Mean-variance Portfolio Selection under Markov Regime: Discrete-time Models and Continuous-time Limits

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Abstract

In this paper, we propose a discrete-time model for mean-variance portfolio selection. One of the distinct features is that the system under consideration is a Markov modulated system. We show that under suitable conditions and scaling, the process of interest goes to a switching diffusion limit. Related issues on optimal strategies and efficient frontier will also be mentioned.

1 Introduction

The purpose of portfolio selection is to find an optimal strategy for allocating wealth among a number of securities. The mean-variance approach initiated in [11, 12] sets up a basis for portfolio selection in a single period. It may be regarded as a multi-objective optimization task, namely, to maximize the terminal wealth and to minimize the risk using the variance as a criterion, which stems from the investors goal of seeking highest return upon specifying their acceptable risk level.

Owing to its practical value, the mean-variance model has drawn continuing attention; see for example, [15, 18, 8, 5, 6, 7, 17, 4, 19] among others. Recently, using the stochastic LQ theory developed in [2], a stochastic linear-quadratic (LQ) control framework for studying mean-variance optimization/hedging problems was introduced in [27], along with a closed-form solution of the optimal portfolio policy and an explicit expression of the efficient frontier for a continuous-time mean-variance portfolio selection problem. To better reflect the market trends and other economic factors, following the approach in [26] (see also related work [1, 3, 25]), we have considered hybrid mean-variance problems, in which the appreciation rate and the volatility depend on a continuous-time Markov chain in [28]. Together with the optimal selection strategy, the efficient frontiers have also been found. Such a hybrid model enables us to have the coexistence of continuous dynamics and discrete events.

In view of the recent advances in mean-variance portfolio selection and hybrid geometric Brownian motion formulation, this work develops a class of discrete-time mean-variance portfolio selection models. For some of the recent development of multi-period, discrete-time portfolio selection problems, see [10], in which optimal strategy was derived together with the efficient frontier. One of the salient features of the problem we propose to study is its expected appreciation rate and volatility are modulated by a discrete-time Markov chain that has a finite state space. The consideration of discrete-time model is because, very often, one needs to deal with discrete-time problems especially owing to the use of digital computers. In addition, to solve many continuous-time problems, one needs to use a discretization technique leading to a discrete-time problem formulation.

With various economic factors being taken into consideration, the state space of the Markov chain is likely to be large. To reduce the complexity, we observe that the transition rates among different states could be quite different. In fact, there is certain hierarchy involved. To highlight the different rates of changes, we introduce a small parameter $\varepsilon > 0$ into the transition matrix resulting in the so-called nearly decomposable model. Then the underlying problem becomes one involving a singular perturbation formulation. Based on the recent progress of singularly perturbed Markov chains (see [20, 23]), we establish the natural connection of the discrete-time problem and its continuous-time limit. Under simple conditions, we show that suitably interpolated processes converges weakly to their limit leading to a hybrid continuous-time mean-variance portfolio selection problem.

The limit mean-variance portfolio selection problem has an optimal solution as was obtained in [28]. Using that solution, we can construct policies that are asymptotically optimal. Our findings indicate that in lieu of examining the more complex original problem, we could use the much simplified limit problem as a guide to obtain portfolio selection policies that are nearly as good as the optimal one from a practical concern. Furthermore, we can also develop near optimality regarding mutual fund theorem and one-fund theorem, and obtain nearly efficient frontier. The advantage of our approach is that the complexity is much reduced. Although mean-variance control problems are treated in this paper, the formulation and techniques can also be employed in hybrid control problems that are modulated by a Markov chain for many other applications.

The paper is arranged as follows. Section 2 gives the formulation of the discrete-time mean-variance portfolio selection problem. Section 3 presents weak convergence results establishing the connection of the discrete-time and continuous-time models. Section 4 concludes paper with additional remarks.

