Simultaneous Robust Regulation and Robust Stabilization with Degree Constraint

Ryozo Nagamune
Division of Optimization and Systems Theory
Royal Institute of Technology
SE 100 44 Stockholm, Sweden

Abstract

This paper characterizes all controllers to a problem of simultaneous robust regulation and robust stabilization, which was left open by Cevik and Schumacher. The characterization is based on a combination of their results, i.e., the controller parameterization of robust regulation with nominal internal stability and the stability margin given by an internally stabilizing controller. The controller set will be represented in terms of a solution set to the boundary Nevanlinna-Pick interpolation problem. It is shown that a certain degree restriction on the solution set leads to a reasonable degree bound of controllers. With this degree bound, an efficient algorithm for calculating each controller is available. Using the freedom in the controller set, other performance specifications than robust regulation and robust stability may be satisfied without increasing the controller degree.

1 Introduction

The regulation problem, which encompasses both disturbance rejection and reference tracking, is one of the most fundamental problems in control theory. Thorough investigations to this problem for linear systems have been done by means of geometric control theory [2, 20]. Robustness of the regulation property has also been considered there. The central result for robust regulation is the *internal model principle*, asserting that, to achieve robust regulation, controllers must incorporate suitably reduplicated dynamics of the exosystem [9]. On the other hand, robust stability is significant from the practical viewpoint. For several types of model sets, the design methods of robust stabilizing controllers have been developed [10, 12, 19].

Simultaneous robust regulation and robust stabilization ($RRRS$), which is one of the multi-objective control problem, was considered in several papers in different settings [1, 7, 15, 17]. Among these, this paper is concerned with a series of papers by Cevik and Schumacher [4, 5, 6, 7]. All the controllers which satisfy robust regulation with nominal internal stability have been parameterized in [6, Proposition 6.2], in a form relevant to the Youla parameterization. In addition, assuming the gap metric as a measure for plant uncertainty, they have given a necessary and sufficient solvability condition for $RRRS$, by constructing a controller for $RRRS$ based on Nehari extension and boundary Nevanlinna-Pick interpolation theory. However, as has been mentioned in the conclusions of [6], the controller construction in the approach is laborious and an efficient software is necessary to be developed. Further, because of the lack of the characterization of all controllers for $RRRS$, it will be difficult to incorporate other specifications than $RRRS$ by this approach, even though some effort toward the incorporation has been made in [6, Lemma 6.4]. The objective of this paper is to cover the drawbacks mentioned above for scalar systems.

We will show that, in the scalar case, the problem of finding controllers for $RRRS$ amounts to the boundary Nevanlinna-Pick interpolation problem. Thus, we can characterize all the controllers for $RRRS$ by means of Nevanlinna-Pick interpolation theory, without involving Nehari extension theory. The expression of all controllers for $RRRS$ involves some "initial" controller and a free but stable polynomial. However, with some normalization assumptions, it will turn out that the values of an interpolant at interpolation points depend only on the given plant and the given exosystem, and the controller depends only on the plant and the interpolant. This fact is not straight-forward from the Nevanlinna-Pick formulation of controller expression for $RRRS$ that we will derive. Consequently, our first main result is that *we express the controller set for RRRS with only given plant and exosystem.*

It is often not enough that the closed-loop system meets the conditions for $RRRS$, and other specifications such as frequency shaping and pole placement are usually required. In practice, we do not want to increase the degree of controller even if these extra requirements are involved. For this purpose, it is better to obtain a set of controllers whose degree is bounded appropriately. Our second main result is that *we obtain a set of controllers fulfilling RRRS conditions as well as appropriate degree constraint.* This is done by the theory of Nevanlinna-Pick interpolation with degree constraint in [3]. Since an efficient algorithm is available for calculating each interpolant with a degree bound [14], the software problem mentioned in the conclusions of [6] will be resolved.

2 System description

In this paper, we shall treat exactly the same structure of systems as the one considered in [4, 5, 6, 7], but only for scalar cases. Consider the following two finite-dimensional linear time-invariant continuous-time systems.

$$
\begin{array}{ll}
\text{(S)} & \left\{ \begin{array}{rcl} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = & \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u \\ y & = & \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{array} \right. \\
\text{(C)} & \left\{ \begin{array}{rcl} \dot{z} & = & Fz + Gy \\ u & = & Hz + Jy \end{array} \right.
\end{array}
\tag{2.1}
$$

The system (S) is a given single-input single-output (SISO) system, and the system (C) is a SISO controller to be designed for some control objectives. The system (S) is divided into two parts, i.e., a "plant" with its state vector x_1 of dimension n_1 and an "exosystem" with its state vector x_2 of dimension n_2. The block diagram of these equations is depicted in Figure 1, where

$$C(s) := J + H(sI - F)^{-1}G. \tag{2.2}$$

Since the state vector x_2 in the exosystem cannot be affected by the input u, it can be interpreted physically as disturbances and/or references. Although the dynamics of the exosystem is assumed to be known in this paper, there are some applications where this assumption is valid (see [13] and the references therein). It can be seen in the figure that the signals generated by the exosystem are added at the plant state and at the plant output.

