbb{Z}_+} (\frac{d^k w_1}{dt^k})^T \Phi_{k\ell} (\frac{d^\ell w_2}{dt^\ell})$$

called the *bilinear differential form (BDF)* on $\mathfrak{B}_1 \times \mathfrak{B}_2$ induced by $\Phi$ and which is denoted by $L_\Phi|_{\mathfrak{B}_1 \times \mathfrak{B}_2}$. When $\mathtt{w}_1 = \mathtt{w}_2 = \mathtt{w}$ and $\mathfrak{B} \in \text{Ł}^{\mathtt{w}}$, $\Phi$ also induces the map $Q_\Phi : \mathfrak{B} \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ with $Q_\Phi(w) := L_\Phi(w, w)$. We call this map the *quadratic differential form (QDF)* on $\mathfrak{B}$ induced by $\Phi$ and denote it by $Q_\Phi|_{\mathfrak{B}}$. Define the $^*$ operator as $(\Phi^*)(\zeta, \eta) := (\Phi(\eta, \zeta))^T$. When considering QDF's, it is sufficient to consider $\Phi$'s that are symmetric, i.e., those that satisfy $\Phi = \Phi^*$.

We are interested in non-negativity of QDF's on behaviors. For $f : A \to \mathbb{R}, f \geq 0$ means $f(t) \geq 0$ for all $t \in A$. We shall use this general definition of non-negativity for QDF's too. Let $\mathfrak{B} \in \text{Ł}^{\mathtt{w}}$ and $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$. We call the QDF $Q_\Phi$ non-negative on $\mathfrak{B}$ (and denote it by $Q_\Phi|_{\mathfrak{B}} \geq 0$) if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{B}$. Extending this notion of non-negativity of a QDF to positive definiteness the usual way, we say $Q_\Phi|_{\mathfrak{B}} > 0$ if for all $w \in \mathfrak{B}$: $Q_\Phi(w) \geq 0$ *and* $Q_\Phi(w) = 0$ implies that $w = 0$. Here $\mathfrak{B}$ is a subset of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ and in the special case

$\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, the subscript $\mathfrak{B}$ is skipped.

Let $\mathfrak{B}_1 \in \text{Ł}^{\mathtt{w}_1}$ and $\mathfrak{B}_2 \in \text{Ł}^{\mathtt{w}_2}$. There is a one-to-one correspondence between the BDF $L_\Phi$ on $\mathfrak{B}_1 \times \mathfrak{B}_2$ and the BF on $\mathfrak{B}_1 \times \mathfrak{B}_2$ defined by $(w_1, w_2) \mapsto L_\Phi(w_1, w_2)(0)$. Given $\mathfrak{B} \in \text{Ł}^{\mathtt{w}}$, there is a similar correspondence between the QDF $Q_\Phi$ on $\mathfrak{B}$ and the QF on $\mathfrak{B}$ defined by $w \mapsto Q_\Phi(w)(0)$. We define the ranks and signatures of a BDF or QDF by this correspondence. Although they act on infinite dimensional spaces, both $L_\Phi|_{\mathfrak{B}_1\times\mathfrak{B}_2}$ and $Q_\Phi|_{\mathfrak{B}}$ have finite rank. If $\mathfrak{B} \in \text{Ł}^{\mathtt{w}}$ and $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ then $\Phi$ can be expressed as $\Phi(\zeta, \eta) = F_+^T(\zeta)F_+(\eta) - F_-^T(\zeta)F_-(\eta)$, with $F = \text{col}(F_+, F_-) \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, such that the rows of $F$ induce (linear) functionals on $\mathfrak{B}$ that are linearly independent over $\mathbb{R}$. A factorization of $\Phi$ as the one above is called a canonical factorization on $\mathfrak{B}$. Such a factorization yields the signature and the rank of $Q_\Phi|_{\mathfrak{B}}$ by $\text{sign}(Q_\Phi|_{\mathfrak{B}}) = (\text{rowdim}(F_-), \text{rowdim}(F_+))$ and $\text{rank}(Q_\Phi|_{\mathfrak{B}}) = \text{rowdim}(F)$, and $Q_\Phi|_{\mathfrak{B}}$ can be expressed canonically as $Q_\Phi(w) = |F_+(\frac{d}{dt})w|^2 - |F_-(\frac{d}{dt})w|^2$. A formal exposition on QDF's can be found in [18].