2 Formulation

Let α_k^ε, for $0 \le k \le T/\varepsilon$, be a discrete-time Markov chain with finite state space $\mathcal{M}$. Suppose that there are $d+1$ assets. One of which is the bond and the rest of them are the stock holdings. Use $S_k^{\varepsilon,0}$ to denote the price of the bond, and $S_k^{\varepsilon,\beta}$, $\beta = 1, \dots, d$, to denote the prices of the stocks at time k, respectively. Then $S_k^{\varepsilon,\beta}$ satisfies the following system of equations:

$$\begin{cases} S_{k+1}^{\varepsilon,0} = S_k^{\varepsilon,0} + \varepsilon r(\varepsilon k, \alpha_k^\varepsilon) S_k^{\varepsilon,0} \\ S_0^{\varepsilon,0} = S^0 > 0, \\ S^{\varepsilon,\beta} = S_k^{\varepsilon,\beta} + \varepsilon b^\beta(\varepsilon k, \alpha_k^\varepsilon) S_k^{\varepsilon,\beta} + \sqrt{\varepsilon} \displaystyle\sum_{\gamma=1}^{d} \sigma^{\beta\gamma}(\varepsilon k, \alpha_k^\varepsilon) \xi_k^\gamma S_k^{\varepsilon,\beta}, \ \beta = 1, \dots, d, \\ S_0^{\varepsilon,\beta} = S^\beta > 0, \end{cases} \tag{2.1}$$

where $r(\cdot,\cdot)$, $b^\beta(\cdot,\cdot)$, $\sigma^{\beta\gamma}(\cdot,\cdot) : \mathbb{R} \times \mathcal{M} \mapsto \mathbb{R}$, for $\beta, \gamma = 1, \dots, d$, are some appropriate functions to be specified later and $\{\xi_k^\beta\}$ $\beta = 1, \dots, d$ are sequences of independent and identically distributed random variables. Note that $r(\cdot)$ represents the interest rate, $b^\beta(\cdot)$ is the return rate, and $\sigma^{\beta\gamma}(\cdot)$ is the volatility. Note that in our model, both the interest rate and the volatility depend on the Markov chain.

At time instant k, the investor's portfolio selection is based on the prior information up to time $k-1$. Thus his or her wealth, $x^\varepsilon_k = \sum_{i=0}^d N^\beta(\varepsilon(k-1))S^{\varepsilon,\beta}_k$ for $0 \le k \le T/\varepsilon$, satisfies

$$\begin{aligned}
x^\varepsilon_{k+1} &= \sum_{\beta=0}^d N^\beta(\varepsilon k) S^{\varepsilon,\beta}_{k+1} \\
&= [S^{\varepsilon,0}_k + \varepsilon r(\varepsilon k, \alpha^\varepsilon_k) S^{\varepsilon,0}_k] + S^{\varepsilon,\beta}_k + \varepsilon \sum_{\beta=1}^d N^\beta(\varepsilon k) b^\beta(\varepsilon k, \alpha^\varepsilon_k) S^{\varepsilon,\beta}_k \\
&\quad + \sqrt{\varepsilon} \sum_{\beta=1}^d N^\beta(\varepsilon k) \sum_{\gamma=1}^d \sigma^{\beta\gamma}(\varepsilon k, \alpha^\varepsilon_k) \xi^\gamma_k S^{\varepsilon,\beta}_k \\
&= x^\varepsilon_k + \varepsilon r(\varepsilon k, \alpha^\varepsilon_k) x^\varepsilon_k + \varepsilon \sum_{\beta=1}^d [b^\beta(\varepsilon k, \alpha^\varepsilon_k) - r(\varepsilon k, \alpha^\varepsilon_k)] u^{\varepsilon,\beta}_k \\
&\quad + \sqrt{\varepsilon} \sum_{\beta=1}^d \sum_{\gamma=1}^d \sigma^{\beta\gamma}(\varepsilon k, \alpha^\varepsilon_k) u^{\varepsilon,\beta}_k \xi^\gamma_k,
\end{aligned} \tag{2.2}$$

where $u^{\varepsilon,\beta}_k = N^\beta(\varepsilon k) S^{\varepsilon,\beta}_k$ for $\beta = 0, \dots, d$.