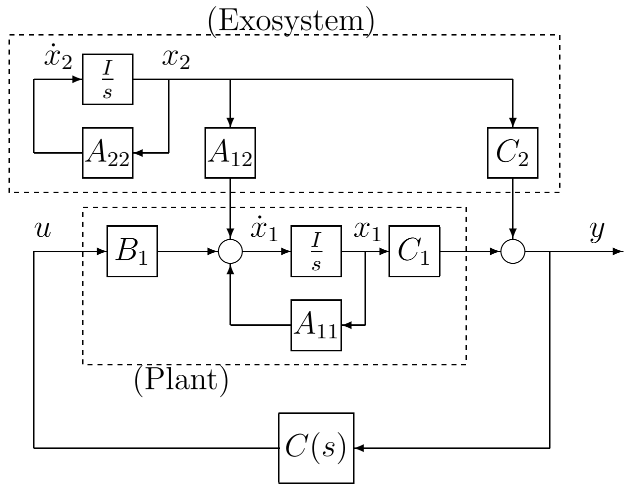


Figure 1: The feedback system with an exosystem

Now, we set several assumptions on the system, which are valid throughout this paper.

Assumption 2.1. The system (2.1) satisfies the following assumptions.

1. (A_{11}, B_1) is stabilizable.
2. $\left(\begin{bmatrix} C_1 & C_2 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \right)$ is detectable.
3. All the eigenvalues of A_{22} lie on the imaginary axis.
4. The matrix $\begin{bmatrix} \lambda I - A_{11} & -B_1 \\ C_1 & 0 \end{bmatrix}$ is full rank for any eigenvalue λ of A_{22}.

These assumptions are standard, and the motivations on these assumptions are explained in [4, 9, 20]. In particular, the last assumption is known as the necessary and sufficient solvability condition for the robust regulation problem with internal stability, which will be formulated later. (see [7, Theorem 3.3]).

Next, for the use in the following sections, we review the coprime factorization and introduce some expressions of the plant $P(s) := C_1(sI - A_{11})^{-1}B_1$, and the controller $C(s)$ in

(2.2). Since the plant P and the controller C are both scalar transfer functions, they can respectively be written as a ratio of two coprime polynomials, and also as a ratio of two coprime[1] stable rational functions:

$$P(s) =: \frac{p_n(s)}{p_d(s)} = \frac{p_n(s)/d_p(s)}{p_d(s)/d_p(s)}, \quad C(s) =: \frac{c_n(s)}{c_d(s)} = \frac{c_n(s)/d_c(s)}{c_d(s)/d_c(s)}, \tag{2.3}$$

where $p_n(s)$, $c_d(s)$ etc. are polynomials with

$$\deg d_p = \deg p_d \quad \text{and} \quad \deg d_c = \deg c_d. \tag{2.4}$$

The latter expressions in (2.3) are called *coprime factorization in RH_∞* (the class of real rational proper stable functions), and has been widely employed in control theory (see e.g. [18]). For given P and C, the coprime factorization is essentially unique, that is, unique up to unimodular functions in RH_∞ (see [18, p. 75]).

Using the factors in (2.3), we define several vectors as

$$\mathtt{P}(s) := \frac{1}{d_p(s)} \begin{bmatrix} p_n(s) \\ p_d(s) \end{bmatrix}, \quad \tilde{\mathtt{P}}(s) := \frac{1}{d_p(s)} \begin{bmatrix} p_d(s) & -p_n(s) \end{bmatrix}, \tag{2.5}$$

$$\mathtt{C}(s) := \frac{1}{d_c(s)} \begin{bmatrix} c_d(s) \\ c_n(s) \end{bmatrix}, \quad \tilde{\mathtt{C}}(s) := \frac{1}{d_c(s)} \begin{bmatrix} c_n(s) & -c_d(s) \end{bmatrix}. \tag{2.6}$$

In this paper, we refer to $\mathtt{P}(s)$ ($\mathtt{C}(s)$) and $\tilde{\mathtt{P}}(s)$ ($\tilde{\mathtt{C}}(s)$) as the *plant (controller) vector* and the *left annihilator of the plant (controller) vector* respectively, and these vectors will play an essential role to present main results. Note that $\tilde{\mathtt{P}}(s)\mathtt{P}(s) \equiv 0$ and $\tilde{\mathtt{C}}(s)\mathtt{C}(s) \equiv 0$. For given P and C, the vectors $\mathtt{P}$ and $\mathtt{C}$ are also determined essentially uniquely due to the essential uniqueness of the coprime factorization. Conversely, for a given 2×1 rational vector $\mathtt{C}$, we can obtain a unique controller $C = c_n/c_d$ by removing the least common multiple of denominators of the two rational functions.

Later, we will often be interested in the *normalized coprime factorization in RH_∞* of the plant P, i.e., the coprime factorization meeting $\mathtt{P}^*(s)\mathtt{P}(s) = 1$, $\tilde{\mathtt{P}}(s)\tilde{\mathtt{P}}^*(s) = 1$, where $\mathtt{P}^*(s) := \mathtt{P}^T(-s)$. In this case, a stable polynomial $d_p(s)$ is chosen to satisfy

$$p_n(s)p_n(-s) + p_d(s)p_d(-s) = d_p(s)d_p(-s).$$

3 Robust regulation with robust stability

This section reviews the results in [4, 6, 7], which will be necessary for our results that will follow. We shall present independently the robust stability condition and the robust regulation condition that the controller vectors $\mathtt{C}$ must satisfy.

First, to formulate robust stability of the feedback system (S) and (C) in (2.1), we need to introduce the concept of internal stability for the nominal system.

[1]Two stable rational functions are called *coprime* if there is no common zero in the extended right half-plane [18].

Definition 3.1. The closed-loop system (2.1) is *internally stable* if, for $x_2 \equiv 0$, the plant state x_1 and the controller state z go to zero asymptotically for any initial condition.

Given a plant vector $\mathtt{P}$ and its annihilator vector $\tilde{\mathtt{P}}$, the internal stability condition for the controller vectors $\mathtt{C}$ is given by the following lemma.