Note the similarity of linear independence over $\mathbb{R}$ of the rows of $F$ and of those of a minimal state map. This similarity lies behind the very appealing result of [15]. We need a related property of a minimal state map which is also satisfied by other polynomial matrices under suitable assumptions. In this context we have the following theorem.

**Theorem 1. :** *Let $\mathfrak{B} \in \text{Ł}^{\mathtt{w}}$, $F \in \mathbb{R}^{\mathtt{q}\times\mathtt{w}}[\xi]$ and $K = K^T \in \mathbb{R}^{q\times q}$ and define $\Phi(\zeta, \eta) := F^T(\zeta)KF(\eta)$. Assume for $\eta \in \mathbb{R}^q$: $\eta^T F(\frac{d}{dt})\mathfrak{B} = 0 \Rightarrow \eta = 0$. Then we have $K > 0$ if and only if (1) $Q_\Phi|_{\mathfrak{B}} \geq 0$ and (2) $\text{rank}(Q_\Phi|_{\mathfrak{B}}) = \mathtt{q}$*

A close connection exists with the assumption in the theorem above and the notion of trimness. A behavior $\mathfrak{B} \in \text{Ł}^{\mathtt{w}}$ is called *trim* if for all $a \in \mathbb{R}^{\mathtt{w}}$, there exists $w \in \mathfrak{B}$ such that $w(0) = a$. It is possible to show that the property that for $\eta \in \mathbb{R}^q$: $\eta^T F(\frac{d}{dt})\mathfrak{B} = 0 \Rightarrow \eta = 0$ is equivalent to the trimness of the behavior $F(\frac{d}{dt})\mathfrak{B}$ (here $F(\frac{d}{dt})\mathfrak{B}$ is an element of $\text{Ł}^{\mathtt{q}}$). In theorem 1 above, $F$ need not be a state map. However, as mentioned above, if $F$ is a minimal state map of $\mathfrak{B}$ then the behavior $F(\frac{d}{dt})\mathfrak{B}$ is always trim.

# 3 Synthesis of strictly dissipative systems

Let $SS \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and $\mathfrak{B} \in \text{Ł}^{\mathtt{w}}_{\text{cont}}$. $\mathfrak{B}$ is said to be *dissipative* with respect to $Q_\Sigma$ (or briefly, $\Sigma$-dissipative) if $\int_{-\infty}^{+\infty} Q_\Sigma(w)\, dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. (In this case $Q_\Sigma(w)$ equals $w^T\Sigma w$.) Further, it is said to be dissipative on $\mathbb{R}_-$ with respect to $Q_\Sigma$ (or briefly, $\Sigma$-dissipative on $\mathbb{R}_-$) if $\int_{-\infty}^{0} Q_\Sigma(w)\, dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. We also use the analogous definition of dissipativity on $\mathbb{R}_+$. A controllable behavior $\mathfrak{B}$ is said to be *strictly dissipative* with respect to $Q_\Sigma$ (or briefly, strictly $\Sigma$-dissipative) if there exists an $\epsilon > 0$ such that $\mathfrak{B}$ is dissipative with respect to $Q_{\Sigma-\epsilon I}$. We have the obvious definitions for strict dissipativity on $\mathbb{R}_-$ and on $\mathbb{R}_+$. Equipped with these definitions, we state below the problem that we solve in this short paper.

**Strict dissipativity synthesis problem formulation:** Let $\mathcal{N}$ and $\mathcal{P} \in \text{Ł}^{\mathtt{v}}_{\text{cont}}$, and let $\Sigma = \Sigma^T \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}$ be non-singular. The problem is to find $\mathcal{K} \in \text{Ł}^{\mathtt{v}}_{\text{cont}}$ such that:

1. $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$ ,

2. $\mathcal{K}$ is strictly $\Sigma$-dissipative on $\mathbb{R}_-$ ,

3. $\mathtt{m}(\mathcal{K}) = \sigma_+(Q_\Sigma)$ .

The constraints that $\mathcal{K}$ has to satisfy have important control-theoretic interpretations. We call $\mathcal{P}$ the manifest plant behavior, $\mathcal{N}$ the hidden behavior and $\mathcal{K}$ the controlled behavior. The condition $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$ is equivalent to implementability of the controlled behavior through a restricted set of variables called control variables. The third condition formalizes the requirement that the controlled behavior should be *live* enough to accept sufficiently many exogenous inputs (which can be interpreted as disturbances). The strict $\Sigma$-dissipativity condition combines various control design specifications depending on $\Sigma$, for example, disturbance attenuation. The dissipativity on $\mathbb{R}_-$ implies stability. We refer to [19] for details. For additional material on strictly dissipative systems, see [6].