Denote by $\mathcal{F}^\varepsilon_k$, the σ-algebra generated by $\{\alpha^\varepsilon_{k_1}, \xi_{k_1} : 0 \le k_1 < k\}$, where $\xi_k = (\xi^1_k, \dots, \xi^d_k)'$, and z' denotes the transpose of z. A portfolio $u^\varepsilon_\cdot$ is admissible if u^ε_k is $\mathcal{F}^\varepsilon_k$-measurable for each $0 \le k \le T/\varepsilon$ and (2.2) has a unique solution x^ε_k corresponding to u^ε_k. We also call $(x^\varepsilon_k, u^\varepsilon_k)$ an admissible wealth portfolio pair. Denote the class of admissible portfolios by $\mathcal{A}^\varepsilon$.

Our objective is to find an admissible portfolio $(x^\varepsilon_k, u^\varepsilon_k) \in \mathcal{A}^\varepsilon$ such that the terminal wealth is $Ex^\varepsilon_{T/\varepsilon} = z$ for some given $z \in \mathbb{R}$ and the risk in terms of the variance of the terminal wealth $E[x^\varepsilon_{T/\varepsilon} - z]^2$ is minimized. In [28], treating a class of continuous-time hybrid mean-variance control problems, we formulated the problem directly as a constrained stochastic optimization problem. Following such an approach, we also formulate the current problem as a constrained optimization problem:

$$\begin{cases} \text{Minimize } J^\varepsilon(x, i, u^\varepsilon(\cdot)) = E[x^\varepsilon_{T/\varepsilon} - z]^2 \\ \text{subject to: } Ex^\varepsilon_{T/\varepsilon} = z \;\text{ and }\; (x^\varepsilon_k, u^\varepsilon_k) \in \mathcal{A}^\varepsilon. \end{cases} \tag{2.3}$$

3 Asymptotic Results

To proceed, we make the following assumptions.

(A1) The transition matrix of α^ε_k is given by

$$P^\varepsilon = P + \varepsilon Q, \tag{3.1}$$

where

$$P = \text{diag}(P^1, \dots, P^l) \tag{3.2}$$

such that P^i for $i = 1, \dots, l$ are transition matrix, and Q is a generator (i.e., $q^{\ell\ell_1} \ge 0$ for $\ell \ne \ell_1$ and $\sum_{\ell_1 \in \mathcal{M}} q^{\ell\ell_1} = 0$ for each $\ell \in \mathcal{M}$). Moreover, P^i, $i = 1, \dots, l$ are irreducible and aperiodic.

(A2) For each $\ell \in \mathcal{M}$, $\beta, \gamma = 1, \dots, d$, $r(\cdot, \ell)$, $b^\beta(\cdot, \ell)$, $\sigma^{\beta\gamma}(\cdot, \ell)$ are real-valued continuous functions defined on $[0, T]$.

(A3) For each $\beta = 1, \ldots, d$, $\{\xi_k^\beta\}$ is a sequence of independent and identically distributed (i.i.d.) random variables that are independent of α_k^ε and that have mean 0 and variance 1. For $\beta \neq \gamma$, ξ_k^β and ξ_k^γ are independent.

To carry out the analysis, define an aggregated process $\overline{\alpha}_k^\varepsilon$ by

$$\overline{\alpha}_k^\varepsilon = i \ \text{ if } \ \alpha_k^\varepsilon \in \mathcal{M}_i.$$

Next define the interpolated processes for $t \in [\varepsilon k, \varepsilon k + \varepsilon)$,

$$\begin{aligned} &S^{\varepsilon,\beta}(t) = S_k^{\varepsilon,\beta}, \ u^{\varepsilon,\beta}(t) = u_k^{\varepsilon,\beta}, \\ &x^\varepsilon(t) = x_k^\varepsilon, \ \text{ and } \ \overline{\alpha}^\varepsilon(t) = \overline{\alpha}_k^\varepsilon. \end{aligned} \tag{3.3}$$

A number of results concerning the asymptotic properties of the discrete-time Markov are available; see [21] (see also the related work [20, 23]). In particular, we will need: For $k \leq T/\varepsilon$, the k-step transition probability matrix $(P^\varepsilon)^k$ satisfies

$$(P^\varepsilon)^k = \Phi(t) + O\left(\varepsilon + \lambda^k\right), \tag{3.4}$$

where

$$\begin{aligned} &\Phi(t) = \widetilde{1\!\!1}\Theta(t)\text{diag}(\nu^1, \ldots, \nu^l) \\ &\frac{d\Theta(t)}{dt} = \Theta(t)\overline{Q}, \quad \Theta(0) = I \\ &\overline{Q} = \text{diag}(\nu^1, \ldots, \nu^l)Q\widetilde{1\!\!1}. \end{aligned} \tag{3.5}$$