Lemma 3.1. *[4, Lemma 2.1][5, p. 332] The following statements are equivalent.*

1. *The closed-loop system (S) and (C) is internally stable.*
2. *The function $\tilde{\mathtt{P}}(s)\mathtt{C}(s)$ is RH_∞ unimodular.*
3. *The following complementarity condition holds:*

$$G(\mathtt{P},\mathtt{C}) := \min_{s\in\mathbb{C}^+} \sin\phi(\mathrm{Im}\mathtt{P}(s), \mathrm{Im}\mathtt{C}(s)) > 0,$$

where $\mathbb{C}^+ := \{s : \ \mathrm{Re}\, s \geq 0\} \cup \{\infty\}$, and for each $s \in \mathbb{C}^+$,

$$\sin\phi(\mathrm{Im}\mathtt{P}(s), \mathrm{Im}\mathtt{C}(s)) := \min\left\{\|p - c\| : \ p \in \mathrm{Im}\mathtt{P}(s), \ c \in \mathrm{Im}\mathtt{C}(s), \ \|p\| = 1\right\}.$$

Here, $\mathrm{Im}\, v$ denotes the image subspace spanned by the vector v.

Due to this lemma, for robustness of stability, it is natural to require $G(\mathtt{P},\mathtt{C}) > \gamma$ for a given value $\gamma \in (0, 1)$ which indicates the stability margin. In fact, this condition implies that the controller obtained by the controller vector $\mathtt{C}$ internally stabilizes all the plants in the set with center the plant obtained by $\mathtt{P}$ and with radius γ measured in the *gap metric* (see [4] and [16]). Formally, the set of all controller vectors for robust stability with margin γ ($RS(\gamma)$) is written by

$$\mathcal{S}_{RS(\gamma)} := \left\{\mathtt{C} \in RH_\infty^{2\times 1} : \ G(\mathtt{P},\mathtt{C}) > \gamma\right\}. \tag{3.7}$$

For given normalized $\mathtt{P}$ and $\tilde{\mathtt{P}}$, if we scale an internally stabilizing $\mathtt{C}$ to satisfy $\tilde{\mathtt{P}}(s)\mathtt{C}(s) \equiv 1$, the computation of $G(\mathtt{P},\mathtt{C})$ can be simplified as $G(\mathtt{P},\mathtt{C}) = \|\mathtt{C}\|_\infty^{-1}$ (see [4]). Using this simple expression of $G(\mathtt{P},\mathtt{C})$, for normalized $\mathtt{P}$ and $\tilde{\mathtt{P}}$, define a subset of $\mathcal{S}_{RS(\gamma)}$:

$$\hat{\mathcal{S}}_{RS(\gamma)} := \left\{\mathtt{C} \in RH_\infty^{2\times 1} : \ \tilde{\mathtt{P}}\mathtt{C} = 1, \ G(\mathtt{P},\mathtt{C}) = \|\mathtt{C}\|_\infty^{-1} > \gamma\right\}. \tag{3.8}$$

It should be noted that the two sets of controllers corresponding to two controller vector sets $\mathcal{S}_{RS(\gamma)}$ and $\hat{\mathcal{S}}_{RS(\gamma)}$ are identical. Therefore, the set of controller vectors meeting robust stability condition can be simply written as in (3.8).

Next, we show the controller vector set which gives controllers for robust regulation with internal stability ($RRIS$) [6]. To this end, we first explain the property of $RRIS$.

Definition 3.2. The system (2.1) satisfies *robust regulation with internal stability* ($RRIS$) requirement if the nominal system is internally stable and if the signal y goes to zero asymptotically for some neighborhood of the data (A_{11}, A_{12}, B_1) in $\mathbb{R}^{n_1\times n_1} \times \mathbb{R}^{n_1\times n_2} \times \mathbb{R}^{n_1\times 1}$.

Existence of controllers for $RRIS$ is guaranteed by the standing assumptions. The set of all controller vectors $\mathtt{C}$ for $RRIS$ has been obtained in [6] as follows.

Proposition 3.1. *[6, Proposition 6.2] Let* $\mathtt{P}$ *be a given plant vector. Then, all the controller vectors* $\mathtt{C}$ *for RRIS can be represented as*

$$\mathcal{S}_{RRIS} := \left\{ \mathtt{C} \in RH_\infty^{2\times 1} : \ \mathtt{C}(s) := \mathtt{C}_0(s) - \mathtt{P}(s)H(s)\Psi(s), \ \Psi \in RH_\infty \right\}, \tag{3.9}$$

where $\mathtt{C}_0$ *is a particular controller vector for RRIS,* $H \in RH_\infty$ *is biproper (both* H *and* H^{-1} *are proper) and has all the zeros exactly at the roots of the minimal polynomial of* A_{22}*, and* Ψ *is arbitrary in* RH_∞*.*

Note that there is no assumption of normalization in the proposition. Later, we will concentrate on a subset of $\mathcal{S}_{RRIS}$ with some normalization assumptions. Also, keep in mind that, in the expression (3.9), we need to determine some “initial” controller vector $\mathtt{C}_0$ and the denominator of H. One may wonder how to find them and which choice of them is appropriate. However, we will show in the next section that, in a problem of simultaneous robust regulation and robust stabilization, the controller vector set can be represented in a form independent of $\mathtt{C}_0$ and the stable denominator of H.

Finally, based on the two concepts given above, the robust regulation with robust stability was described in [6, 7], with its solvability condition, as follows.

Definition 3.3. For a given $\gamma \in (0,1)$, the system (2.1) satisfies *robust regulation with robust stability margin* γ *(*$RRRS(\gamma)$*)* if the system satisfies both $RS(\gamma)$ and $RRIS$.