For a behavior $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}_{\text{cont}}$ and a $\Sigma = \Sigma^T \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, we say that $\Psi = \Psi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ induces a storage function $Q_\Psi$ for $\mathfrak{B}$ with respect to $Q_\Sigma$ if the dissipation inequality $\frac{d}{dt}Q_\Psi(w) \leq Q_\Sigma(w)$ is satisfied for all $w \in \mathfrak{B}$. It has been shown that such a storage function exists if and only if $\mathfrak{B}$ is $\Sigma$-dissipative. Moreover, $\mathfrak{B}$ is $\Sigma$-dissipative on $\mathbb{R}_-$ if and only if there exists a $\Psi$ such that $Q_\Psi|_\mathfrak{B} \geq 0$. Analogously, $\mathfrak{B}$ is $\Sigma$-dissipative on $\mathbb{R}_+$ if and only if there exists a $\Psi$ such that $Q_\Psi|_\mathfrak{B} \leq 0$. It is also known (see for instance, [15]) that such a storage function is always a state function, i.e., if $X \in \mathbb{R}^{\mathtt{n}\times\mathtt{w}}[\xi]$ induces a state map for $\mathfrak{B}$, then associated with this $\Psi$ there exists a $K \in \mathbb{R}^{n\times n}$ such that $Q_\Psi(w) = |X(\frac{d}{dt})w|^2_K$. Thus we often speak of the matrix associated with a storage function (and a state map). A storage function is not unique. However, there exists a maximal and a minimal one between which every other storage function lies. We denote the largest and the smallest storage functions by $\Psi^+$ and $\Psi^-$, and their associated matrices by $K^+$ and $K^-$ respectively. Further, corresponding to each storage function, we have a *dissipation function* which is the QDF $Q_\Delta$ defined by $Q_\Delta(w) := Q_\Sigma(w) - \frac{d}{dt}Q_\Psi(w)$ for all $w \in \mathfrak{B}$.

Given a BDF induced by a constant matrix we have a notion of the orthogonal complement of a controllable behavior with respect to this BDF. Let $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}^{\mathtt{w}}_{\text{cont}}$; $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are said to be *orthogonal with respect to $L_\Sigma$ (briefly, $\Sigma$-orthogonal)* if $\int_{-\infty}^{+\infty} L_\Sigma(w_1, w_2)dt = 0$ for all $w_1 \in \mathfrak{B}_1 \cap \mathfrak{D}$ and $w_2 \in \mathfrak{B}_2 \cap \mathfrak{D}$. This orthogonality relation between $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is denoted by $\mathfrak{B}_1 \perp_\Sigma \mathfrak{B}_2$. For $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}_{\text{cont}}$ we define the $\Sigma$-orthogonal complement $\mathfrak{B}^{\perp_\Sigma}$ of $\mathfrak{B}$ as

$$\mathfrak{B}^{\perp_\Sigma} := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \int_{-\infty}^{+\infty} L_\Sigma(w, w')dt = 0 \text{ for all } w' \in \mathfrak{B} \cap \mathfrak{D}\}.$$

When $\Sigma = I$, we use $\perp$ instead of $\perp_\Sigma$. The following identities are easily verified: $\mathfrak{B}^{\perp_\Sigma} = (\Sigma\mathfrak{B})^\perp = ((\Sigma^T)^{-1})\mathfrak{B}^\perp$. (Here $^{-1}$ denotes set-theoretic inverse.) Further, if $\Sigma$ is nonsingular, $\mathfrak{B} = (\mathfrak{B}^{\perp_\Sigma})^{\perp_\Sigma}$. In the context of behaviors that are $\Sigma$-orthogonal we have the following result.