In addition, as $\varepsilon \to 0$, $\overline{\alpha}^\varepsilon(\cdot)$ converges weakly to $\overline{\alpha}(\cdot)$, which is a continuous-time Markov chain with state space $\overline{\mathcal{M}} = \{1, \ldots, l\}$ and generator $\overline{Q}$. Moreover, for the occupation measures defined by

$$o_{k,ij}^\varepsilon = \varepsilon \sum_{k_1=0}^{k} [I_{\{\alpha_{k_1}^\varepsilon = \zeta_{ij}\}} - \nu_j^i I_{\{\alpha_k^\varepsilon \in \mathcal{M}_i\}}], \quad \text{for} \ \ i = 1, \ldots, l, \ j = 1, \ldots, m_i, \ 0 \leq k \leq T/\varepsilon$$

the following mean square estimates hold

$$\sup_{0 \leq k \leq T/\varepsilon} E|o_{k,ij}^\varepsilon|^2 = O(\varepsilon). \tag{3.6}$$

Next, we present the main limit results of this paper. The detailed proofs can be found in [22].

Proposition 3.1 *Under assumptions* (A1)–(A3)*, for each* $\beta = 0, \ldots, d$, $\{S^{\varepsilon,\beta}(\cdot)\}$ *is tight on* $D[0,T]$*, where* $D[0,T]$ *is the space of functions that are right continuous, have left limits, endowed with the Skorohod topology.*

Proposition 3.2 *Assume* (A1)–(A3)*. Then for each* $\beta = 0, \ldots, d$, $(S^\beta(\cdot), \overline{\alpha}(\cdot))$*, the limit of* $(S^{\varepsilon,\beta}(\cdot), \overline{\alpha}^\varepsilon(\cdot))$ *is the unique solution of the martingale problem with operator* $(\partial/\partial t) + \mathcal{L}^\beta$*, given by*

$$\begin{aligned} &\mathcal{L}^0 f(t,y,i) = f_y(t,y,i)\overline{r}(t,i)y, \\ &\mathcal{L}^\beta f(t,y,i) = f_y(t,y,i)\overline{b}^\beta(t,i)y + \frac{1}{2}f_{yy}(t,y,i)\text{tr}[\overline{\Sigma}(t,i)\overline{\Sigma}'(t,i)]y^2 \\ &\qquad\qquad\qquad + \overline{Q}f(t,y,\cdot)(i), \ 1 \leq \beta \leq d, \end{aligned} \tag{3.7}$$

where for each i, $f(\cdot,\cdot,i)$ *is a suitable function defined on* $\mathbb{R} \times \mathbb{R}$, $\text{tr}(B)$ *denotes the trace of the matrix* B.

Proposition 3.3. *Under the conditions of Theorem* 3.2, $(x^\varepsilon(\cdot), u^\varepsilon(\cdot))$ *converges weakly to* $(x(\cdot), u(\cdot))$ *that belongs to* $\mathcal{A}$ *and that is a solution of*

$$\begin{cases} \text{minimize} \quad J(x, i, u(\cdot)) = E[x(T) - z]^2 \\ \text{subject to:} \quad Ex(T) = z \text{ and } (x(\cdot), u(\cdot)) \in \mathcal{A}. \end{cases} \tag{3.8}$$

4 Further Remarks

This work has focused on a class of mean-variance portfolio selection problems. We proposed a class of discrete-time models that are modulated by a Markov chain taking into consideration of Market trend and other factors. We used nearly completely decomposable transition matrices and weak convergence methods to derive the limit mean-variance portfolio selection problems.

When the limit system is obtained, one can design optimal control and derive efficient frontiers of the limit system using the framework of linear quadratic control with indefinite control weights [2]. Then using the optimal control of the limit system, one can construct controls of the original system and show that such controls lead to near optimality.

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