Theorem 3.1. *[7, Theorem 3.6 and Proposition 3.8] Under the standing assumptions, the necessary and sufficient condition for the problem of* $RRRS(\gamma)$ *to be solvable is*

$$\gamma < \min \left\{ \min_{\lambda \in \sigma(A_{22})} \left| \frac{p_n(\lambda)}{d_p(\lambda)} \right|, \sqrt{1 - \|\Gamma_{\tilde{\mathtt{P}}^*}\|_H^2} \right\}, \tag{3.10}$$

where $\sigma(A_{22})$ *is the spectrum of* A_{22}*, and* $\|\Gamma_{\tilde{\mathtt{P}}^*}\|_H$ *is the Hankel norm of normalized* $\tilde{\mathtt{P}}^*$*.*

This theorem has been proven in a way of constructing a controller vector for $RRRS(\gamma)$, with both the Nehari extension and the boundary Nevanlinna-Pick interpolation. However, the construction by hand is apparently laborious, as can be seen in the example in [6]. In the next section, by only using the boundary Nevanlinna-Pick interpolation theory, we characterize all the controller vectors for $RRRS(\gamma)$ with γ satisfying (3.10).

4 All controller vectors to $RRRS(\gamma)$

Our idea to derive a characterization of all controller vectors that solve the problem of $RRRS(\gamma)$ is simply to take the intersection of $\hat{\mathcal{S}}_{RS(\gamma)}$ and $\mathcal{S}_{RRIS}$. However, because of normalization conditions in $\hat{\mathcal{S}}_{RS(\gamma)}$, we have to focus on a subset of $\mathcal{S}_{RRIS}$. For this purpose, the next lemma plays an important role.

Lemma 4.1. *If RRIS is solvable, then there exists a controller vector* $\mathtt{C}_0$ *which solves the RRIS problem and fulfills* $\tilde{\mathtt{P}}(s)\mathtt{C}_0(s) \equiv 1$.

For controller vectors $\mathtt{C}$ in $\mathcal{S}_{RRIS}$, the relation $\tilde{\mathtt{P}}\mathtt{C} = \tilde{\mathtt{P}}\mathtt{C}_0$ holds, due to $\mathtt{C} = \mathtt{C}_0 - \mathtt{P}H\Psi$ and $\tilde{\mathtt{P}}\mathtt{P} = 0$. Hence, by selecting one $\mathtt{C}_0$ such that $\tilde{\mathtt{P}}\mathtt{C}_0 = 1$, we can focus on a subset of $\mathcal{S}_{RRIS}$:

$$\hat{\mathcal{S}}_{RRIS} := \left\{ \mathtt{C} \in \mathcal{S}_{RRIS} : \ \tilde{\mathtt{P}}\mathtt{C} = 1 \right\}.$$

Then, the controller vectors for $RRRS(\gamma)$ are characterized in the following theorem.

Theorem 4.1. *Let* $\mathtt{P}$ *and* $\tilde{\mathtt{P}}$ *be the normalized plant vector and the normalized left annihilator of* $\mathtt{P}$ *respectively. Also, let* $\mathtt{C}_0 \in RH_\infty^{2\times 1}$ *be a particular controller vector for RRIS which satisfies* $\tilde{\mathtt{P}}(s)\mathtt{C}_0(s) = 1$. *Denote by* H *a biproper function in* RH_∞ *whose numerator is the minimal polynomial of* A_{22}. *Then, the controller vectors which give all the controllers for* $RRRS(\gamma)$ *can be characterized by*

$$\begin{aligned}\mathcal{S}_{RRRS(\gamma)} &:= \hat{\mathcal{S}}_{RRIS} \cap \hat{\mathcal{S}}_{RS(\gamma)} \\ &= \left\{ \mathtt{C}(s) := \mathtt{C}_0(s) - \mathtt{P}(s)H(s)\Psi(s), \ \Psi \in RH_\infty, \ \|B(\mathtt{P}^*\mathtt{C}_0 - H\Psi)\|_\infty < \frac{\sqrt{1-\gamma^2}}{\gamma} \right\},\end{aligned}$$

where B *is an inner function in* RH_∞ *which cancels all the unstable poles of* $\mathtt{P}^*$ *and does not introduce any extra zero.*

Proof. By the assumptions in this theorem, we can substitute $\mathtt{C} = \mathtt{C}_0 - \mathtt{P}H\Psi$ into (3.8):

$$\mathcal{S}_{RRRS(\gamma)} = \left\{ \mathtt{C} := \mathtt{C}_0 - \mathtt{P}H\Psi, \ \Psi \in RH_\infty, \ \|\mathtt{C}_0 - \mathtt{P}H\Psi\|_\infty^{-1} > \gamma \right\},$$

since $\tilde{\mathtt{P}}\mathtt{C} = 1$ for any $\Psi \in RH_\infty$. If we take $\tilde{\mathtt{P}}$ as a normalized plant left annihilator vector, then the square matrix function $\begin{bmatrix} B\mathtt{P}^*(s)^T & \tilde{\mathtt{P}}(s)^T \end{bmatrix}^T$ is inner. Therefore, using the invariance of the H_∞ norm with respect to the left multiples of inner functions, the robust stability condition can be transformed as follows:

$$\gamma < \|\mathtt{C}_0 - \mathtt{P}H\Psi\|_\infty^{-1} = \left\| \begin{bmatrix} B\mathtt{P}^* \\ \tilde{\mathtt{P}} \end{bmatrix} (\mathtt{C}_0 - \mathtt{P}H\Psi) \right\|_\infty^{-1} = (1 + \|B(\mathtt{P}^*\mathtt{C}_0 - H\Psi)\|_\infty^2)^{-1/2}.$$