**Proposition 2. :** *Let $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}^{\mathtt{w}}_{\text{cont}}$. There exists a $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that $\frac{d}{dt}L_\Psi(w_1,w_2) = w_1^T\Sigma w_2$ for all $(w_1,w_2) \in \mathfrak{B}_1 \times \mathfrak{B}_2$ if and only if $\mathfrak{B}_1 \perp_\Sigma \mathfrak{B}_2$. Moreover, $\Psi$ is essentially unique, i.e., if $\Psi_1, \Psi_2 \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ both satisfy the above equality, then $L_{\Psi_1}(w_1,w_2) = L_{\Psi_2}(w_1,w_2)$ for all $(w_1,w_2) \in \mathfrak{B}_1 \times \mathfrak{B}_2$.*

We call this BDF $L_\Psi$ on $\mathfrak{B}_1 \times \mathfrak{B}_2$, the $[(\mathfrak{B}_1,\mathfrak{B}_2);\Sigma]$-adapted bilinear differential form. Here also $L_\Psi$ can be written as a function of the states of $\mathfrak{B}_1$ and $\mathfrak{B}_2$, i.e., given $X_1$ and $X_2$ that induce minimal state maps for $\mathfrak{B}_1$ and for $\mathfrak{B}_2$ respectively, there exists a matrix $L \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}_1)\times\mathtt{n}(\mathfrak{B}_2)}$ such that $L_\Psi(w_1,w_2) = (X_1(\frac{d}{dt})w_1)^T L X_2(\frac{d}{dt})w_2$ . For the case of $\Sigma = I$ and for behaviors $\mathfrak{B}$ and $\mathfrak{B}^\perp$, $L$ happens to be invertible and we can modify one of the two (minimal) state maps to obtain a *matched pair* of state maps. $(X,Z)$ is said to be a matched pair of minimal state maps for $(\mathfrak{B},\mathfrak{B}^\perp)$ if $\frac{d}{dt}(X(\frac{d}{dt})w_1)^T Z(\frac{d}{dt})w_2 = w_1^T w_2$ for all $(w_1,w_2) \in \mathfrak{B}\times\mathfrak{B}^\perp$. More on this can be found in [18] (section 10).

We are now ready to state the main result of the paper, which is a solution to the strict dissipativity problem described above. Since $\mathcal{N} \subset \mathcal{P}$, we have that $\mathcal{N} \perp_\Sigma \mathcal{P}^{\perp_\Sigma}$. Let $\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ induce the $[(\mathcal{N},\mathcal{P}^{\perp_\Sigma});\Sigma]$-adapted BDF. It turns out that the existence of a controlled behavior $\mathcal{K}$ as described in our problem formulation involves, in addition to a non-negativity requirement, a rank condition on the coupling QDF.

**Theorem 3. :** *A controlled behavior $\mathcal{K} \in \mathcal{L}^{\mathtt{v}}_{\text{cont}}$ as described in the problem formulation exists if and only if the following conditions are satisfied:*

1. *$\mathcal{N}$ is strictly $\Sigma$-dissipative on $\mathbb{R}_-$,*
2. *$\mathcal{P}^{\perp_\Sigma}$ is strictly $(-\Sigma)$-dissipative on $\mathbb{R}_+$,*
3. *the* coupling QDF *$Q_{\text{cpl}}$ on $\mathcal{N}\times\mathcal{P}^{\perp_\Sigma}$ defined by:*

$$Q_{\text{cpl}}(v_1,v_2) := Q_{\Psi^+_{\mathcal{N}}}(v_1) - Q_{\Psi^-_{\mathcal{P}^{\perp_\Sigma}}}(v_2) + 2L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}}(v_1,v_2) \tag{1}$$

*satisfies the following two properties:*

*(i)* $Q_{\text{cpl}}|_{\mathcal{N}\times\mathcal{P}^{\perp_\Sigma}} \geq 0$ *and*

*(ii)* $\boxed{\operatorname{rank}(Q_{\text{cpl}}|_{\mathcal{N}\times\mathcal{P}^{\perp_\Sigma}}) = \mathtt{n}(\mathcal{N}) + \mathtt{n}(\mathcal{P})\ .}$

*Here, $\Psi^+_{\mathcal{N}}$ induces the largest storage function for $\mathcal{N}$ as a $\Sigma$-dissipative system and $\Psi^-_{\mathcal{P}^{\perp_\Sigma}}$ induces the smallest storage function for $\mathcal{P}^{\perp_\Sigma}$ as a $(-\Sigma)$-dissipative system.*

We note here the importance of the last statement in the theorem above. Since $Q_{\text{cpl}}$ is a sum of three terms that are themselves functions of the states of the behaviors concerned, it cannot have rank *more than* $\mathtt{n}(\mathcal{N}) + \mathtt{n}(\mathcal{P})$. So the existence of a strictly dissipative controlled behavior as in the problem formulation, in fact, requires the existence of a non-negative coupling QDF of *maximal rank*. It is in this way that the strictness of the dissipativity in the problem formulation affects the theorem. But unlike here, the McMillan degrees of the hidden behavior and the plant behavior played no role in the nonstrict synthesis result of [19].

# 4 Application to the state space strictly suboptimal $\mathcal{H}_\infty$ control problem

We will now apply theorem 3 to the special case that the the behaviors $\mathcal{N}$ and $\mathcal{P}$ are given as the hidden behavior and manifest plant behavior associated with a given to be controlled plant $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{v}+\mathtt{c}}$, i.e.

$$\mathcal{N} = \{v \mid (v,0) \in \mathcal{P}_{\text{full}}\}$$

and

$$\mathcal{P} = \{v \mid \text{ there exists } c \text{ such that } (v,c) \in \mathcal{P}_{\text{full}}\}.$$

We will assume that the plant $\mathcal{P}_{\text{full}}$ is given in input/state/output representation. Our results on the general problem set-up will lead to a solution for the state space case, analogous to those on the standard $\mathcal{H}_\infty$ problem obtained in [1]. This double Riccati equation solution and its variations have been the subject of very intensive research, see e.g. [5, 14] and generalizations in [10, 11], [6, 7], [3, 4] and [2].

Assume the plant $\mathcal{P}_{\text{full}}$ is given in input/state/output representation by

$$\begin{array}{rclllll} \frac{d}{dt}x & = & Ax & + & Bu & + & Gd \\ y & = & Cx & + & & + & Dd \\ f & = & Hx & + & Ju & & \end{array} \tag{2}$$

The state variable $x$ is assumed to take its values in $\mathbb{R}^{\mathtt{n}}$. The following three regularity conditions are assumed to hold:

A.1: $D$ is surjective and $J$ is injective,

A.2: $(A - GD^T(DD^T)^{-1}C, G(I_{\mathtt{d}} - D^T(DD^T)^{-1}D))$ is a controllable pair of matrices,

A.3: $(A - B(J^TJ)^{-1}J^TH, (I_{\mathtt{f}} - J(J^TJ)^{-1}J^T)H)$ is an observable pair of matrices.

The problem is to find a controller acting on the control variables $(u,y)$ such that the controlled system meets certain specifications. We want the controller to be also in state representation, more exactly, in input/state/output representation, with $y$ the input, $u$ the output, and with the controller state denoted as $x_c$:

$$\begin{array}{rclll} \frac{d}{dt}x_c & = & A_cx_c & + & B_cy \\ u & = & C_cx_c & + & D_cy \end{array} \tag{3}$$

Our aim is to derive conditions for the existence and algorithms for the computation of the controller parameter matrices $(A_c, B_c, C_c, D_c)$ such that the controlled system has the following properties:

1. *strict disturbance attenuation* with gain factor normalized to 1, i.e., for all $(d,f) \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{f}})$ for which there exist $(u,y,x,x_c)$ satisfying both the plant equations (2) and the controller equations (3), there should hold $||f||_{\mathcal{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{f}})} < ||d||_{\mathcal{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{d}})}$;

2. *internal stability*, meaning that in the controlled system $d = 0$ should imply that the signals $(x, x_c, u, f)$ all go to zero as $t \to \infty$.

Note that conditions 1 and 2 are equivalent to the condition that the controlled system is internally stable and has transfer function $G_{d\mapsto f}$ satisfying $\|G_{d\mapsto f}\|_{\mathcal{H}_\infty} < 1$.

In terms of the notation used in the previous section, we have $v = (d, f)$ as the to-be-controlled variables, $c = (u, y)$ as the control variables, and $\Sigma = \mathrm{diag}(I_{\mathtt{d}}, -I_{\mathtt{f}})$ as the weighting matrix. In this section, hence, $\Sigma = \mathrm{diag}(I_{\mathtt{d}}, -I_{\mathtt{f}})$.