From the last expression, we obtain

$$\|B(\mathtt{P}^*\mathtt{C}_0 - H\Psi)\|_\infty < \frac{\sqrt{1-\gamma^2}}{\gamma}, \tag{4.11}$$

which completes the proof. □

Next, we will show that finding the RH_∞ functions Ψ which meet (4.11) amounts to solving the boundary Nevanlinna-Pick interpolation problem. Define a set of Ψ characterizing $\mathcal{S}_{RRRS(\gamma)}$ by

$$\mathcal{S}_\Psi := \left\{ \Psi \in RH_\infty : \ \|B\mathtt{P}^*\mathtt{C}_0 - BH\Psi\|_\infty < \frac{\sqrt{1-\gamma^2}}{\gamma} \right\}. \tag{4.12}$$

Denote the unstable zeros of BH as z_i, $i = 1, \dots, n_p + n_h$, which are actually the poles of $\mathtt{P}^*$ and of the exosystem. Here, we have introduced notation

$$n_p := \deg P(\le n_1) \text{ and } n_h := \deg H(\le n_2). \tag{4.13}$$

Note that all the zeros of B are in the open right half-plane, and that those of H are on the imaginary axis. In addition, define

$$\begin{gathered} T_1(s) := B(s)\mathtt{P}^*(s)\mathtt{C}_0(s) \in RH_\infty, \quad T_2(s) := B(s)H(s) \in RH_\infty, \\ T(s) := T_1(s) - T_2(s)\Psi(s). \end{gathered} \tag{4.14}$$

Then, the set $\mathcal{S}_\Psi$ can be regarded as a solution set of the *model matching problem* (see [8]). Obviously, $\Psi \in RH_\infty$ implies $T \in RH_\infty$. Conversely, for the function Ψ to be in RH_∞, the function T must satisfy the following interpolation conditions: $T(z_i) = T_1(z_i)$, $i = 1, \dots, n_p + n_h$, if all the zeros are distinct, while if a zero z_i with multiplicity n is contained, then we have instead $T^{(k)}(z_i) = T_1^{(k)}(z_i)$, $k = 0, 1, \dots, n-1$, where the superscript (k) means the k-th derivative.

Consequently, supposing for simplicity that the unstable zeros of BH are distinct, the set $\mathcal{S}_\Psi$ in (4.12) is bijective, with respect to the map $T = T_1 - T_2\Psi$, to the set

$$\mathcal{T} := \left\{ T \in RH_\infty : \ T(z_i) = T_1(z_i), \ i = 1, \dots, n_p + n_h, \ \|T\|_\infty < \frac{\sqrt{1-\gamma^2}}{\gamma} \right\}. \tag{4.15}$$

The set $\mathcal{T}$ is exactly the real rational solution set of the *boundary Nevanlinna-Pick interpolation problem*. The solvability condition for the problem is well-known (see e.g. [11]).

In calculating controller vectors $\mathtt{C}$ by this approach, it seems at the first glance that we need to specify some "initial" controller vector $\mathtt{C}_0$ for *RRIS* meeting $\tilde{\mathtt{P}}\mathtt{C}_0 = 1$ and some stable denominator of the function H. However, this is not the case, and we can get each $\mathtt{C}$ in the set $\mathcal{S}_{RRRS(\gamma)}$ directly, using neither $\mathtt{C}_0$ nor the denominator of H. This is shown by the following two lemmas. First lemma states that the values of T_1 at the interpolation points do not depend on the choice of $\mathtt{C}_0$ in Theorem 4.1, even if the function T_1 in (4.14) does.

Lemma 4.2. *Suppose that $z_i, i = 1, \dots, n_p + n_h$ are distinct. Then, $T_1(z_i)$ is obtained by*

$$T_1(z_i) = \begin{cases} -\frac{p_d^*(z_i)}{p_n(z_i)}, & \text{if } p_n(z_i) \neq 0, \\ \frac{p_n^*(z_i)}{p_d(z_i)}, & \text{if } p_n(z_i) = 0. \end{cases}$$

If z_i has multiplicity n, then $T_1^{(k)}(z_i)$, $k = 0, 1, \dots, n-1$ are given by

$$T_1^{(k)}(z_i) = \begin{cases} -\left(\frac{p_d^*}{p_n}\right)^{(k)}(z_i), & \text{if } p_n(z_i) \neq 0, \\ \left(\frac{p_n^*}{p_d}\right)^{(k)}(z_i), & \text{if } p_n(z_i) = 0. \end{cases}$$

Proof. We will use the same notation as in (2.5) and $\mathtt{C}_0(s) := \frac{1}{d_{c0}(s)}\begin{bmatrix} c_d(s) \\ c_n(s) \end{bmatrix}$. By the assumptions $\mathtt{P}^*\mathtt{P} = 1$ and $\tilde{\mathtt{P}}\mathtt{C}_0 = 1$, we have

$$p_n p_n^* + p_d p_d^* = d_p d_p^*, \; p_d c_d - p_n c_n = d_p d_{c0}. \tag{4.16}$$

Since $\mathtt{C}_0$ is a controller vector for *RRIS*, due to the internal model principle,

$$c_d(z_i) = 0 \text{ for } z_i \text{ such that } H(z_i) = 0. \tag{4.17}$$

Due to (4.16), the function T_1 can be written in two ways as

$$T_1 := B\mathtt{P}^*\mathtt{C}_0 = -\frac{p_d^*}{p_n} + \frac{d_p^* c_d}{p_n d_{c0}} \tag{4.18}$$

$$= \frac{p_n^*}{p_d} + \frac{d_p^* c_n}{p_d d_{c0}}. \tag{4.19}$$

If z_i is zero of H, then $c_d(z_i) = 0$ (including multiplicities) from (4.17). In this case, since $p_n(z_i) \neq 0$ from Assumption 2.1, the second term of (4.18) vanishes at z_i (until the derivatives of the multiplicity minus one).