Assume now that a feedback controller (3) exists that achieves strict disturbance attenuation and internal stability. This leads to a controlled behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}$, represented in i/s/o representation by

$$\mathcal{K} = \{(d, f) \mid \text{ there exist } x, x_c, u \text{ and } y \text{ such that (2) and (3) hold}\}.$$

This system is internally stable and its transfer matrix $G_{d\mapsto f}$ satisfies $\|G_{d\mapsto f}\|_{\mathcal{H}_\infty} < 1$. It is clear that $\mathtt{m}(\mathcal{K}) = \mathtt{d} = \sigma_+(\Sigma)$ and $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$ (since $\mathcal{K}$ is implemented by a controller acting on the control variable $(u, y)$). Now let $\mathcal{K}_{\text{cont}}$ be the controlable part of $\mathcal{K}$. Since $\mathcal{K}_{\text{cont}}$ has the same transfer matrix as $\mathcal{K}$ (which satisfies $\|G_{d\mapsto f}\|_{\mathcal{H}_\infty} < 1$), we have that $\mathcal{K}_{\text{cont}}$ is strictly dissipative on $\mathbb{R}_-$. Also, $\mathtt{m}(\mathcal{K}_{\text{cont}}) = \mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$ and $\mathcal{N} = \mathcal{N}_{\text{cont}} \subset \mathcal{K}_{\text{cont}} \subset \mathcal{P}_{\text{cont}} = \mathcal{P}$. Thus we find that $\mathcal{K}_{\text{cont}}$ satisfies the three conditions of the strict dissipativity synthesis problem, and we immediately conclude that the conditions of theorem 3 must hold. We will investigate which form these conditions take for the plant $\mathcal{P}_{\text{full}}$ given in i/s/o representation (2).

We first derive the various behaviors that are involved. In particular, for the full plant behavior $\mathcal{P}_{\text{full}}$ represented by (2), we will derive specific representations for the manifest plant behavior $\mathcal{P}$ and its $\Sigma$-orthogonal complement $\mathcal{P}^{\perp_\Sigma}$, and the hidden behavior $\mathcal{N}$. Subsequently, we will derive conditions such that $\mathcal{P}$ and $\mathcal{N}$ satisfy the conditions of theorem 3.

Eliminating $(u, y)$ from (2), yields the following driving variable representation for $\mathcal{P}$:

$$\begin{aligned} \frac{d}{dt}x_{\mathcal{P}} &= Ax_{\mathcal{P}} + \begin{bmatrix} B & G \end{bmatrix} \begin{bmatrix} d'_{\mathcal{P}} \\ d''_{\mathcal{P}} \end{bmatrix}, \\ v_{\mathcal{P}} &= \begin{bmatrix} 0 \\ H \end{bmatrix} x_{\mathcal{P}} + \begin{bmatrix} 0 & I \\ J & 0 \end{bmatrix} \begin{bmatrix} d'_{\mathcal{P}} \\ d''_{\mathcal{P}} \end{bmatrix}, \end{aligned}$$

Putting $(u, y) = (0, 0)$ in (2) yields the following output nulling representation for $\mathcal{N}$:

$$\begin{aligned} \frac{d}{dt}x_{\mathcal{N}} &= Ax_{\mathcal{N}} + \begin{bmatrix} G & 0 \end{bmatrix} \begin{bmatrix} v'_{\mathcal{N}} \\ v''_{\mathcal{N}} \end{bmatrix}, \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} C \\ H \end{bmatrix} x_{\mathcal{N}} + \begin{bmatrix} D & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} v'_{\mathcal{N}} \\ v''_{\mathcal{N}} \end{bmatrix}. \end{aligned}$$

Assumptions A.2 and A.3 ensure that $\mathcal{N}$ and $\mathcal{P}$ are controllable. Moreover, their state space representations obtained above are controllable and observable.

From the relations between an output nulling representation and driving variable representation of a behavior and its orthogonal complement (see section 6 of [19]), we obtain the following output nulling representation for $\mathcal{P}^{\perp_\Sigma}$:

$$\begin{aligned}\frac{d}{dt}z_{\mathcal{P}} &= -A^T z_{\mathcal{P}} + \begin{bmatrix}0 & -H^T\end{bmatrix}\begin{bmatrix}v'_{\mathcal{P}^{\perp_\Sigma}}\\ v''_{\mathcal{P}^{\perp_\Sigma}}\end{bmatrix},\\ \begin{bmatrix}0\\0\end{bmatrix} &= \begin{bmatrix}B^T\\G^T\end{bmatrix}z_{\mathcal{P}} + \begin{bmatrix}0 & J^T\\ -I & 0\end{bmatrix}\begin{bmatrix}v'_{\mathcal{P}^{\perp_\Sigma}}\\ v''_{\mathcal{P}^{\perp_\Sigma}}\end{bmatrix}.\end{aligned}$$

The next step, after computing these behaviors, is to verify $\Sigma$-dissipativity of $\mathcal{N}$ on $\mathbb{R}_-$ and $(-\Sigma)$-dissipativity of $\mathcal{P}^{\perp_\Sigma}$ on $\mathbb{R}_+$. Using [6], theorem 5.3.4, it can be shown that $\mathcal{N}$ is strictly $\Sigma$-dissipative on $\mathbb{R}_-$ if and only if the algebraic Riccati equation

$$-A^T K_{\mathcal{N}} - K_{\mathcal{N}}A - H^T H - K_{\mathcal{N}}GG^T K_{\mathcal{N}} + (C^T + K_{\mathcal{N}}GD^T)(DD^T)^{-1}(C + DG^T K_{\mathcal{N}}) = 0 \quad (4)$$

has a real symmetric solution $K_{\mathcal{N}} > 0$ such that

$$-A + GD^T(DD^T)^{-1}(C + DG^T K_{\mathcal{N}}) - GG^T K_{\mathcal{N}}$$

has all its eigenvalues in the open left half complex plane $\mathbb{C}^-$. Similarly, strict $(-\Sigma)$-dissipativity on $\mathbb{R}_+$ of $\mathcal{P}^{\perp_\Sigma}$ is equivalent to the existence of a real symmetric $K_{\mathcal{P}} \in \mathbb{R}^{n\times n}$ such that the algebraic Riccati equation

$$AK_{\mathcal{P}} + K_{\mathcal{P}}A^T - GG^T - K_{\mathcal{P}}H^T HK_{\mathcal{P}} + (B - K_{\mathcal{P}}H^T J)(J^T J)^{-1}(B^T - J^T HK_{\mathcal{P}}) = 0 \quad (5)$$

has a real symmetric solution $K_{\mathcal{P}} < 0$ such that

$$A^T - H^T J(J^T J)^{-1}(B^T - J^T HK_{\mathcal{P}}) - H^T HK_{\mathcal{P}}$$

has all its eigenvalues in the open right half complex plane $\mathbb{C}^+$. In fact, by using well-known properties of the algebraic Riccati equation, we may conclude that the largest storage function of $\mathcal{N}$ as a $\Sigma$-dissipative system is equal to $x_{\mathcal{N}}^T K_{\mathcal{N}}^+ x_{\mathcal{N}}$, where $K_{\mathcal{N}}^+$ is the largest real symmetric solution of the ARE (4). This solution $K_{\mathcal{N}}^+$ satisfies $K_{\mathcal{N}}^+ > 0$ and $-A + GD^T(DD^T)^{-1}(C + DG^T K_{\mathcal{N}}^+) - GG^T K_{\mathcal{N}}^+$ has all its eigenvalues in $\mathbb{C}^-$. Similarly, the smallest storage function of $\mathcal{P}^{\perp_\Sigma}$ as a $(-\Sigma)$-dissipative system is equal to $z_{\mathcal{P}}^T K_{\mathcal{P}}^- z_{\mathcal{P}}$, where $K_{\mathcal{P}}^-$ is the smallest real symmetric solution of the ARE (5). This solution satisfies $K_{\mathcal{P}}^- < 0$ and $A^T - H^T J(J^T J)^{-1}(B^T - J^T HK_{\mathcal{P}}^-) - H^T HK_{\mathcal{P}}^-$ has all its eigenvalues in $\mathbb{C}^+$.

We now study the coupling QDF $Q_{\text{cpl}}$. Since $\frac{d}{dt}x_{\mathcal{N}}^T z_{\mathcal{P}} = v_{\mathcal{N}}^T \Sigma v_{\mathcal{P}}$ it is clear that the $[(\mathcal{N}, \mathcal{P}^{\perp_\Sigma}); \Sigma]$-adapted BDF can represented as $x_{\mathcal{N}}^T z_{\mathcal{P}}$, so the coupling QDF is given by

$$\left[\begin{array}{c} x_{\mathcal{N}} \\ z_{\mathcal{P}} \end{array}\right]^T \left[\begin{array}{cc} K_{\mathcal{N}}^+ & I \\ I & -K_{\mathcal{P}}^- \end{array}\right] \left[\begin{array}{c} x_{\mathcal{N}} \\ z_{\mathcal{P}} \end{array}\right]. \quad (6)$$