If z_i is zero of B, then $d_p^*(z_i) = 0$ (including multiplicities). The case of $p_n(z_i) \neq 0$ can be proven as above. If $p_n(z_i) = 0$, then $p_d(z_i) \neq 0$ by the coprimeness of p_n/p_d. Thus, the second term of (4.19) vanishes at z_i (until the derivatives of the multiplicity minus one). □

Second lemma states that the controller vectors in $\mathcal{S}_{RRRS(\gamma)}$ can explicitly be written in terms of only the plant and the interpolant T in $\mathcal{T}$, and it includes neither $\mathtt{C}_0$ nor the denominator of H.

Lemma 4.3. *Write each element in $\mathcal{T}$ as a ratio of coprime polynomials as $T(s) = t_n(s)/t_d(s)$. Then, the corresponding controller vector can be expressed by*

$$\mathtt{C}(s) = \frac{1}{t_d(s)d_p^*(s)}\begin{bmatrix} t_n(s)p_n(s) + t_d(s)p_d^*(s) \\ t_n(s)p_d(s) - t_d(s)p_n^*(s) \end{bmatrix}.$$

Proof. From (4.14), $H\Psi = B^{-1}(T_1 - T) = \mathtt{P}^*\mathtt{C}_0 - B^{-1}T$. Therefore, due to (4.16), the controller vector is derived as

$$\mathtt{C} = \mathtt{C}_0 - \mathtt{P}H\Psi = \mathtt{C}_0 - \mathtt{P}\left(\mathtt{P}^*\mathtt{C}_0 - B^{-1}T\right) = \frac{1}{t_d d_p^*}\begin{bmatrix} t_n p_n + t_d p_d^* \\ t_n p_d - t_d p_n^* \end{bmatrix}.$$

This completes the proof. □

Hence, we have obtained our first main result as follows.

Theorem 4.2. *The controller vector set for $RRRS(\gamma)$ can be simply written by*

$$\mathcal{S}_{RRRS(\gamma)} = \left\{ \mathtt{C} := \frac{1}{t_d d_p^*} \begin{bmatrix} t_n p_n + t_d p_d^* \\ t_n p_d - t_d p_n^* \end{bmatrix},\ t_n/t_d \in \mathcal{T} \right\}, \tag{4.20}$$

where $\mathcal{T}$ is defined in (4.15) (in the distinct case).

In other words, the set $\mathcal{S}_{RRRS(\gamma)}$ is characterized completely by the real rational solution set of the boundary Nevanlinna-Pick interpolation problem, and independent of the choice of $\mathtt{C}_0$ and the denominator of H.

Define two functions:

$$\hat{c}_n := \frac{t_n p_d - t_d p_n^*}{d_p^*},\ \ \hat{c}_d := \frac{t_n p_n + t_d p_d^*}{d_p^*}.$$

Then, it can be proven that both $\hat{c}_n$ and $\hat{c}_d$ are polynomials due to the cancellations caused by interpolation conditions (see Lemma 4.2). The cancellations are done numerically since we cannot write explicitly these polynomials. For each controller vector $\mathtt{C}$ in $\mathcal{S}_{RRRS(\gamma)}$, the corresponding controller for $RRRS(\gamma)$ is obtained by $C(s) = \hat{c}_n(s)/\hat{c}_d(s)$.

5 Degree constraint on controllers

From the practical point of view, controllers with low complexities are desirable than controllers with high complexities. We shall give here one method to derive a controller set for $RRRS(\gamma)$ with a reasonably low degree bound.

Intuitively, we can expect that, if we bound the degree of T in $\mathcal{T}$, we can have a certain bound of the degree of a controller $C = \hat{c}_n/\hat{c}_d$. In fact, the degree of T itself becomes the degree bound of C, which is the second main result in this paper.

Proposition 5.1. *Let C be any controller for $RRRS(\gamma)$ obtained by the approach in the previous section. Then, the degree relation $\deg C \le \deg T$ holds.*

Proof. Noting the possibility of pole-zero cancellations between $\hat{c}_n$ and $\hat{c}_d$, we have

$$\deg C \le \deg \hat{c}_d = \deg(t_n p_n + t_d p_d^*) - \deg d_p^* = \deg t_d = \deg T.$$

Here, we have used the relation in (2.4). □

Owing to this proposition, it is meaningful to impose degree constraint on T. To be more specific, we focus on the following subset of $\mathcal{T}$:

$$\hat{\mathcal{T}} := \mathcal{T} \cap \{T :\ \deg T \le n_p + n_h - 1\},$$

where n_p and n_h are defined in (4.13). The number n_h is the degree of the minimal polynomial of A_{22}, which is the "A–matrix" of the exosystem. Note that the number of interpolation constraints in $\mathcal{T}$ is $n_p + n_h$. The reason why we choose this bound is that we can guarantee the nonempty-ness of the set $\hat{\mathcal{T}}$ whenever $\mathcal{T}$ is nonempty.

The controller vector set for $RRRS(\gamma)$ with this degree bound can be written, by using $\hat{\mathcal{T}}$ instead of $\mathcal{T}$, as

$$\hat{\mathcal{S}}_{RRRS(\gamma)} = \left\{ \mathtt{C} := \frac{1}{t_d d_p^*} \begin{bmatrix} t_n p_n + t_d p_d^* \\ t_n p_d - t_d p_n^* \end{bmatrix}, \ t_n/t_d \in \hat{\mathcal{T}} \right\}. \tag{5.21}$$

Due to Proposition 5.1, each controller which is generated by each controller vector in $\hat{\mathcal{S}}_{RRRS(\gamma)}$ satisfies not only $RRRS(\gamma)$ conditions but also the degree bound

$$\deg C \le n_p + n_h - 1. \tag{5.22}$$

This bound is reasonable in the sense that (1) it is necessary for the controller to have degree at least n_h in order to achieve $RRIS$, due to the internal model principle, and (2) there always exists a controller of degree $n_p - 1$ which achieves $RS(\gamma)$ if γ is chosen appropriately.

The set $\hat{\mathcal{T}}$ is completely characterized by the theory of Nevanlinna-Pick interpolation with degree constraint presented in [3][2], where each element in the set $\hat{\mathcal{T}}$ is determined by means of the $n_p + n_h - 1$ self-conjugate spectral zeros in the open left half-plane of the function $\frac{1-\gamma^2}{\gamma^2} - T(s)T(-s)$. Also, an efficient algorithm for calculating each interpolant in $\hat{\mathcal{T}}$ is available (see [14]). Therefore, by searching for an appropriate controller vector in the set $\hat{\mathcal{S}}_{RRRS(\gamma)}$, we may obtain a controller with a reasonable degree bound satisfying other performance specifications than robust regulation and robust stability. This will be demonstrated with a numerical example in the next section.

6 A numerical example

Consider the standard feedback system depicted in Figure 2. As usual, P is a given plant and C is a controller to be designed. Exosystems 1, 2 and 3 generate reference signal to be tracked, disturbance to the states of P and disturbance at the output of P, respectively. In our setting, frequency information of these signals has to be available.

Let us denote the state vectors of the plant $P(s) := C_p(sI - A_p)^{-1}B_p$ and of the exosystems 1, 2 and 3 by x_p and r, w and v, respectively. Using these vectors, the system equations are

[2]Here, we need slight modification of the theory in [3], since the theory does not cover the interpolation problems with boundary interpolation constraints. We will take the analytic region slightly larger than the right half-plane by a variable change, apply the computational procedure in [14] to obtain the interpolant, and change again a variable reversely.

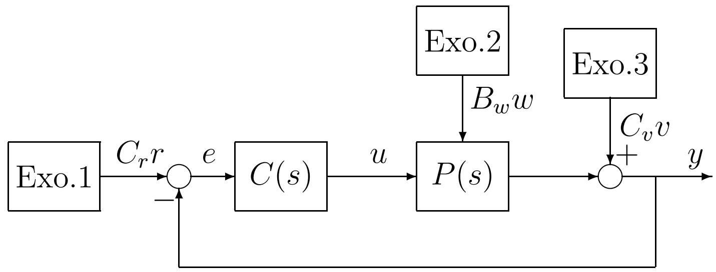


Figure 2: Feedback system

assumed to be expressed by

$$\begin{cases} \dot{x}_p = A_p x_p + B_p u + B_w w, \\ \begin{bmatrix} \dot{r} \\ \dot{w} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} A_r & & \\ & A_w & \\ & & A_v \end{bmatrix} \begin{bmatrix} r \\ w \\ v \end{bmatrix}, \\ y = C_p x_p + C_v v, \\ e = C_r r - y, \end{cases}$$

where the matrices (A_r, B_w etc.) have appropriate sizes. The system can be rewritten in a form corresponding to (2.1) as

$$\begin{cases} \begin{bmatrix} \dot{x}_p \\ \begin{bmatrix} \dot{r} \\ \dot{w} \\ \dot{v} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} A_p & \begin{bmatrix} 0 & B_w & 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} A_r & & \\ & A_w & \\ & & A_v \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_p \\ \begin{bmatrix} r \\ w \\ v \end{bmatrix} \end{bmatrix} + \begin{bmatrix} B_p \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} u \\ e = \begin{bmatrix} -C_p & \begin{bmatrix} C_r & 0 & -C_v \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_p \\ \begin{bmatrix} r \\ w \\ v \end{bmatrix} \end{bmatrix} \end{cases}$$

Now, our control requirements are *robust regulation*: the error signal e in Figure 2 goes to zero asymptotically in the face of small perturbations of A_p, B_w and B_p, and *robust stability*: C stabilizes the closed-loop system robustly, that is, C internally stabilizes each plant in the set with radius γ in the gap metric for some $\gamma \in (0,1)$.

We set the plant to

$$P(s) = -\frac{s+2}{s^2+s-2} = \underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}}_{C_p} \left(sI - \underbrace{\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}}_{A_p} \right)^{-1} \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{B_p},$$

and Exosystem 1, 2 and 3 respectively to a generator of the step function, the sinusoidal signal with frequency one rad/sec and the sinusoidal signal with frequency two rad/sec, i.e.,

$$A_r = 0,\ C_r = 1,\ A_w = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix},\ B_w = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix},\ A_v = \begin{bmatrix} & -2 \\ 2 & \end{bmatrix},\ C_v = \begin{bmatrix} 3 & 1 \end{bmatrix}.$$