Condition 3 of theorem 3 then requires the matrix

$$\left[\begin{array}{cc} K_{\mathcal{N}}^+ & I \\ I & -K_{\mathcal{P}}^- \end{array}\right] \quad (7)$$

to be non-negative definite, and its rank should be equal to $\mathtt{n}(\mathcal{N})+\mathtt{n}(\mathcal{P})$. Since both McMillan degrees are equal to $\mathtt{n}$, the dimension of the state space of (2), condition 3 of theorem 3 is equivalent to the requirement that the matrix (7) is *positive definite*. This positive definiteness can be seen to be equivalent to the combined conditions

1. $K_{\mathcal{N}}^{+} > 0$,
2. $K_{\mathcal{P}}^{-} < 0$,
3. $K_{\mathcal{N}}^{+} > (-K_{\mathcal{P}}^{-})^{-1}$.

The last condition is easily seen to be equivalent to $\rho(K_{\mathcal{N}}^{+}K_{\mathcal{P}}^{-}) > 1$, where $\rho$ denotes the spectral radius.

We conclude that the following conditions are necessary for the existence of an internally stabilizing, strictly disturbance attenuating feedback controller:

1. both the algebraic Riccati equations (4) and (5) have at least one real symmetric solution,
2. the largest solution $K_{\mathcal{N}}^{+}$ of (4) is positive definite, and the smallest solution $K_{\mathcal{P}}^{-}$ of (5) is negative definite,
3. $-A + GD^T(DD^T)^{-1}(C + DG^TK_{\mathcal{N}}^{+}) - GG^TK_{\mathcal{N}}^{+}$ has all its eigenvalues in $\mathbb{C}^-$, $A^T - H^TJ(J^TJ)^{-1}(B^T - J^THK_{\mathcal{P}}^{-}) - H^THK_{\mathcal{P}}^{-}$ has all its eigenvalues in $\mathbb{C}^+$,
4. $K_{\mathcal{N}}^{+} > (-K_{\mathcal{P}}^{-})^{-1}$.

It can be proven that this set of conditions is also *sufficient* for the existence of an internally stabilizing, strictly disturbance attenuating feedback controller and formulas for such controllers can be given analogous to those obtained for the non-strict problem in [17]. We note that our conditions are equivalent to those obtained in [1]. In fact, by pre- and post-multiplying (4) and (5) by $(K_{\mathcal{N}}^{+})^{-1}$ and $(K_{\mathcal{P}}^{-})^{-1}$, respectively, we find that $P := -(K_{\mathcal{P}}^{-})^{-1}$ and $Q := (K_{\mathcal{N}}^{+})^{-1}$ satisfy the 'mixed sign' algebraic Riccati equations

$$\begin{aligned} A^TP + PA + PGG^TP - (PB + H^TJ)(J^TJ)^{-1}(B^TP + J^TH) + H^TH &= 0, \\ AQ + QA^T + QH^THQ - (QC^T + GD^T)(DD^T)^{-1}(CQ + DG^T) + GG^T &= 0, \end{aligned}$$

that $A + QH^TH - (QC^T + GD^T)(DD^T)^{-1}C$ and $A + GG^TP - B(J^TJ)^{-1}(B^TP + J^TH)$ have all their eigenvalues in $\mathbb{C}^-$, and that $P > 0$, $Q > 0$, and $Q^{-1} > P$.

# 5 Conclusions and remarks

As expected, the solution to the strictly dissipative synthesis problem differs from that of the nonstrict synthesis result of [19]. We have shown that it is the coupling QDF (which was just non-negative in the nonstrict case) that has to be suitably strict, namely, it should have maximal rank. In this context the McMillan degrees of the hidden behavior and the plant behavior also come into picture. But while a minimal state map for a behavior is not unique, the McMillan degree of a behavior does not depend on a particular representation. We remark that both the problem formulation and the main theorem let themselves be treated in a representation-free manner. This makes it possible to apply theorem 3 when the to-be-controlled plant is given by any particular representation, like for example in input/state/output representation. Analogously as in [17, section 5], by applying theorem 3 we re-obtained for this case the well-known conditions in terms of two Riccati equations and a coupling condition that first appeared in [1], and was later studied in various forms in, for example, [12, 13].

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