With $\gamma = 0.2$, the central controller and the controller in the set $\hat{\mathcal{S}}_{RRRS(\gamma)}$ in (5.21) obtained by specifying all the spectral zeros of $\frac{1-\gamma^2}{\gamma^2} - T(s)T(-s)$ at -1 are respectively calculated as

$$C_0(s) := \frac{2.55s^6 + 5.27s^5 + 12.54s^4 + 24.72s^3 + 10.23s^2 + 19.04s + 0.10}{s^6 + 1.99s^5 + 5.00s^4 + 9.96s^3 + 4.00s^2 + 7.97s}, \qquad (6.23)$$

$$C_1(s) := \frac{4.67s^6 + 13.44s^5 + 26.80s^4 + 44.48s^3 + 24.70s^2 + 21.37s + 2.25}{s^6 + 2.00s^5 + 5.00s^4 + 9.98s^3 + 4.00s^2 + 7.99s}. \qquad (6.24)$$

We can verify that both controllers have poles at $s = 0, \pm 1i, \pm 2i$, which is consistent with the internal model principle. In addition, the degree of these controllers is six, which satisfies (5.22) since $\deg P = 2$ and the degree sum of exosystems is five.

For these two controllers, nominal performances are compared by drawing the error signals e, as shown in Figure 3. Here, the initial states of the plants and the controllers were set to zero, while all of the initial states of the exosystems were set to one. As can be seen in this figure, although the central controller makes the closed-loop system satisfy nominal regulation property, the speed of convergence is very slow. On the other hand, the controller C_1 provides a much better nominal performance. Of course, the nominal performance is determined by the location of closed-loop poles. The closed-loop pole locations are also shown in Figure 3, where we can see that, with the central controller, some of the closed-loop poles are almost on the imaginary axis, which causes the slow convergence of the error signal e. It can be verified that the closed-loop poles are the roots of

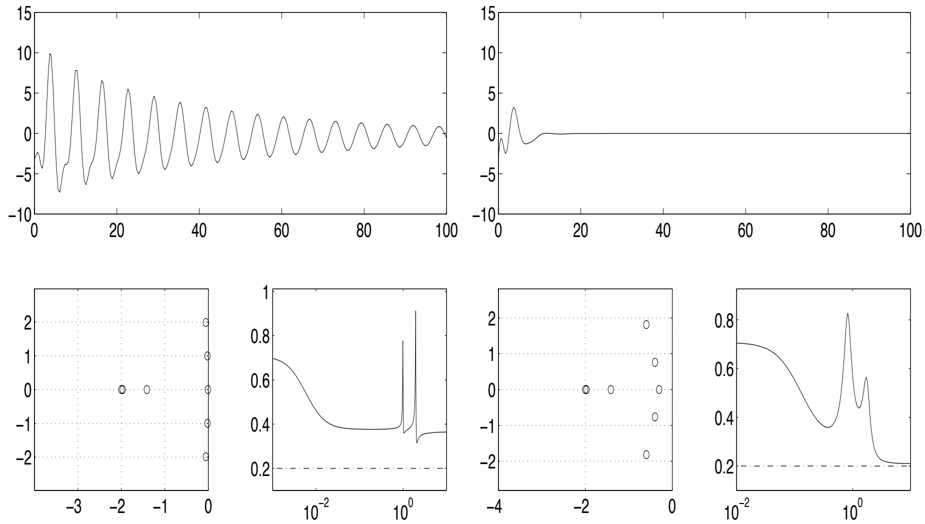
$$p_d(s)\hat{c}_d(s) - p_n(s)\hat{c}_n(s) = \frac{t_d(s)}{d_p^*(s)}(p_d(s)p_d^*(s) + p_n(s)p_n^*(s)) = t_d(s)d_p(s).$$

Since d_p is determined from the given plant P, what we can adjust for enhancement of nominal regulation property is the roots of t_d, or equivalently, the poles of the solution T of the Nevanlinna-Pick interpolation problem. However, at present, the pole placement problem based on the Nevanlinna-Pick interpolation with degree constraint is open.

Finally, we depict the frequency-dependent stability margin $|\mathtt{C}(i\omega)|^{-1}$ in Figure 3. It can be verified that the stability margin requirement $\|\mathtt{C}\|_\infty^{-1} > 0.2$ is satisfied.

7 Conclusions

This paper has dealt with a problem of simultaneous robust regulation and robust stabilization ($RRRS$), in the context of the one considered by Cevik and Schumacher in [6]. There are two contributions of this paper. First, we have given, in a scalar case, a complete parameterization of the set of the controller vectors for $RRRS$, without involving any initial



(a) Central controller (6.23) (b) Better controller (6.24)

Figure 3: The error signals e (upper figures), the closed-loop pole locations (lower-left figures) and stability margins (lower-right figures)

controller or any free stable polynomial which are necessary for the approach proposed in [6]. The set is characterized by the solution set of the boundary Nevanlinna-Pick interpolation problem. Secondly, we have restricted the set of controller vectors for $RRRS$ with a reasonable degree bound. The complete characterization of this set attributes to the theory of Nevanlinna-Pick interpolation with degree constraint by Byrnes, Georgiou and Lindquist in [3]. Each element in the set is computable by the algorithm developed in [14]. With this complete characterization, we may incorporate other specifications than $RRRS$ without increasing the degree of controllers. Concerning this point, we have given one numerical example to illustrate the potential of the performance enhancement by choosing different controllers with the same degree bound.

Although we have characterized a controller set whose elements satisfy both $RRRS$ condition and degree constraint in terms of spectral zeros of a certain function, tuning rules of the design parameters are unclear. In the example, we have revealed that, for the improvement of nominal performance, we need to develop the pole placement in the context of the theory of Nevanlinna-Pick interpolation with degree constraint. In addition, the extension of the results in this paper to multivariable cases are also future research subject.

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